TWISTED TORIC STRUCTURES

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Abstract. This paper introduces the notion of twisted toric manifolds which is a generalization of one of symplectic toric manifolds, and proves the weak Delzant type classification theorem for them. Their fundamental groups are investigated. Methods of computing their cohomology groups in general dimensional cases and signatures in four-dimensional cases are also given.

Contents

1. Introduction

By Delzant's classification theorem [4], there is a one-to-one correspondence between a symplectic toric manifold which is one of the special objects in the theory of Hamiltonian torus actions and a Delzant polytope which is a combinatorial object. Through this correspondence, various researches on the relationship between symplectic geometry, topology, and transformation groups with combinatorics have been done [2, 4, 7].

On the other hand, there exists a manifold such that it may not be itself a symplectic toric manifold, but it has a toric structure in a neighborhood of each point all of which are patched together in some weak condition. In this paper, as a formulation of such manifolds, we shall introduce the notion of twisted toric

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manifolds and generalize the weak version of Delzant's classification theorem to them. Recently, some generalizations are also considered [11, 12, 17, 18, 19]. We also investigate the topology of twisted toric manifolds. As a result, we can see that there are examples of twisted toric manifolds which are not complete non-singular toric varieties in the original algebro-geometric sense. In particular, these are not symplectic toric manifolds.

In general, a twisted toric manifold no longer has a global torus action like that of a original symplectic toric manifold, but it has a torus action on a neighborhood of each point which comes from a local toric structure and they are patched together in certain sense. One of our motivation is to generalize the topological theory of transformation group to such a *twisted* torus action. Some invariants for transformation groups such as equivariant cohomology groups can be generalized to this case and we are investigating their properties, in particular, localizations. Unfortunately we could not describe this topic in this paper. This will appear later on.

This paper is organized as follows. First, we recall Hamiltonian torus actions in Section 2 and symplectic toric manifolds in Section 3 in order that the paper is self-contained. Then we shall give the definition of the twisted toric manifold and some examples in Section 4. Section 5 is devoted to the classification of twisted toric manifolds. In Section 6, We shall investigate their fundamental groups and compute their cohomology groups. We shall also compute their signatures in four-dimensional cases.

In the rest of this paper, we shall assume that all manifolds are compact, connected, and oriented and all maps preserve orientations, unless otherwise stated.

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2. Hamiltonian torus actions

2.1. **Moment maps.** A *symplectic manifold* (X, ω) is a smooth manifold X equipped with a non-degenerate closed 2-form ω. Let us assume that a k-dimensional torus T^k acts on X which preserves ω. In this paper, we identify T^k with $\mathbb{R}^k/\mathbb{Z}^k$, and its Lie algebra $\mathfrak{t}$ with $\mathbb{R}^k$. By the natural inner product $\langle , \rangle$ on $\mathbb{R}^k$, we also identify the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ with $\mathfrak{t}$ itself.

Definition 2.1 ([2, 9])**.** A *moment map* for the T^k-action is a map $\mu : X \to \mathfrak{t}^*$ which is T^k-invariant with respect to the given T^k-action on X and satisfies the condition

$$\iota(v_\xi)\omega = \langle d\mu, \xi \rangle$$

for $\xi \in \mathfrak{t}$, where v_ξ is the infinitesimal action, that is, the vector field which is defined by

$$v_\xi(x) = \frac{d}{d\tau}\Big|_{\tau=0} e^{2\pi\sqrt{-1}\tau\xi} \cdot x.$$

Note that a moment map for a T^k-action is determined up to an additive constant. The torus action which has a moment map is said to be *Hamiltonian.*

Although these are not compact, the following examples are fundamental in this paper.

Example 2.2. Let $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ be the n-dimensional complex vector space with the symplectic form $\omega_{\mathbb{C}^n} = \dfrac{-\sqrt{-1}}{2\pi}\displaystyle\sum_{i=1}^n dz_i \wedge d\bar{z}_i$. T^n acts on $\mathbb{C}^n$ by

$$t \cdot z = (e^{2\pi\sqrt{-1}t_1} z_1, \ldots, e^{2\pi\sqrt{-1}t_n} z_n)$$

for $t = (t_1, \ldots, t_n) \in T^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. This action is Hamiltonian and a moment map $\mu_{\mathbb{C}^n} : \mathbb{C}^n \to \mathfrak{t}^*$ is defined by

$$\mu_{\mathbb{C}^n}(z) = (|z_1|^2, \ldots, |z_n|^2).$$

In particular, the image of $\mu_{\mathbb{C}^n}$ is

$$\mathfrak{t}^*_{\geq 0} = \{\xi = (\xi_1, \ldots, \xi_n) \in \mathfrak{t}^* = \mathbb{R}^n : \xi_i \geq 0 \text{ for } i = 1, \ldots, n\}.$$

Example 2.3. Let T^*T^n be the cotangent bundle of T^n. On T^*T^n, we fix the natural trivialization $T^*T^n \cong \mathfrak{t}^* \times T^n$ and identify T^*T^n with $\mathfrak{t}^* \times T^n$ by this trivialization. T^*T^n has a symplectic form $\omega_{T^*T^n} = \displaystyle\sum_{i=1}^n d\theta_i \wedge d\xi_i$, where $\theta = (\theta_1, \ldots, \theta_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ denote the standard coordinates of T^n and $\mathfrak{t}^* \cong \mathbb{R}^n$, respectively. T^n acts on T^*T^n by

$$t \cdot (\xi, \theta) = (\xi, \theta + t)$$

for $t \in T^n$ and $(\xi, \theta) \in T^*T^n$. This action is Hamiltonian and a moment map $\mu_{T^*T^n} : T^*T^n \to \mathfrak{t}^*$ is defined by

$$\mu_{T^*T^n}(\xi, \theta) = \xi.$$

2.2. **Symplectic reduction.** Let (X, ω) be a $2n$-dimensional symplectic manifold equipped with a Hamiltonian T^k-action with a moment map $\mu : X \to \mathfrak{t}^*$. There is a method, so called a *symplectic reduction*, to construct a new symplectic manifold which we shall explain. See [2, 9] for more details.

Proposition 2.4. *Let $T^k_x (\subset T^k)$ be the stabilizer of $x \in X$. Then the annihilator $(\operatorname{Im} d\mu_x)^\perp$ of the image of $d\mu_x : T_xX \to \mathfrak{t}^*$ is isomorphic to the Lie algebra $\mathfrak{t}_x$ of T^k_x.*

Proof. It is clear from the condition in Definition 2.1 and the non-degeneracy of ω. □

Let $\varepsilon \in \mathfrak{t}^*$. Since μ is invariant under the action, T^k-action preserves $\mu^{-1}(\varepsilon)$. Suppose that T^k-action on $\mu^{-1}(\varepsilon)$ is free. Then Proposition 2.4 implies that the level set $\mu^{-1}(\varepsilon)$ is smooth, and the quotient space $\mu^{-1}(\varepsilon)/T^k$ is a $2(n-k)$-dimensional smooth manifold. In this case, the following proposition is well known.

Proposition 2.5 ([16])**.** *The quotient space $\mu^{-1}(\varepsilon)/T^k$ carries a natural symplectic form ω_ε such that the equality $\iota^*\omega = \pi^*\omega_\varepsilon$ holds, where ι is a natural inclusion and π is a projection*

$$\begin{array}{ccc} (\mu^{-1}(\varepsilon), \iota^*\omega) & \xhookrightarrow{\iota} & (X, \omega) \\ \downarrow{\scriptstyle \pi} & & \\ (\mu^{-1}(\varepsilon)/T^k, \omega_\varepsilon). & & \end{array}$$

$(\mu^{-1}(\varepsilon)/T^k, \omega_\varepsilon)$ is called a *symplectic quotient*.

2.3. **Symplectic cutting.** Let us recall the symplectic cutting by Lerman [13]. Suppose that (X, ω) is a symplectic manifold equipped with a Hamiltonian S^1-action with a moment map $\mu : X \to \mathbb{R}$. Define the S^1-action on the product space $(X \times \mathbb{C}, \omega \oplus \omega_{\mathbb{C}})$ by

$$t \cdot (x, z) = (t \cdot x, e^{-2\pi\sqrt{-1}t} z)$$

for $t \in S^1$ and $(x, z) \in X \times \mathbb{C}$. This action is Hamiltonian and the moment map $\Phi : X \times \mathbb{C} \to \mathbb{R}$ is

$$\Phi(x, z) = \mu(x) - \mu_{\mathbb{C}}(z) = \mu(x) - |z|^2.$$

Proposition 2.6. *Let $\varepsilon \in \mathbb{R}$. the S^1-action on $\Phi^{-1}(\varepsilon)$ is free, if and only if the S^1-action on $\mu^{-1}(\varepsilon)$ is free.*

Proof. Let $(x, z) \in \Phi^{-1}(\varepsilon)$. If $z \neq 0$, then the stabilizer of (x, z) for the S^1-action on $X \times \mathbb{C}$ only consists of the unit element since the stabilizer of z for the S^1-action on $\mathbb{C}$ only consists of the unit element. In the case where $z = 0$, the stabilizer of (x, z) for the S^1-action on $X \times \mathbb{C}$ is equal to that of x for the S^1-action on X. This proves the proposition. □

Assume that the S^1-action on $\Phi^{-1}(\varepsilon)$ is free. Then the reduced space $\Phi^{-1}(\varepsilon)/S^1$ is a smooth manifold whose dimension is equal to that of X. Let us consider the reduced space $\Phi^{-1}(\varepsilon)/S^1$. The level set $\Phi^{-1}(\varepsilon)$ is a disjoint union of two S^1 invariant parts

$$\Phi^{-1}(\varepsilon) = \left\{(x, z) \in X \times \mathbb{C} \colon \mu(x) > \varepsilon, \quad |z|^2 = \mu(x) - \varepsilon\right\} \amalg \mu^{-1}(\varepsilon) \times \{0\}.$$

The first part is equivariantly diffeomorphic to the product $\{x \in X \colon \ \mu(x) > \varepsilon\} \times S^1$, and the second part is naturally identified with $\mu^{-1}(\varepsilon)$. Then as a set, the quotient space $\Phi^{-1}(\varepsilon)/S^1$ is the disjoint union

$$\Phi^{-1}(\varepsilon)/S^1 \cong \{x \in X \colon \mu(x) > \varepsilon\} \amalg \mu^{-1}(\varepsilon)/S^1.$$

We denote $\Phi^{-1}(\varepsilon)/S^1$ by $\overline{X}_{\mu \geq \varepsilon}$. Topologically, $\overline{X}_{\mu \geq \varepsilon}$ is the quotient of the manifold $X_{\mu \geq \varepsilon} = \{x \in X \colon \mu(x) \geq \varepsilon\}$ with the boundary $\mu^{-1}(\varepsilon)$ by the relation $\sim$, where $x \sim x'$ if and only if $x, x' \in \mu^{-1}(\varepsilon)$ and $x' = t \cdot x$ for some $t \in S^1$. For this reason, we would like to call the operation that produces $\overline{X}_{\mu \geq \varepsilon}$ from the Hamiltonian S^1-action on (X, ω) *symplectic cutting* and $\overline{X}_{\mu \geq \varepsilon}$ is called a *cut space*.

Remark 2.7. Suppose that (X, ω) has another Hamiltonian T^k-action which commutes with the S^1-action. Then the T^k-action on X induces the Hamiltonian T^k-action on $\overline{X}_{\mu \geq 0}$ simply by letting it act on the first factor.

Example 2.8. Let us consider Example 2.3 for $n = 1$. By Remark 2.7, the cut space $\overline{T^*S^1}_{\mu_{T^*S^1} \geq 0}$ has the Hamiltonian circle action which is induced by the original circle action on T^*S^1. In this case, $\overline{T^*S^1}_{\mu_{T^*S^1} \geq 0}$ is equivariantly symplectomorphic to $(\mathbb{C}, \omega_{\mathbb{C}})$ with the circle action in Example 2.2 by the symplectomorphism $\varphi : \mathbb{C} \to \overline{T^*S^1}_{\mu_{T^*S^1} \geq 0}$ which is defined by

$$\varphi(z) = [|z|^2, \frac{\arg z}{2\pi}, |z|].$$

Example 2.9. Let $u \in \mathbb{Z}^n \subset \mathfrak{t}$, and consider the Hamiltonian S^1-action on $(T^*T^n, \omega_{T^*T^n})$ which is defined by

$$t \cdot (\xi, \theta) = (\xi, \theta + tu)$$

for $t \in S^1 = \mathbb{R}/\mathbb{Z}$ and $(\xi, \theta) \in T^*T^n$. The moment map $\mu_u : T^*T^n \to \mathbb{R}$ is obtained by

$$\mu_u(\xi, \theta) = \langle u, \xi \rangle .$$

In this case,

$$\overline{(T^*T^n)}_{\mu_u \geq \varepsilon} \cong \{\xi \in \mathfrak{t}^* \colon \langle u, \xi\rangle > \varepsilon\} \amalg \{\xi \in \mathfrak{t}^* \colon \langle u, \xi\rangle = \varepsilon\} \times T^n/S^1_u,$$

where S^1_u is the circle subgroup of T^n generated by u. Note that $\overline{(T^*T^n)}_{\mu_u \geq \varepsilon}$ is smooth, if and only if u is primitive in the sense of Definition 3.3. For more details, see Appendix A. In this case, since the Hamiltonian T^n-action in example 2.3 commutes with the S^1-action, by Remark 2.7, the Hamiltonian T^n-action is induced to $\overline{(T^*T^n)}_{\mu_u \geq \varepsilon}$ with the moment map

$$\overline{\mu_{T^*T^n}}([\xi, \theta, z]) = \xi.$$

Figure 1 shows the change of the image of the moment map under symplectic cutting for $n = 2$.

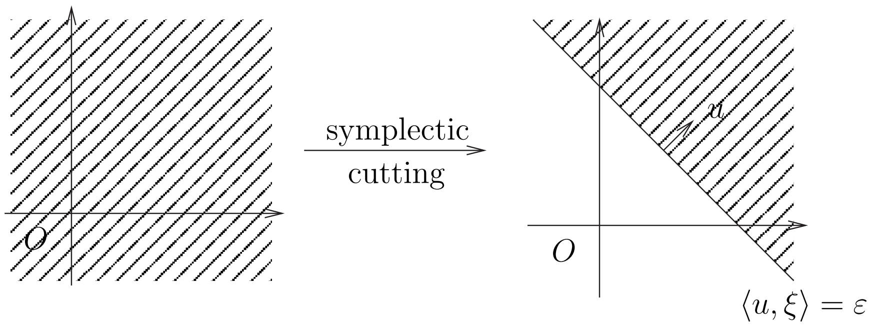


FIGURE 1. the change of the moment image by symplectic cutting

Remark 2.10 (Simultaneous symplectic cuttings)**.** Suppose that (X, ω) is equipped with two commutative Hamiltonian S^1-actions on (X, ω) with moment maps μ_1 and μ_2. Then corresponding symplectic cutting operations also commute each other, and the both cut spaces $\overline{(\overline{X}_{\mu_1 \geq \varepsilon_1})}_{\mu_2 \geq \varepsilon_2}$ and $\overline{(\overline{X}_{\mu_2 \geq \varepsilon_2})}_{\mu_1 \geq \varepsilon_1}$ are naturally symplectomorphic to the cut space

$$\overline{X}_{\{\mu_i \geq \varepsilon_i\}_{i=1,2}} = \{(x, z) \in X \times \mathbb{C}^2 \colon \mu_i(x) - |z_i|^2 = \varepsilon_i,\ i = 1, 2\}/T^2$$

of the simultaneous symplectic cuttings, that is, the symplectic quotient of the Hamiltonian T^2-action on $(X \times \mathbb{C}^2, \omega \oplus \omega_{\mathbb{C}^2})$ with the moment map

$$\Phi(x, z) = (\mu_1(x) - |z_1|^2, \mu_2(x) - |z_2|^2)$$

which is obtained by putting two S^1-actions together.

More generally, for the case where (X, ω) is equipped with k commutative Hamiltonian S^1-actions with moment maps μ_i for $i = 1, \dots, k$, the argument goes similar way, and the cut spaces are naturally symplectomorphic to the simultaneous cut space

$$\overline{X}_{\{\mu_i \geq \varepsilon_i\}_{i=1,\dots,k}} = \{(x, z) \in X \times \mathbb{C}^k \colon \mu_i(x) - |z_i|^2 = \varepsilon_i,\ i = 1, \dots, k\}/T^k.$$

3. Symplectic toric manifolds

For Hamiltonian torus actions, the following fact is well known.

Theorem 3.1 ([7])**.** *If a k-dimensional torus T^k acts effectively on a $2n$-dimensional symplectic manifold (X, ω) in a Hamiltonian fashion, then $k \leq n$.*

In particular, in the maximal case of Theorem 3.1, that is, a closed, connected $2n$-dimensional symplectic manifold (X, ω) equipped with an effective Hamiltonian T^n-action is called a *symplectic toric manifold*.

For a general Hamiltonian torus action, Atiyah and Guillemin-Sternberg show the following convexity theorem.

Theorem 3.2 ([1, 8])**.** *Let (X, ω) be a closed, connected $2n$-dimensional symplectic manifold equipped with a Hamiltonian action of k-dimensional torus. (We do not require $k = n$.) Then the image of a moment map is a convex hull of images of fixed points.*

In the theory of symplectic toric manifolds, the image of a moment map plays a crucial role.

Definition 3.3. Let $\{u_1, \dots, u_k\}$ is a tuple of vectors of $\mathbb{Z}^n$. $\{u_1, \dots, u_k\}$ is said to be *primitive*, if the sub-lattice $\mathrm{span}_{\mathbb{Z}}\{u_1, \dots, u_k\}$ spanned by $u_1, \dots, u_k$ is a rank k direct summand of the free $\mathbb{Z}$-module $\mathbb{Z}^n$. We also say that the tuple $\{L_1, \dots, L_k\}$ of rank one sub-lattices in $\mathbb{Z}^n$ is *primitive*, if there exists a primitive tuple $\{u_1, \dots, u_k\}$ of vectors in $\mathbb{Z}^n$ such that each L_i is spanned by u_i for $i = 1, \dots, k$.

Remark 3.4. The notion of the primitivity of a tuple $\{u_1, \dots, u_k\}$ of vectors (hence the tuple $\{\mathbb{Z}_1, \dots, \mathbb{Z}_k\}$ of rank one sub-lattices) in $\mathbb{Z}^n$ is invariant under the action of $GL_n(\mathbb{Z})$.

Let Δ be a convex polytope in $\mathfrak{t}^*(= \mathbb{R}^n)$ in $\mathfrak{t}^*$ which is written by

$$\Delta = \bigcap_{i=1}^{d} \{\xi \in \mathfrak{t}^* \colon \langle u_i, \xi \rangle \geq \lambda_i\} \tag{3.1}$$

for $u_1, \dots, u_d \in \mathbb{R}^n \subset \mathfrak{t}$ and $\lambda_1, \dots, \lambda_d \in \mathbb{R}$. Without loss of generality, we may assume that for $i = 1, \dots, d$, each intersection $\Delta \cap \{\xi \in \mathfrak{t}^* \colon \langle u_i, \xi \rangle = \lambda_i\}$ of Δ and the hyperplane defined by $\langle u_i, \xi \rangle = \lambda_i$ is a *facet*, that is, a codimension one face of Δ. We set

$$\mathfrak{J} = \left\{ I \subset \{1, \dots, d\} \colon \bigcap_{i \in I} \{\xi \in \Delta \colon \langle u_i, \xi \rangle = \lambda_i\} \neq \emptyset \right\}.$$

Definition 3.5. The convex polytope Δ is said to be *Delzant*, if Δ satisfies the following conditions

(i) Δ is *rational*, that is, $u_1, \dots, u_d \in \mathbb{Z}^n \subset \mathfrak{t}$,
(ii) Δ is *simple*, that is, exactly n facets meet at each vertex of Δ,
(iii) Δ is *non-singular*, that is, $u_1, \dots, u_d$ (hence $\lambda_1, \dots, \lambda_d$) in (3.1) can be taken so that $\{u_i\}_{i \in I}$ is primitive for each $I \in \mathfrak{J}$.

In the rest of this paper, when we say that a convex polytope Δ written as in (3.1) is Delzant, we assume that $u_1, \dots, u_d$ are taken so that $\{u_i\}_{i \in I}$ is primitive for each $I \in \mathfrak{J}$.

Delzant shows that a symplectic toric manifold is exactly determined by the image of its moment map.

Theorem 3.6 ([4])**.** (1) *The image of a moment map of a symplectic toric manifold is a Delzant polytope.*
(2) *By associating the image of a moment map to a symplectic toric manifold, the set of equivariantly symplectomorphism classes of $2n$-dimensional symplectic toric manifolds corresponds one-to-one to the set of Delzant polytopes in $\mathfrak{t}^* \cong \mathbb{R}^n$ up to parallel transport in $\mathfrak{t}^*$.*

Theorem 3.6 says that a symplectic toric manifold is recovered from a Delzant polytope. It is done as follows. Let Δ be an n-dimensional Delzant polytope defined by (3.1). As in Example 2.9, for $i = 1, \ldots, d$, each vector u_i in (3.1) defines the circle actions on $(T^*T^n, \omega_{T^*T^n})$ with the moment maps $\mu_{u_i}(\xi, \theta) = \langle u_i, \xi \rangle$. Since these actions commute each other, we can obtain the simultaneous cut space $X_\Delta = \overline{(T^*T^n)}_{\{\mu_{u_i} \geq \lambda_i\}_{i=1,\ldots,d}}$ as we described in Remark 2.10. By the definition of the Delzant polytope and Theorem A.3 in Appendix A, X_Δ is a $2n$-dimensional smooth symplectic manifold. Moreover by Remark 2.7, X_Δ is equipped with the Hamiltonian T^n-action with the moment map $\mu : X_\Delta \to \mathfrak{t}^*$ which is induced from the natural T^n-action on T^*T^n in Example 2.3. It is clear that this action is effective and the image of μ is Δ. Hence the cut space X_Δ is the symplectic toric manifold which we want.

From this construction, we can see that a symplectic toric manifold is locally identified with the Hamiltonian T^n-action on $\mathbb{C}^n$ in Example 2.2 in the following sense.

Definition 3.7. Let ρ be an automorphism of T^n. Two $2n$-dimensional symplectic toric manifolds (X_1, ω_1) and (X_2, ω_2) are *ρ-equivariantly symplectomorphic*, if there exists a symplectomorphism $\varphi : (X_1, \omega_1) \to (X_2, \omega_2)$ such that φ satisfies the two conditions

(i) $\varphi(t \cdot x) = \rho(t) \cdot \varphi(x)$ for $x \in X_1$ and $t \in T^n$,

(ii) the following diagram is commutative

$$\begin{tikzcd} X_1 \arrow[r, "\varphi"] \arrow[d, "\mu_1"'] \arrow[dr, phantom, "\circlearrowleft"] & X_2 \arrow[d, "\mu_2"] \\ \mathfrak{t}^* & \mathfrak{t}^*, \arrow[l, "(d\rho)^*"'] \end{tikzcd}$$

where μ_1 and μ_2 are moment maps of X_1 and X_2, respectively.

For a vertex $v \in \Delta$ which is defined by the exactly n equalities $\langle u_{i_a}, \xi \rangle = \lambda_{i_a}$ for $a = 1, \ldots, n$, we define the open set $U_v \subset \Delta$ by

$$U_v = \{\xi \in \Delta \colon \langle u_i, \xi \rangle > \lambda_i \text{ for } i \neq i_1, \ldots, i_n\}.$$

Then $\{\mu^{-1}(U_v)\}_{v:\text{vertex of }\Delta}$ is a open covering of X_Δ. Moreover, for each vertex v, there exists an automorphism ρ_v of T^n such that $(d\rho_v^{-1})^* : \mathfrak{t}^* \to \mathfrak{t}^*$ sends U_v diffeomorphically to the open set $(d\rho_v^{-1})^*(U_v)$ in $\mathfrak{t}^*_{\geq 0}$ and $\mu^{-1}(U_v)$ is ρ_v-equivariantly symplectomorphic to $\mu_{\mathbb{C}^n}^{-1}((d\rho_v^{-1})^*(U_v))$

$$\begin{tikzcd} X_\Delta \supset \arrow[d, "\mu"'] & \mu^{-1}(U_v) \arrow[d, "\mu"'] & \cong \arrow[d, phantom, "\circlearrowleft"] & \mu_{\mathbb{C}^n}^{-1}((d\rho_v^{-1})^*(U_v)) \arrow[d, "\mu_{\mathbb{C}^n}"] & \subset \mathbb{C}^n \arrow[d, "\mu_{\mathbb{C}^n}"] \\ \Delta \supset & U_v & \cong & (d\rho_v^{-1})^*(U_v) & \subset \mathfrak{t}^*_{\geq 0}. \end{tikzcd}$$

We shall show this claim. For a vertex v which is defined as above, $\mu^{-1}(U_v)$ is identified with the open subset of the simultaneous cut space $\overline{(T^*T^n)}_{\{\mu_{u_{i_a}} \geq \lambda_{i_a}\}_{a=1,\ldots,n}}$ of the commutative Hamiltonian circle actions on (T^*T^n) defined by u_{i_j} with the moment maps $\mu_{u_{i_a}}(\xi, \theta) = \langle u_{i_j}, \xi \rangle$ for $a = 1, \ldots, n$ naturally. Now define the parallel transport p_v of $T^*T^n \times \mathbb{C}^k$ by

$$p_v(\xi, \theta, z) = (\xi - v, \theta, z)$$

for $(\xi, \theta, z) \in T^*T^n \times \mathbb{C}^k$. Then p_v induces the equivariantly symplectomorphism from $\overline{(T^*T^n)}_{\{\mu_{u_{i_a}} \geq \lambda_{i_a}\}_{a=1,\ldots,n}}$ to $\overline{(T^*T^n)}_{\{\mu_{u_{i_a}} \geq 0\}_{a=1,\ldots,n}}$. Then Theorem A.4 in Appendix A with the above equivariantly symplectomorphisms implies the claim.

4. TWISTED TORIC MANIFOLDS

By the topological construction, a $2n$-dimensional symplectic toric manifold X_Δ is obtained from the trivial T^n-bundle on a Delzant polytope Δ by collapsing each fiber on the face $\{\xi \in \Delta\colon \langle u_i, \xi\rangle = \lambda_i\}$ by the circle subgroup $S^1_{u_i}$ generated by u_i. By replacing the trivial T^n-bundle on Δ to a T^n-bundle on an n-dimensional manifold with corners which may be non-trivial, we can obtain the notion of $2n$-dimensional twisted toric manifolds.

4.1. **The definition and examples.** Let B be an n-dimensional manifold with corners, $\pi_P : P \to B$ a principal $SL_n(\mathbb{Z})$-bundle on B. The T^n-bundle and the $\mathbb{Z}^n$-bundle associated with P by the natural action of $SL_n(\mathbb{Z})$ on T^n and on $\mathbb{Z}^n$ are denoted by $\pi_T : T^n_P \to B$ and $\pi_{\mathbb{Z}} : \mathbb{Z}^n_P \to B$, respectively. Consider a $2n$-dimensional manifold X, surjective maps $\nu : T^n_P \to X$ and $\mu : X \to B$ such that the following diagram is commutative

$$\begin{tikzcd}
T^n_P \arrow[rr, two heads, "\nu"] \arrow[dr, "\pi_T"'] & \arrow[d, phantom, "\circlearrowleft"] & X \arrow[dl, "\mu"] \\
& B. &
\end{tikzcd}$$

Definition 4.1. The above tuple $\{X, \nu, \mu\}$ is called a $2n$-dimensional *twisted toric manifold* (or often called a *twisted toric structure* on B) associated with the principal $SL_n(\mathbb{Z})$-bundle $\pi_P : P \to B$, if for arbitrary $b \in B$, there exist

(i) a coordinate neighborhood (U, φ^B) of $b \in B$, that is, U is an open neighborhood of b in B and φ^B is an orientation consistent diffeomorphism from U to the intersection $\mathfrak{t}^*_{\geq 0} \cap D^n_\epsilon(\xi_0)$ of $\mathfrak{t}^*_{\geq 0}$ and the open disc $D^n_\epsilon(\xi_0)$ in $\mathfrak{t}^* = \mathbb{R}^n$ with a center $\xi_0 \in \mathfrak{t}^*_{\geq 0}$ and a radius $\epsilon > 0$ which sends b to ξ_0, (for the definition of $\mathfrak{t}^*_{\geq 0}$, see Example 2.2,)

(ii) a local trivialization $\varphi^P : \pi_P^{-1}(U) \cong U \times SL_n(\mathbb{Z})$ of P, (then φ^P induces local trivializations $\varphi^T : \pi_T^{-1}(U) \cong U \times T^n$ and $\varphi^{\mathbb{Z}} : \pi_{\mathbb{Z}}^{-1}(U) \cong U \times \mathbb{Z}^n$ of T^n_P and $\mathbb{Z}^n_P$, respectively,)

(iii) an orientation preserving diffeomorphism $\varphi^X : \mu^{-1}(U) \cong \mu^{-1}_{\mathbb{C}^n}(D^n_\epsilon(\xi_0))$

such that the following diagram commutes

$$\begin{tikzcd}
\pi_T^{-1}(U) \arrow[rrr, "\nu"] \arrow[drr, "\pi_T"] \arrow[ddr, "(\varphi^B \times \mathrm{id}_{T^n}) \circ \varphi^T"'] & & & \mu^{-1}(U) \arrow[dl, "\mu"'] \arrow[ddr, "\varphi^X"] & \\
& & U \arrow[ddd, "\varphi^B"] & & \\
& \varphi^B(U) \times T^n \arrow[rrr, "\nu_{\mathbb{C}^n}"] \arrow[ddr, "\mathrm{pr}_1"'] & & & \mu^{-1}_{\mathbb{C}^n}(D^n_\epsilon(\xi_0)) \arrow[ddll, "\mu_{\mathbb{C}^n}"] \\
& & & & \\
& & \varphi^B(U), & &
\end{tikzcd}$$

where $\mu_{\mathbb{C}^n}$ is the moment map of T^n-action on $\mathbb{C}^n$ in Example 2.2 and $\nu_{\mathbb{C}^n}$ is the map which is defined by

$$\nu_{\mathbb{C}^n}(\xi, \theta) = (\sqrt{\xi_i} e^{2\pi\sqrt{-1}\theta_i}). \tag{4.1}$$

Note that $\nu_{\mathbb{C}^n}$ is smooth only for (ξ, θ) with all $\xi_i > 0$. The tuple $(U, \varphi^P, \varphi^X, \varphi^B)$ is called a *locally toric chart*. If there are no confusions, we call simply X a twisted toric manifold.

Remark 4.2 (Orientations)**.** We fix the orientations of $\mathfrak{t}^*_{\geq 0} \times T^n$ (or T^*T^n) and $\mathbb{C}^n$ so that $d\xi_1 \wedge d\theta_1 \wedge \cdots \wedge d\xi_n \wedge d\theta_n (= (-1)^n \frac{(\omega_{T^*T^n})^n}{n!})$ and $(-1)^n \frac{(\omega_{\mathbb{C}^n})^n}{n!}$ are the positive volume forms, respectively. Then the map $\nu_{\mathbb{C}^n}$ in (4.1) preserves the orientations. (c.f. Example 2.8)

Example 4.3 (Torus bundle)**.** Let $\pi_P : P \to B$ be a principal $SL_n(\mathbb{Z})$-bundle on a closed oriented n-dimensional manifold B. Then the associated T^n-bundle $\pi_T : T^n_P \to B$ itself is an example of a twisted toric manifold associated with $\pi_P : P \to B$. In particular, the even dimensional torus T^{2n} is a twisted toric manifold, which is a T^n-bundle on T^n.

Example 4.4 (Symplectic toric manifold)**.** A $2n$-dimensional symplectic toric manifold X with a Delzant polytope Δ has a structure of a twisted toric manifold associated with the trivial $SL_n(\mathbb{Z})$-bundle on Δ. In fact, as we described in Section 3, X is obtained from the trivial T^n-bundle on Δ by collapsing each fiber on the facet of Δ by the circle subgroup which is generated by the inward pointing normal vector in $\mathbb{Z}^n$ of the facet.

Example 4.5 ($S^{2n-1} \times T^{2n-3}$)**.** For $n \geq 2$, let S^{2n-1} be the unit sphere in $\mathbb{C}^n$. The product $X = S^{2n-1} \times T^{2n-3}$ is a twisted toric manifold associated with the trivial $SL_{2n-2}(\mathbb{Z})$-bundle P on the $(2n-2)$-dimensional unit disk $B = \overline{D^{2n-2}} = \{z \in \mathbb{C}^{n-1} \colon \|z\| \leq 1\}$ in $\mathbb{C}^{n-1}$. The maps $\nu : T^{2n-2}_P = B \times T^{2n-2} \to X$ and $\mu : X \to B$ are given by

$$\begin{aligned}
\nu(z, \theta) &= \left((\sqrt{1 - \|z\|^2} e^{2\pi\sqrt{-1}\theta_1}, z), (\theta_2, \dots, \theta_{2n-2}) \right), \\
\mu(w, \tau) &= (w_2, \dots, w_{n-1})
\end{aligned}$$

for $(z, \theta) \in B \times T^{2n-2}$, and $(w, \tau) \in S^{2n-1} \times T^{2n-3}$.

Example 4.6 (Even dimensional sphere)**.** Let Δ^n be the n-simplex and $\partial\Delta^n_0$ the facet of Δ^n which are defined by

$$\begin{aligned}
\Delta^n &= \{\xi = (\xi_i) \in \mathfrak{t}^* = \mathbb{R}^n \colon \xi_i \geq 0, \ \sum_{i=1}^n \xi_i \leq 1\}, \\
\partial\Delta^n_0 &= \{\xi = (\xi_i) \in \Delta^n \colon \sum_{i=1}^n \xi_i = 1\}.
\end{aligned}$$

The $2n$-dimensional sphere S^{2n} in $\mathbb{C}^n \times \mathbb{R}$ is equipped with a twisted toric structure associated with the trivial $SL_n(\mathbb{Z})$-bundle P on the quotient space $B = \Delta^n / \partial\Delta^n_0$ of Δ^n by $\partial(\Delta^n)_0$ which is explained as follows. S^{2n} can be thought of as the one point compactification $\overline{D^{2n}} / \{z \in \overline{D^{2n}} \colon \|z\| = 1\}$ of the $2n$-dimensional unit disk $\overline{D^{2n}} = \{z \in \mathbb{C}^n \colon \|z\| \leq 1\}$ in $\mathbb{C}^n$. With this identification, the maps $\overline{\nu} : \Delta^n \times T^n \to \overline{D^{2n}}$ and $\overline{\mu} : \overline{D^{2n}} \to \Delta^n$ defined by

$$\begin{aligned}
\overline{\nu}(\xi, \theta) &= (\sqrt{\xi_i} e^{2\pi\sqrt{-1}\theta_i}), \\
\overline{\mu}(z) &= (|z_i|^2)
\end{aligned}$$

for $(\xi, \theta) \in \Delta^n \times T^n$ and $z = (z_i) \in \overline{D^{2n}}$ induce the maps $\nu : T^n_P = B \times T^n \to S^{2n}$ and $\mu : S^{2n} \to B$, respectively.

Example 4.7. Consider the Hamiltonian T^2-action on $(T^*T^2 \times \mathbb{C}^2, \omega_{T^*T^2} \oplus \omega_{\mathbb{C}^2})$ defined by

$$t \cdot (\xi, \theta, z) = (\xi, (\theta_1, \theta_2 + t_1 - t_2), (e^{-2\pi\sqrt{-1}t_1} z_1, e^{-2\pi\sqrt{-1}t_2} z_2))$$

with the moment map $\Phi : T^*T^2 \times \mathbb{C}^2 \to \mathbb{R}^2$

$$\Phi(\xi, \theta, z) = (\xi_2 - |z_1|^2, -\xi_2 - |z_2|^2 + 1).$$

We should remark that the second component of Φ is added the constant 1. (cf. the end of Definition 2.1.) We denote by $\overline{X}$ its symplectic quotient $\Phi^{-1}(0)/T^2$. Define the right action of $\mathbb{Z}$ on $T^*T^2 \times \mathbb{C}^2$ by

$$(\xi, \theta, z) \cdot n = ((\xi_1 + n, \xi_2), \rho(-n)\theta, \varphi_n(z))$$

for $(\xi, \theta, z) \in T^*T^2 \times \mathbb{C}^2$ and $n \in \mathbb{Z}$, where $\rho : \mathbb{Z} \to SL_2(\mathbb{Z})$ is the homomorphism

$$\rho(n) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^n$$

and

$$\varphi_n(z) = \begin{cases} z & n : \text{even} \\ \bar{z} & n : \text{odd} \end{cases}.$$

It is easy to see that the action descends to the action of $\mathbb{Z}$ on $\overline{X}$, and its quotient space $\overline{X}/\mathbb{Z}$ is denoted by X. We shall explain that X is a twisted toric manifold associated with the principal $SL_2(\mathbb{Z})$-bundle P on the cylinder $B = S^1 \times [0, 1]$ which is determined by the representation $\rho : \mathbb{Z} = \pi_1(B) \to SL_2(\mathbb{Z})$. Let $\widetilde{B} = \mathbb{R} \times [0, 1]$ be the universal covering of B on which the fundamental group $\pi_1(B) = \mathbb{Z}$ acts as a deck transformation by

$$\xi \cdot n = (\xi_1 + n, \xi_2).$$

Since $\overline{X}$ is a simultaneous cut space by two S^1-actions on T^*T^2 corresponding to the positive and negative second fundamental vectors $e_2 = (0, 1)$, $-e_2 = (0, -1)$ in $\mathbb{Z}^2$, the natural T^2-action on T^*T^2 induces the Hamiltonian action of T^2 on $\overline{X}$ with the moment map $\overline{\mu}([\xi, \theta, z]) = \xi$ whose image is equal to $\widetilde{B}$. Consider the following commutative triangle of maps

$$\begin{tikzcd} \widetilde{B} \times T^2 \arrow[rr, "\overline{\nu}"] \arrow[dr, "\mathrm{pr}_1"'] & \arrow[d, phantom, "\circlearrowleft"] & \overline{X} \arrow[dl, "\overline{\mu}"] \\ & \widetilde{B}, & \end{tikzcd} \tag{4.2}$$

where $\overline{\nu} : \widetilde{B} \times T^2 \to \overline{X}$ is given by

$$\overline{\nu}(\xi, \theta) = [\xi, \theta, (\sqrt{\xi_2}, \sqrt{1 - \xi_2})].$$

$\mathbb{Z}$ acts on $\widetilde{B} \times T^2$ by

$$(\xi, \theta) \cdot n = ((\xi_1 + n, \xi_2), \rho(-n)\theta)$$

for $(\xi, \theta) \in \widetilde{B} \times T^2$ and $n \in \mathbb{Z}$. Since the maps pr_1, $\overline{\nu}$, and $\overline{\mu}$ are equivariant with respect to the above $\mathbb{Z}$-actions, these descend to the maps $\pi_T : T^2_P \to B$, $\nu : T^2_P \to X$, and $\mu : X \to B$.

Example 4.8. Let B be a compact, connected, and oriented surface of genus $g \geq 1$ with one boundary component and $k(\neq 1)$ corner points, B_1 the open subset of B which is obtained by removing a sufficiently small closed neighborhood of the boundary from B, and B_2 an open neighborhood of the boundary such that $B_1 \cap B_2 \cong S^1 \times [0, 1]$, see Figure 2. B has the oriented boundary loop which we denote by γ. Moreover, for $k > 0$, the boundary ∂B of B consists of exactly k edge arcs which are denoted by $\gamma_1, \ldots, \gamma_k$ as in Figure 2. The fundamental group $\pi_1(B)$ of B can be identified with the quotient group F_{2g+1}/N of the free group F_{2g+1} of

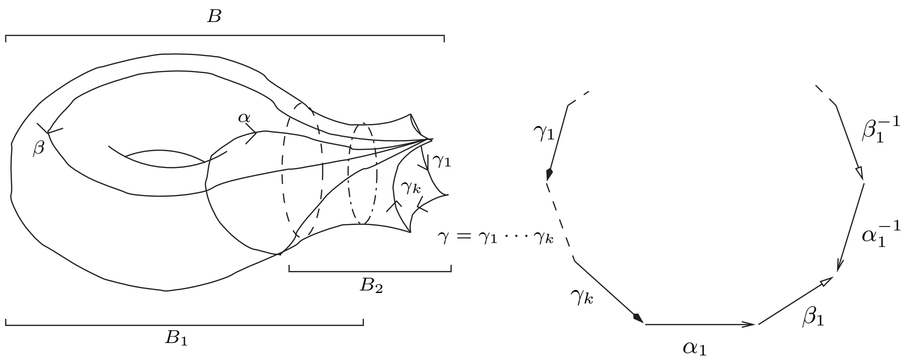


FIGURE 2. B, B_1, and B_2

$2g + 1$ generators α_1, β_1, …, α_g, β_g, γ by the least normal subgroup N containing $\prod_{i=1}^{g}[\alpha_i, \beta_i]\gamma$, and define the representation $\rho : \pi_1(B) \to SL_2(\mathbb{Z})$ by

$$\rho(\alpha_i) = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix},\ \rho(\beta_i) = \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix},\ \rho(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

ρ determines the principal $SL_2(\mathbb{Z})$-bundle P on B as the quotient space $\widetilde{B} \times_\rho SL_n(\mathbb{Z})$ of the action of $\pi_1(B)$ on the product $\widetilde{B} \times SL_n(\mathbb{Z})$ of the universal cover $\widetilde{B}$ of B and $SL_n(\mathbb{Z})$ which is defined by

$$(\widetilde{b}, g) \cdot \tau = (\widetilde{b} \cdot \tau, \rho(\tau^{-1})g)$$

for $(\widetilde{b}, g) \in \widetilde{B} \times SL_n(\mathbb{Z})$ and $\tau \in \pi_1(B)$, where $\widetilde{b} \cdot \tau$ means the deck transformation. Its associated T^2-bundle by the natural action of $SL_2(\mathbb{Z})$ on T^2 is denoted by T^2_P.

For $k \geq 3$, we fix the 4-dimensional symplectic toric manifold X_Δ corresponding to the Delzant polytope Δ with k vertices, and let X_2 denote the subspace of X_Δ which is obtained by removing from X_Δ the inverse image $\mu^{-1}(\overline{D^2})$ of a small closed disk $\overline{D^2}$ in the interior of Δ by the moment map $\mu : X_\Delta \to \mathfrak{t}^* = \mathbb{R}^2$. We take a small open disk D'^2 including $\overline{D^2}$ in the interior of Δ. Since $T^2_P\big|_{B_1 \cap B_2} \to B_1 \cap B_2$ is trivial, it can be identified with $\mu^{-1}(D'^2 \backslash \overline{D^2}) \to D'^2 \backslash \overline{D^2}$. Then we can glue $T^2_P\big|_{B_1} \to B_1$ with $X_\Delta \backslash \mu^{-1}(\overline{D^2}) \to \Delta \backslash \overline{D^2}$ by this identification to obtain a 4-dimensional twisted toric manifold associated with P.

For $k = 0$ (resp. $k = 2$), by replacing X_Δ by $S^3 \times S^1$ in Example 4.5 (resp. S^4 in Example 4.6) in the above construction, we can obtain a 4-dimensional twisted toric manifold in this case.

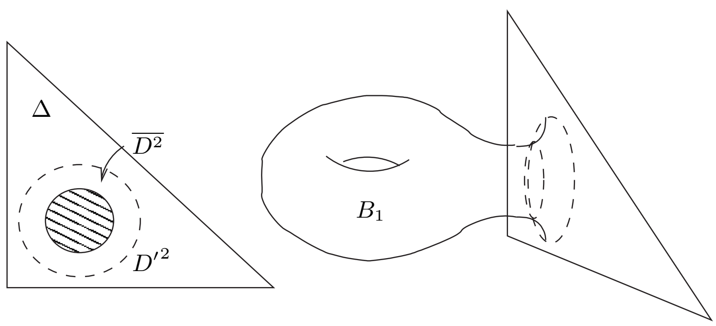


FIGURE 3. gluing B_1 and $\Delta \backslash \overline{D^2}$

Example 4.9. Let B be a compact, connected, and oriented surface of genus one with one boundary component and one corner points, B_1 the set of interior points of B, that is, $B_1 = B \backslash \partial B$. In this case, consider the principal $SL_2(\mathbb{Z})$-bundle P on B which is determined by the representation $\rho : \pi_1(B) \to SL_2(\mathbb{Z})$

$$\rho(\alpha) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\ \rho(\beta) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},\ \rho(\gamma) = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}.$$

As above, we also denote by $\pi_T : T^2_P \to B$ its associated T^2-bundle by the natural action of $SL_2(\mathbb{Z})$ on T^2. In this case, $\pi_T : T^2_P \to B$ is no more trivial near the boundary ∂B of B, and let us construct the twisted toric structure near ∂B as follows. We define the subset $\overline{B}_2$ of $\mathfrak{t}^*_{\geq 0}$ by

$$\overline{B}_2 = \{\xi \in \mathfrak{t}^*(= \mathbb{R}^2) \colon 0 \leq \xi_1 < 4,\ 0 \leq \xi_2 < 1\} \cup \{\xi \in \mathfrak{t}^* \colon 0 \leq \xi_1 < 1,\ 0 \leq \xi_2 < 4\}.$$

The restrictions $\nu_{\mathbb{C}^2}|_{\overline{B}_2 \times T^2} : \overline{B}_2 \times T^2 \to \mu_{\mathbb{C}^2}^{-1}(\overline{B}_2)$ and $\mu_{\mathbb{C}^2}|_{\mu_{\mathbb{C}^2}^{-1}(\overline{B}_2)} : \mu_{\mathbb{C}^2}^{-1}(\overline{B}_2) \to \overline{B}_2$ of $\nu_{\mathbb{C}^2}$ and $\mu_{\mathbb{C}^2}$ defined in (4.1) and Example 2.2, respectively, form a commutative triangle of surjective maps

$$\begin{tikzcd} \overline{B}_2 \times T^2 \arrow[rr, twoheadrightarrow, "\nu_{\mathbb{C}^2}"] \arrow[dr, "\mathrm{pr}_1"'] & \arrow[d, phantom, "\circlearrowleft"] & \mu_{\mathbb{C}^2}^{-1}(\overline{B}_2) \arrow[dl, "\mu_{\mathbb{C}^2}"] \\ & \overline{B}_2. & \end{tikzcd} \tag{4.3}$$

Let U_1 and U_2 be open sets of $\overline{B}_2$ which are defined by

$$U_1 = \{\xi \in \mathfrak{t}^* \colon 3 < \xi_1 < 4,\ 0 \leq \xi_2 < 1\},$$
$$U_2 = \{\xi \in \mathfrak{t}^* \colon 0 \leq \xi_1 < 1,\ 3 < \xi_2 < 4\}.$$

We define diffeomorphisms $\varphi^B : U_1 \to U_2$, $\varphi^T : U_1 \times T^2 \to U_2 \times T^2$, and $\varphi^X : \mu_{\mathbb{C}^2}^{-1}(U_1) \to \mu_{\mathbb{C}^2}^{-1}(U_2)$ by

$$\varphi^B(\xi) = (\xi_2, 7 - \xi_1),$$
$$\varphi^T(\xi, \theta) = (\varphi^B(\xi), \rho(\gamma)\theta),$$
$$\varphi^X(z) = (\frac{z_1^3 z_2}{|z_1|^3}, \sqrt{7 - |z_1|^2} \left(\frac{z_1}{|z_1|}\right)^{-1}).$$

We denote by X_2 the manifold which is obtained from $\mu_{\mathbb{C}^2}^{-1}(\overline{B}_2)$ by gluing $\mu_{\mathbb{C}^2}^{-1}(U_1)$ and $\mu_{\mathbb{C}^2}^{-1}(U_2)$ with φ^X and denote by B_2 the surface with one corner which is obtained from $\overline{B}_2$ by gluing U_1 and U_2 with φ^B. B_2 can be naturally identified

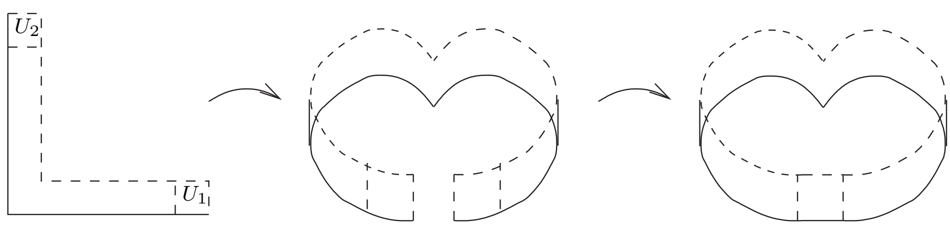


Figure 4. B_2

with a neighborhood of the boundary of B. Since the following diagram commutes

$$\begin{tikzcd}
U_1 \times T^2 \arrow[rr, "\varphi^T"] \arrow[dr, "\nu_{\mathbb{C}^2}"] \arrow[dd, "\mathrm{pr}_1"'] & & U_2 \times T^2 \arrow[dr, "\nu_{\mathbb{C}^2}"] \arrow[dd, "\mathrm{pr}_1" near start] & \\
& \mu_{\mathbb{C}^2}^{-1}(U_1) \arrow[rr, "\varphi^X" near start] \arrow[dl, "\mu_{\mathbb{C}^2}"] & & \mu_{\mathbb{C}^2}^{-1}(U_2) \arrow[dl, "\mu_{\mathbb{C}^2}"] \\
U_1 \arrow[rr, "\varphi^B"] & & U_2, &
\end{tikzcd}$$

under the identification of B_2 with a neighborhood of ∂B, the restriction $\pi_T\big|_{B_2} : T_P^2\big|_{B_2} \to B_2$ of the torus bundle $\pi_T : T_P^2 \to B$ to B_2 is obtained from the trivial bundle $\mathrm{pr}_1 : \overline{B}_2 \times T^2 \to \overline{B}_2$ by gluing $U_1 \times T^2$ and $U_2 \times T^2$ by φ^T, and the diagram (4.3) induces the following commutative triangle of surjective maps

$$\begin{tikzcd}
T_P^2\big|_{B_2} \arrow[rr, two heads, "\nu_2"] \arrow[dr, "\pi_T"'] \arrow[rr, phantom, shift right=6ex, "\circlearrowright"] & & X_2 \arrow[dl, "\mu_2"] \\
& B_2, &
\end{tikzcd}$$

where ν_2 and μ_2 are maps induced by $\nu_{\mathbb{C}^2}$ and $\mu_{\mathbb{C}^2}$, respectively. It is easy to see that the restriction $\mu_2\big|_{\mu_2^{-1}(B_1\cap B_2)} : \mu_2^{-1}(B_1 \cap B_2) \to B_1 \cap B_2$ is a T^2-bundle with the structure group $SL_2(\mathbb{Z})$ and the restriction of ν_2 to $T_P^2\big|_{B_1\cap B_2}$ is a bundle isomorphism from $T_P^2\big|_{B_1\cap B_2}$ to $\mu_2^{-1}(B_1 \cap B_2)$. Thus we can patch $\pi_T\big|_{B_1} : T_P^2\big|_{B_1} \to B_1$ with $\mu_2 : X_2 \to B_2$ by this isomorphism to get the twisted toric manifold $\mu : X \to B$ associated with $\pi : P \to B$.

4.2. **A remark on compatible symplectic forms.** Let B be an n-dimensional manifold and X a $2n$-dimensional twisted toric manifold associated with a principal $SL_n(\mathbb{Z})$-bundle $P \to B$ on B. For each locally toric chart $(U, \varphi^P, \varphi^X, \varphi^B)$ of X, we have a symplectic form on $\mu^{-1}(U)$ which comes from $\omega_{\mathbb{C}^n}$ on $(\mu_{\mathbb{C}^n}^{-1})(\varphi^B(U))$. In the definition of twisted toric manifolds, we do not assume that these local symplectic forms are patched together to a global symplectic form of X. In this section, we shall investigate the condition for these forms to be patched together to a global symplectic form on X. Let $(U_\alpha, \varphi_\alpha^P, \varphi_\alpha^X, \varphi_\alpha^B)$, $(U_\beta, \varphi_\beta^P, \varphi_\beta^X, \varphi_\beta^B)$ be two locally toric charts of X which satisfies $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$. We denote by $g_{\beta\alpha} : U_{\alpha\beta} \to SL_n(\mathbb{Z})$ the transition function of P and set

$$\begin{aligned}
\varphi_{\beta\alpha}^B &= (\varphi_\beta^B\big|_{U_{\alpha\beta}}) \circ (\varphi_\alpha^B\big|_{U_{\alpha\beta}})^{-1} : \varphi_\alpha^B(U_{\alpha\beta}) \to \varphi_\beta^B(U_{\alpha\beta}),\\
\varphi_{\beta\alpha}^T &= \varphi_\beta^T\big|_{\pi_T^{-1}(U_{\beta\alpha})} \circ (\varphi_\alpha^T\big|_{\pi_T^{-1}(U_{\beta\alpha})})^{-1} : U_{\alpha\beta} \times T^n \to U_{\alpha\beta} \times T^n,\\
\varphi_{\beta\alpha}^X &= (\varphi_\beta^X\big|_{\mu^{-1}(U_{\alpha\beta})}) \circ (\varphi_\alpha^X\big|_{\mu^{-1}(U_{\alpha\beta})})^{-1} : (\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_{\alpha\beta}))) \to (\mu_{\mathbb{C}^n}^{-1}(\varphi_\beta^B(U_{\alpha\beta}))).
\end{aligned}$$

We also define the maps $\widetilde{g}_{\beta\alpha} : \varphi_\alpha^B(U_{\alpha\beta}) \to SL_n(\mathbb{Z})$ and $\widetilde{\varphi}_{\beta\alpha}^T : \varphi_\alpha^B(U_{\alpha\beta}) \times T^n \cong \varphi_\beta^B(U_{\alpha\beta}) \times T^n$ by

$$\begin{aligned}
\widetilde{g}_{\beta\alpha} &= g_{\beta\alpha} \circ (\varphi_\alpha^B\big|_{U_{\alpha\beta}})^{-1},\\
\widetilde{\varphi}_{\beta\alpha}^T &= (\varphi_\beta^B\big|_{U_{\alpha\beta}} \times \mathrm{id}_{T^n}) \circ \varphi_{\beta\alpha}^T \circ (\varphi_\alpha^B\big|_{U_{\alpha\beta}} \times \mathrm{id}_{T^n})^{-1}.
\end{aligned}$$

Note that they satisfy the following conditions

$$\begin{aligned}
\varphi_{\beta\alpha}^T(b, \theta) &= (b, g_{\beta\alpha}(b)\theta),\\
\widetilde{\varphi}_{\beta\alpha}^T(\xi, \theta') &= (\varphi_{\beta\alpha}^B(\xi), \widetilde{g}_{\beta\alpha}(\xi)\theta')
\end{aligned}$$

for $(b,\theta) \in U_{\beta\alpha} \times T^n$ and $(\xi, \theta') \in \varphi^B_\alpha(U_{\alpha\beta}) \times T^n$

$$\begin{tikzcd}
& \pi_T^{-1}(U_{\alpha\beta}) \arrow[dl, "(\varphi^B_\alpha \times \mathrm{id}_{T^n})\circ\varphi^T_\alpha"'] \arrow[dr, "(\varphi^B_\beta \times \mathrm{id}_{T^n})\circ\varphi^T_\beta"] \arrow[dd, "\nu"] & \\
\varphi^B_\alpha(U_{\alpha\beta}) \times T^n \arrow[rr, "\widetilde{\varphi}^T_{\beta\alpha}"] \arrow[dd, "\nu_{\mathbb{C}^n}"] & & \varphi^B_\beta(U_{\alpha\beta}) \times T^n \arrow[dd, "\nu_{\mathbb{C}^n}"] \\
& \mu^{-1}(U_{\alpha\beta}) \arrow[dl, "\varphi^X_\alpha"'] \arrow[dr, "\varphi^X_\beta"] \arrow[dd, "\mu"] & \\
\mu_{\mathbb{C}^n}^{-1}(\varphi^B_\alpha(U_{\alpha\beta})) \arrow[rr, "\varphi^X_{\beta\alpha}"] \arrow[dd, "\mu_{\mathbb{C}^n}"] & & \mu_{\mathbb{C}^n}^{-1}(\varphi^B_\beta(U_{\alpha\beta})) \arrow[dd, "\mu_{\mathbb{C}^n}"] \\
& U_{\alpha\beta} \arrow[dl, "\varphi^B_\alpha"'] \arrow[dr, "\varphi^B_\beta"] & \\
\varphi^B_\alpha(U_{\alpha\beta}) \arrow[rr, "\varphi^B_{\beta\alpha}"] & & \varphi^B_\beta(U_{\alpha\beta}).
\end{tikzcd}$$

Proposition 4.10. *$\varphi^X_{\beta\alpha}$ preserves the symplectic form $\omega_{\mathbb{C}^n}$ on $\mu_{\mathbb{C}^n}^{-1}(\varphi^B_\alpha(U_{\alpha\beta}))$, if and only if, up to additive constant, $\varphi^B_{\beta\alpha}$ is of the form $\varphi^B_{\beta\alpha}(\xi) = {}^t(\widetilde{g}_{\beta\alpha}(\xi))^{-1}\xi$.*

Proof. Since for $k = \alpha$, β, the symplectic form $\omega_{\mathbb{C}^n}$ on $(\mu_{\mathbb{C}^n}^{-1}(\varphi^B_k(U_k)))$ comes from $\omega_{T^*T^n}$ on $\varphi^B_k(U_k) \times T^n \subset T^*T^n$ by symplectic cutting, it is sufficient to see the condition that $\widetilde{\varphi}^T_{\beta\alpha}$ preserves $\omega_{T^*T^n}$. Whereas

$$(\widetilde{\varphi}^T_{\beta\alpha})^*\omega_{T^*T^n} = \sum_{j,k=1}^{n} \left(\sum_{i=1}^{n} (\widetilde{g}_{\beta\alpha}(\xi))_{ij} \frac{\partial(\varphi^B_{\beta\alpha})_i}{\partial \xi_k} \right) d\theta_j \wedge dx_k,$$

$\widetilde{\varphi}^T_{\beta\alpha}$ preserves $\omega_{T^*T^n}$, if and only if

$$\sum_{i=1}^{n} (\widetilde{g}_{\beta\alpha}(\xi))_{ij} \frac{\partial(\varphi^B_{\beta\alpha})_i}{\partial \xi_k} = \delta_{jk},$$

that is, the Jacobi matrix $(\dfrac{\partial(\varphi^B_{\beta\alpha})_i}{\partial \xi_j})$ is equal to ${}^t(\widetilde{g}_{\beta\alpha}(\xi))^{-1}$. Since $\widetilde{g}_{\beta\alpha}$ is locally constant, this implies the lemma. □

Definition 4.11. Let ω be a symplectic form on a twisted toric manifold X. ω is said to be *compatible* with respect to the twisted toric structure of X, if for each locally toric chart $(U, \varphi^P, \varphi^X, \varphi^B)$, the restriction of ω to $\mu^{-1}(U)$ is equal to ${\varphi^X}^*\omega_{\mathbb{C}^n}$.

Example 4.12. A $2n$-dimensional torus T^{2n} is a twisted toric manifold, because T^{2n} itself is the trivial n-dimensional torus bundle on a n-dimensional torus. The symplectic structure on T^{2n} which is induced from that of $\mathbb{R}^{2n}$ is compatible.

Example 4.13. The symplectic structure of every symplectic toric manifold is compatible with respect to the natural twisted toric structure.

Example 4.14. The product of an even dimensional torus and a symplectic toric manifold has a compatible symplectic structure.

Question 4.15. *Are there any twisted toric manifold associated with a non trivial principal $SL_n(\mathbb{Z})$-bundle which has a compatible symplectic form?*

5. The classification theorem

In this section, we shall prove the classification theorem for twisted toric manifolds. Let B be an n-dimensional manifold and $\pi_P : P \to B$ a principal $SL_n(\mathbb{Z})$-bundle $P \to B$ on B. We continue to use same notations for associated T^n- and $\mathbb{Z}^n$-bundles as in Section 4. In this section, we assume that $\partial B \neq \emptyset$. For arbitrary $b \in B$, define

$$n(b) = \#\{i \colon \varphi^B(b)_i = \langle e_i, \varphi^B(b) \rangle = 0\},$$

where e_i is the ith fundamental vector $e_i = {}^t(0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$ of $\mathbb{Z}^n$ and φ^B is a local coordinate on a neighborhood of b defined as in (i) of Definition 4.1. Note that $n(b)$ does not depend on the choice of φ^B. Let $\mathcal{S}^{(k)}B$ be the k-dimensional strata of B which is defined by

$$\mathcal{S}^{(k)}B = \{b \in B \colon n(b) = n - k\}.$$

Let $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(n-1)}B$ be a rank one sub-lattice bundle of the restriction $\pi_{\mathbb{Z}}\big|_{\mathcal{S}^{(n-1)}B} : \mathbb{Z}^n_P\big|_{\mathcal{S}^{(n-1)}B} \to \mathcal{S}^{(n-1)}B$ of the associated $\mathbb{Z}^n$-bundle $\pi_{\mathbb{Z}} : \mathbb{Z}^n_P \to B$ to the codimension one strata $\mathcal{S}^{(n-1)}B$.

Definition 5.1. $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(n-1)}B$ is said to be *primitive*, if for arbitrary point b of the k-dimensional strata $\mathcal{S}^{(k)}B$, there exist

(i) an open neighborhood U of $b \in B$ whose intersection $U \cap \mathcal{S}^{(n-1)}B$ with $\mathcal{S}^{(n-1)}B$ has exactly $n - k$ connected components $(U \cap \mathcal{S}^{(n-1)}B)_1$, …, $(U \cap \mathcal{S}^{(n-1)}B)_{n-k}$,

(ii) a local trivialization $\varphi^P : P\big|_U \cong U \times SL_n(\mathbb{Z})$,

(iii) a primitive tuple $\{L_1, \dots, L_{n-k}\}$ of rank one sub-lattices in $\mathbb{Z}^n$

such that the associated local trivialization $\varphi^{\mathbb{Z}} : \mathbb{Z}^n_P\big|_U \cong U \times \mathbb{Z}^n$ of φ^P maps each $\mathcal{L}\big|_{(U\cap\mathcal{S}^{(n-1)}B)_a}$ to $(U \cap \mathcal{S}^{(n-1)}B)_a \times L_a$

$$\begin{array}{ccc}
\mathbb{Z}^n_P\big|_U & \overset{\varphi^{\mathbb{Z}}}{\cong} & U \times \mathbb{Z}^n \\
\cup & & \cup \\
\mathbb{Z}^n_P\big|_{(U\cap\mathcal{S}^{(n-1)}B)_a} & \cong & (U \cap \mathcal{S}^{(n-1)}B)_a \times \mathbb{Z}^n \\
\cup & & \cup \\
\mathcal{L}\big|_{(U\cap\mathcal{S}^{(n-1)}B)_a} & \cong & (U \cap \mathcal{S}^{(n-1)}B)_a \times L_a
\end{array}$$

for $a = 1, \dots, n - k$.

Remark 5.2. (1) Definition 5.1 does not depend on the choice of a neighborhood U in (i) of Definition 5.1, since the notion of primitivity is invariant under the action of $SL_n(\mathbb{Z})$. (cf. Remark 3.4)
(2) The automorphism group $\mathrm{Aut}(P)$ of P acts on the set of primitive rank one sub-lattice bundles of $\pi_{\mathbb{Z}} : \mathbb{Z}^n_P\big|_{\mathcal{S}^{(n-1)}B} \to \mathcal{S}^{(n-1)}B$ as the automorphisms of the restriction of the associated lattice bundle $\pi_{\mathbb{Z}} : \mathbb{Z}^n_P \to B$ to $\mathcal{S}^{(n-1)}B$.

Theorem 5.3. *For any twisted toric manifold X associated with a principal $SL_n(\mathbb{Z})$-bundle $\pi_P : P \to B$, there exists a primitive rank one sub-lattice bundle $\pi_{\mathcal{L}_X} : \mathcal{L}_X \to \mathcal{S}^{(n-1)}B$ of $\pi_{\mathbb{Z}} : \mathbb{Z}^n_P\big|_{\mathcal{S}^{(n-1)}B} \to \mathcal{S}^{(n-1)}B$ which is determined uniquely by X. $\pi_{\mathcal{L}_X} : \mathcal{L}_X \to \mathcal{S}^{(n-1)}B$ is called a characteristic bundle of X.*

To prove Theorem 5.3, we need some preliminaries. Let $(U_\alpha, \varphi^P_\alpha, \varphi^X_\alpha, \varphi^B_\alpha)$, $(U_\beta, \varphi^P_\beta, \varphi^X_\beta, \varphi^B_\beta)$ be two locally toric charts of X with non empty intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$. We also use the notations defined in Section 4.2. Let $\mathbb{Z}_i$ be the rank one sub-lattice of the free $\mathbb{Z}$-module $\mathbb{Z}^n$ which is spanned by e_i. For $j = \alpha$, β, we define

$$\mathfrak{I}_j = \left\{ i \in \{1, \ldots, n\} \colon \{\xi \in \varphi_j^B(U_{\alpha\beta}) \colon \langle \xi, e_i \rangle = \xi_i = 0\} \neq \emptyset \right\}.$$

For any $i_\alpha \in \mathfrak{I}_\alpha$, there exists a unique $i_\beta \in \mathfrak{I}_\beta$ such that

$$\varphi_{\beta\alpha}^B(\{\xi \in \varphi_\alpha^B(U_{\alpha\beta}) \colon \langle \xi, e_{i_\alpha} \rangle = \xi_{i_\alpha} = 0\}) = \{\xi \in \varphi_\beta^B(U_{\alpha\beta}) \colon \langle \xi, e_{i_\beta} \rangle = \xi_{i_\beta} = 0\}.$$

Lemma 5.4. *For such $i_\alpha \in \mathfrak{I}_\alpha$ and $i_\beta \in \mathfrak{I}_\beta$,*

$$g_{\beta\alpha}(b)\mathbb{Z}_{i_\alpha} = \mathbb{Z}_{i_\beta}$$

for all $b \in (\varphi_\alpha^B)^{-1}(\{\xi \in \varphi_\alpha^B(U_{\alpha\beta}) \colon \langle \xi, e_{i_\alpha} \rangle = \xi_{i_\alpha} = 0\}) \subset U_{\alpha\beta}$.

Proof. Since $g_{\beta\alpha}$ is locally constant, it is sufficient to show the lemma on the set

$$(\varphi_\alpha^B)^{-1}(\{\xi \in \varphi_\alpha^B(U_{\alpha\beta}) \colon \xi_{i_\alpha} = 0, \ \xi_i > 0, \ \text{for } i \neq i_\alpha\}).$$

Let $b \in (\varphi_\alpha^B)^{-1}(\{\xi \in \varphi_\alpha^B(U_{\alpha\beta}) \colon \xi_{i_\alpha} = 0, \ \xi_i > 0, \ \text{for } i \neq i_\alpha\}$. The T^n-action on $\mathbb{C}^n$ in Example 2.2 preserves $\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(b))$ which is naturally equivariantly diffeomorphic to $T^n/S^1_{i_\alpha}$ with the natural T^n-action, where $S^1_{i_\alpha}$ is the circle subgroup generated by e_{i_α}. Under this identification, the restriction $(\nu_{\mathbb{C}^n})_{\varphi_\alpha^B(b)} : \mathrm{pr}_1^{-1}(\varphi_\alpha^B(b)) \to \mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(b))$ of $\nu_{\mathbb{C}^n}$ to the fiber at $\varphi_\alpha^B(b)$ can be thought of as the natural projection from T^n to $T^n/S^1_{i_\alpha}$. The same argument allows us to identify $(\nu_{\mathbb{C}^n})_{\varphi_\beta^B(b)} : \mathrm{pr}_1^{-1}(\varphi_\beta^B(b)) \to \mu_{\mathbb{C}^n}^{-1}(\varphi_\beta^B(b))$ with the natural projection from T^n to $T^n/S^1_{i_\beta}$. The commutativity $\nu_{\mathbb{C}^n} \circ (\varphi_\beta^B \times \mathrm{id}_{T^n}) \circ \varphi_{\beta\alpha}^T \circ (\varphi_\alpha^B \times \mathrm{id}_{T^n})^{-1} = \varphi_{\beta\alpha}^X \circ \nu_{\mathbb{C}^n}$ implies $g_{\beta\alpha}(b)(S^1_{i_\alpha}) = S^1_{i_\beta}$, and since the integral lattice of $\mathrm{Lie}(S^1_{i_j})$ is equal to $\mathbb{Z}_{i_j}$ and $g_{\beta\alpha}(b) \in SL_n(\mathbb{Z}) \subset \mathrm{Aut}(T^n)$, by taking the derivative $g_{\beta\alpha}(b) : \mathrm{Lie}(S^1_{i_\alpha}) \to \mathrm{Lie}(S^1_{i_\beta})$, $g_{\beta\alpha}(b)$ sends $\mathbb{Z}_{i_\alpha}$ to $\mathbb{Z}_{i_\beta}$. □

Proof of Theorem 5.3. Let U_α be a locally toric chart with $U_\alpha \cap \mathcal{S}^{(n-1)}B \neq \emptyset$. If necessary, by shrinking U_α, we can assume that there exists a unique i_α such that

$$U_\alpha \cap \mathcal{S}^{(n-1)}B = (\varphi_\alpha^B)^{-1}(\varphi_\alpha^B(U_\alpha) \cap \{\xi \in \mathfrak{t}^* \colon \xi_{i_\alpha} = 0\}).$$

Let us consider the trivial rank one sub-lattice bundle $U_\alpha \cap \mathcal{S}^{(n-1)}B \times \mathbb{Z}_{i_\alpha} \to U_\alpha \cap \mathcal{S}^{(n-1)}B$ of $\mathbb{Z}_P^n\big|_{U_\alpha \cap \mathcal{S}^{(n-1)}B}$ on $U_\alpha \cap \mathcal{S}^{(n-1)}B$, where $\mathbb{Z}_{i_\alpha}$ is the rank one sub-lattice of $\mathbb{Z}^n$ which is generated by i_αth fundamental vector e_{i_α}. Lemma 5.4 implies that these sub-lattice bundles on locally toric charts which satisfy the above conditions are patched together to the rank one sub-lattice bundle $\pi_{\mathcal{L}_X} : \mathcal{L}_X \to \mathcal{S}^{(n-1)}B$ of $\mathbb{Z}_P^n\big|_{\mathcal{S}^{(n-1)}B}$ on $\mathcal{S}^{(n-1)}B$. From the construction, the primitivity of $\pi_{\mathcal{L}_X} : \mathcal{L}_X \to \mathcal{S}^{(n-1)}B$ is obvious. □

Example 5.5. Consider the characteristic bundle of the twisted toric manifold in Example 4.7. Let $\pi : \partial\widetilde{B}(= \mathbb{R} \times \{0,1\}) \to \partial B(= S^1 \times \{0,1\})$ be the universal covering space of the boundary ∂B of B. The restriction of $\pi_{\mathbb{Z}} : \mathbb{Z}_P^2 \to B$ to ∂B is the $\mathbb{Z}^2$-bundle $\partial\widetilde{B} \times_\rho \mathbb{Z}^2$ associated with $\pi : \partial\widetilde{B} \to \partial B$ by the homomorphism $\rho : \mathbb{Z} \to SL_2(\mathbb{Z})$ defined in Example 4.7. The primitive rank one sub-lattice $\{0\} \times \mathbb{Z}$ of $\mathbb{Z}^2$ is preserved by the action ρ, and in this case, the characteristic bundle is the associated bundle $\pi : \partial\widetilde{B} \times_\rho (\{0\} \times \mathbb{Z}) \to \partial B$ by the induced $\mathbb{Z}$-action on $\{0\} \times \mathbb{Z}$.

Example 5.6. In the case of Example 4.9, the restriction of the associated $\mathbb{Z}^2$-bundle $\pi_{\mathbb{Z}} : \mathbb{Z}_P^2 \to B$ to the neighborhood B_2 of ∂B is obtained from the trivial bundle $\mathrm{pr}_1 : \overline{B}_2 \times \mathbb{Z}^2 \to \overline{B}_2$ by the similar way as in the case of the construction of the restriction $T_P^2\big|_{B_2}$, that is, by gluing $U_1 \times \mathbb{Z}^2$ and $U_2 \times \mathbb{Z}^2$ with the diffeomorphism $\varphi^{\mathbb{Z}} : U_1 \times \mathbb{Z}^2 \to U_2 \times \mathbb{Z}^2$

$$\varphi^{\mathbb{Z}}(\xi, l) = (\varphi^B(\xi), \rho(\gamma)l).$$

The characteristic bundle is obtained by gluing two trivial sub-lattice bundles

$$\mathrm{pr}_1 : \{\xi \in \overline{B}_2 \colon 0 < \xi_1,\ \xi_2 = 0\} \times (\{0\} \times \mathbb{Z}) \to \{\xi \in \overline{B}_2 \colon 0 < \xi_1,\ \xi_2 = 0\},$$
$$\mathrm{pr}_1 : \{\xi \in \overline{B}_2 \colon \xi_1 = 0,\ 0 < \xi_2\} \times (\mathbb{Z} \times \{0\}) \to \{\xi \in \overline{B}_2 \colon \xi_1 = 0,\ 0 < \xi_2\}$$

of $\mathrm{pr}_1 : \overline{B}_2 \times \mathbb{Z}^2 \to \overline{B}_2$ with restrictions

$$\varphi^{\mathbb{Z}}\big|_{\{\xi \in U_1 \colon \xi_2 = 0\} \times (\{0\} \times \mathbb{Z})} : \{\xi \in U_1 \colon \xi_2 = 0\} \times (\{0\} \times \mathbb{Z}) \to \{\xi \in U_2 \colon \xi_1 = 0\} \times (\mathbb{Z} \times \{0\}),$$
$$\varphi^{B}\big|_{\{\xi \in U_1 \colon \xi_2 = 0\}} : \{\xi \in U_1 \colon \xi_2 = 0\} \to \{\xi \in U_2 \colon \xi_1 = 0\}$$

of diffeomorphisms $\varphi^{\mathbb{Z}}$, φ^{B}, respectively.

Fix a principal $SL_n(\mathbb{Z})$-bundle $\pi_P : P \to B$ on an n-dimensional manifold B with corners.

Definition 5.7. Two twisted toric manifolds $\{X_1, \nu_1, \mu_1\}$ and $\{X_2, \nu_2, \mu_2\}$ associated with $\pi_P : P \to B$ are *topologically isomorphic*, if there exist an automorphism ψ^P of $\pi_P : P \to B$ which covers identity map of B (then ψ^P induces the automorphism ψ^T of $\pi_T : T^2_P \to B$), and a homeomorphism ψ^X from X_1 to X_2 such that the following diagram commutes

$$\begin{tikzcd}
T^2_P \arrow[rr, "\psi^T"] \arrow[dr, "\nu_1"] \arrow[dd, "\pi_T"'] & & T^2_P \arrow[dr, "\nu_2"] \arrow[dd, "\pi_T"] & \\
& X_1 \arrow[rr, "\psi^X"] \arrow[dl, "\mu_1"] & & X_2 \arrow[dl, "\mu_2"] \\
B \arrow[rr, equal, "\mathrm{id}_B"] & & B. &
\end{tikzcd}$$

Note that since the restriction $\nu_i\big|_{\pi_T^{-1}(B \backslash \partial B)} : \pi_T^{-1}(B \backslash \partial B) \to \mu_i^{-1}(B \backslash \partial B)$ is a diffeomorphism for $i = 1, 2$, so is $\psi^X\big|_{\mu_1^{-1}(B \backslash \partial B)} : \mu_1^{-1}(B \backslash \partial B) \to \mu_2^{-1}(B \backslash \partial B)$.

Let $\{(U_\alpha, \varphi^B_\alpha)\}_{\alpha \in \Gamma}$ be a coordinate neighborhood system of B. In order to show the classification theorem, we assume the following technical conditions

(i) each coordinate neighborhood $(U_\alpha, \varphi^B_\alpha)$ satisfies the condition (i) of Definition 4.1,

(ii) on a overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$ with $U_{\alpha\beta} \cap \partial B \neq \emptyset$, if $\varphi^B_{\beta\alpha}$ sends $\{\xi \in \varphi_\alpha(U_{\alpha\beta}) \colon \xi_i = 0\}$ to $\{\zeta \in \varphi_\beta(U_{\alpha\beta}) \colon \zeta_j = 0\}$ for some i, j, then $\varphi^B_{\beta\alpha}$ satisfies
$$\varphi^B_{\beta\alpha}(\xi)_j = \langle e_j, \varphi^B_{\beta\alpha}(\xi) \rangle = \xi_i$$
on a sufficiently small neighborhood of $\{\xi \in \varphi_\alpha(U_{\alpha\beta}) \colon \xi_i = 0\}$ of $\varphi_\alpha(U_{\alpha\beta})$.

In the case where B is a surface, B has such a coordinate neighborhood system. In fact, by taking a coordinate neighborhood near ∂B as in Example 4.9, we can adopt the composition of a rotation and a parallel transport in $\mathfrak{t}^* = \mathbb{R}^2$ as a coordinate changing function.

Theorem 5.8. *Assume B has a coordinate neighborhood system with the above properties. Then by associating the characteristic bundle to a twisted toric manifold, the set of topologically isomorphism classes of twisted toric manifolds associated with $\pi_P : P \to B$ corresponds one-to-one to the set of equivalent classes of primitive rank one sub-lattice bundles on $\mathcal{S}^{(n-1)}B$ of $\mathbb{Z}^n_P\big|_{\mathcal{S}^{(n-1)}B}$ by the action of the automorphism group of P.*

Proof. It is easy to see that the map which associates to an isomorphism class of a twisted toric manifold the equivalence class of the characteristic bundle is well-defined. For two twisted toric manifolds X_1 and X_2 whose characteristic bundles are in the same equivalent class, there exists an automorphism ψ^P of P such that induced automorphism $\psi^{\mathbb{Z}}$ of $\mathbb{Z}_P^n$ sends the characteristic bundle of X_1 to that of X_2. Then it is easy to see from the construction of the characteristic bundle that the automorphism ψ^T of T_P^n induced by ψ^P descends to the topologically isomorphism ψ^X from X_1 to X_2. This implies the injectivity.

Conversely, for each primitive rank one sub-lattice bundle $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(n-1)}B$ of $\mathbb{Z}_P^n\big|_{\mathcal{S}^{(n-1)}B}$ on $\mathcal{S}^{(n-1)}B$, we can construct a twisted toric manifold whose characteristic bundle is equal to $\mathcal{L}$ in the following way. Let (U, φ^B) be a coordinate neighborhood of B which satisfies the condition (i) of Definition 4.1. We set

$$k = \#\{i \colon \exists \xi \in \varphi^B(U) \subset \mathfrak{t}^*_{\geq 0} \ s.t.\ \xi_i = 0\}.$$

If $k = 0$, just define $\mu^{-1}(U) = \pi_T^{-1}(U)$. If $k > 0$, let

$$\{i_1, \dots, i_k\} = \{i \colon \exists \xi \in \varphi^B(U) \ s.t.\ \xi_i = 0\}$$

and we denote by $(U \cap \mathcal{S}^{(n-1)}B)_a$ the connected component of $U \cap \mathcal{S}^{(n-1)}B$ which satisfies

$$\varphi^B((U \cap \mathcal{S}^{(n-1)}B)_a) = \{\xi \in \varphi^B(U) \colon \xi_{i_a} = 0,\ \xi_i > 0 \text{ for } i \neq i_a\}$$

for $a = 1, \dots, k$. Since $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(n-1)}B$ is primitive and U is contractible, there is a local trivialization $\varphi^P : \pi_P^{-1}(U) \to U \times SL_n(\mathbb{Z})$ (hence φ^P induces $\varphi^{\mathbb{Z}} : \pi_{\mathbb{Z}}^{-1}(U) \to U \times \mathbb{Z}^n$) and a primitive tuple $\{L_1, \dots, L_k\}$ of rank one sub-lattices of $\mathbb{Z}^n$ such that $\varphi^{\mathbb{Z}}$ trivialize $\pi_{\mathcal{L}}^{-1}((U \cap \mathcal{S}^{(n-1)}B)_a) \cong (U \cap \mathcal{S}^{(n-1)}B)_a \times L_a$. If necessary, by applying an element of $SL_n(\mathbb{Z})$ to $\{L_1, \dots, L_k\}$, we may assume that L_a is equal to the rank one sub-lattice $\mathbb{Z}_{i_a}$ in $\mathbb{Z}^n$ spanned by the i_ath fundamental vector e_{i_a} for $a = 1, \dots, k$. This is possible because $\{L_1, \dots, L_k\}$ is primitive. Let $\widetilde{\varphi}^T : \pi_T^{-1}(U) \cong \varphi^B(U) \times T^n$ be the composition of the trivialization $\varphi^T : \pi_T^{-1}(U) \cong U \times T^n$ which is induced by φ^P and $\varphi^B \times \mathrm{id}_{T^n}$. Consider the composition of $\widetilde{\varphi}^T$ and the map $\nu_{\mathbb{C}^n} : \varphi^B(U) \times T^n \to \mu_{\mathbb{C}^n}^{-1}(\varphi^B(U))$ which is defined in (4.1). On $U \backslash \partial B$, this gives the orientation preserving diffeomorphism from $\pi_T^{-1}(U \backslash \partial B)$ to $\mu_{\mathbb{C}^n}^{-1}(\varphi^B(U \backslash \partial B)$. We define $\mu^{-1}(U)$ to be the space obtained by gluing $\pi_T^{-1}(U \backslash \partial B)$ and $\mu_{\mathbb{C}^n}^{-1}(\varphi^B(U))$ by this map.

Let $\{(U_\alpha, \varphi_\alpha^B)\}_{\alpha \in \Gamma}$ be a coordinate neighborhood system of B which satisfies the assumption before the theorem. We apply the above construction to each $(U_\alpha, \varphi_\alpha^B)$. For each overlap with $U_\alpha \cap U_\beta \cap \partial B \neq \emptyset$, it is easy to check from the second condition in the assumption, Lemma B.2, and Lemma A.4 that the transition map $\widetilde{\varphi}_{\beta\alpha}^T : \varphi_\alpha^B(U_{\alpha\beta}) \times T^n \cong \varphi_\beta^B(U_{\alpha\beta}) \times T^n$ defined in Section 4.2 descends to the orientation preserving diffeomorphism from $\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_{\alpha\beta}))$ to $\mu_{\mathbb{C}^n}^{-1}(\varphi_\beta^B(U_{\alpha\beta}))$. Then we can glue $\mu^{-1}(U_\alpha)$ for all U_α to obtain the twisted toric manifold X whose characteristic bundle is equal to $\mathcal{L}$. This implies the surjectivity. □

6. Topology

Let $\pi_P : P \to B$ be a principal $SL_n(\mathbb{Z})$-bundle on a n-dimensional manifold B, $\pi_T : T_P^n \to B$ the T^n-bundle associated with P by the natural action of $SL_n(\mathbb{Z})$ on T^n, X a $2n$-dimensional twisted toric manifold associated with P.

6.1. Fundamental groups.

In this section, we shall investigate the fundamental group of X. Let $b \in B$ be the base point which is located in the interior of B. Fix the base point $y_0 \in \pi_T^{-1}(b)$ and set $x_0 = \nu(y_0) \in X$. Since T^n is identified with $\mathbb{R}^n/\mathbb{Z}^n$, $\pi_T : T_P^n \to B$ has the zero-section s_T. Then the homotopy exact sequence for T^n_P splits into the short exact sequence, and $\pi_1(T^n_P, y_0)$ is isomorphic to the semi-direct product $\pi_1(\pi_T^{-1}(b), y_0) \rtimes \pi_1(B, b)$ of $\pi_1(\pi_T^{-1}(b), y_0)$ and $\pi_1(B, b)$. The section s_T also defines the section s_X of $\mu : X \to B$ by $s_X = \nu \circ s_T$ which also gives the identification of $\pi_1(X, x_0)$ with the semi-direct product $\ker \mu_* \rtimes \pi_1(B, b)$.

Let us consider the following commutative diagram of homomorphisms

$$\begin{tikzcd}
\{1\} \arrow[r] & \pi_1(\pi_T^{-1}(b), y_0) \arrow[r, "\iota_*"] \arrow[d, "\kappa"] & \pi_1(T^n_P, y_0) \arrow[r, "\pi_{T*}"] \arrow[d, "\nu_*"] & \pi_1(B, b) \arrow[r] \arrow[d, equal] & \{1\} \quad \text{(exact)} \\
\{1\} \arrow[r] & \ker \mu_* \arrow[r] & \pi_1(X, x_0) \arrow[r, "\mu_*"] & \pi_1(B, b) \arrow[r] & \{1\} \quad \text{(exact)},
\end{tikzcd}$$

where $\iota : \pi_T^{-1}(b) \hookrightarrow T^n_P$ is the inclusion map and $\kappa : \pi_1(\pi_T^{-1}(b), y_0) \to \ker \mu_*$ is defined by $\kappa = \nu_* \circ \iota_*$.

Lemma 6.1. *The map $\kappa : \pi_1(\pi_T^{-1}(b), y_0) \to \ker \mu_*$ is surjective.*

Note that the surjectivity of the map κ is equivalent to that of ν_* since $s_{X*} = \nu_* \circ s_{T*}$.

Proof. Since b is in the interior of B, the composition map $\nu \circ \iota : \pi_T^{-1}(b) \to \mu^{-1}(b)$ is a diffeomorpshims which sends y_0 to x_0. Then it is sufficient to show that every element of $\ker \mu_*$ is represented by the loop in $\mu^{-1}(b)$. Let $\alpha \in \ker \mu_*$ and take its representative $a' : I \to X$ with $a'(0) = a'(1) = x_0$. This means that the map $\mu \circ a' : I \to B$ is homotopic to the constant map b. Then if necessary, by replacing a representative of α, we can assume that there exists a contractible locally toric chart U located in the interior of B such that the image of a' is included in $\mu^{-1}(U)$. Since U is a locally toric chart which is included in the interior of B, $\mu^{-1}(U)$ is diffeomorphic to $U \times T^n$, and the fact that U is contractible implies that $\mu^{-1}(b)$ is a deformation retract of $\mu^{-1}(U)$. We take a deformation map $h : I \times \mu^{-1}(U) \to \mu^{-1}(U)$ which satisfies $h(0, \cdot) = id_{\mu^{-1}(U)}$ and $h(1, \cdot) : \mu^{-1}(U) \to \mu^{-1}(b)$. Then the map $h(s, a'(t))$ is the homotopy connecting a' with the loop $a = h(1, a'(t))$ in $\mu^{-1}(b)$. $\square$

Since $SL_n(\mathbb{Z})$ is discrete, P is a flat $SL_n(\mathbb{Z})$-bundle. Then, T^n_P also has the flat connection which is induced from that of P. Let $\gamma : I \to B$ be a path of B which starts from the base point b, and $\mathrm{Hol}^T_\gamma : \pi_T^{-1}(b) \to \pi_T^{-1}(\gamma(1))$ the parallel transport of T^n_P with respect to the induced flat connection along γ. For $\gamma : I \to B$, define the subset K_γ of $\pi_1(\pi_T^{-1}(b), y_0)$ by

$$K_\gamma = \{\alpha \in \pi_1(\pi_T^{-1}(b), y_0) \colon (\nu \circ \mathrm{Hol}^T_\gamma)_*(\alpha) = 1\}.$$

We denote by K the subgroup of $\pi_1(\pi_T^{-1}(b), y_0)$ which is generated by $\cup_\gamma K_\gamma$ where γ runs over all paths of B which start at b. Then the following lemma is obvious.

Lemma 6.2. *The subgroup K is included in the kernel of κ.*

Theorem 6.3. *If B has at least one corner point, then $\mu : X \to B$ induces the isomorphism $\mu_* : \pi_1(X, x_0) \cong \pi_1(B, b)$.*

Proof. Let us show $K = \pi_1(\pi_T^{-1}(b), y_0)$. Let $\gamma : I \to B$ be a path which starts at b to the corner point b' of B. Since $\mu^{-1}(b')$ consists of only one point, the composition map $\nu \circ \mathrm{Hol}^T_\gamma : \pi_T^{-1}(b) \to \mu^{-1}(b')$ sends every loop of $\pi_T^{-1}(b)$ to the constant map. This implies $K = \pi_1(\pi_T^{-1}(b), y_0)$. $\square$

Remark 6.4. From facts that the homomorphism $\mu_* : \pi_1(X, x_0) \to \pi_1(B, b)$ of fundamental groups induced by $\mu : X \to B$ is surjective and that complete non-singular toric varieties in the original algebro-geometric sense are simply connected [6], we can see that twisted toric manifolds associated with principal $SL_n(\mathbb{Z})$-bundles on non-simply connected manifolds, such as those in Example 4.7 and 4.9, are not toric varieties. In particular, these are not symplectic toric manifolds.

6.2. **Cohomology groups.** Since $\mu : X \to B$ has a section, as described in the last subsection, the induced homomorphism $\mu^* : H^1(B;\mathbb{Z}) \to H^1(X;\mathbb{Z})$ is injective. In particular, if B has at least one corner point, μ^* is isomorphism.

In this section, we shall give the method to calculate not only the first cohomology but also the full cohomology group of a twisted toric manifold. For topological tools which we use in this section, see [10]. Let X be a $2n$-dimensional twisted toric manifold associated with a principal $SL_n(\mathbb{Z})$-bundle on a n-dimensional manifold B, and T^n_P the T^n-bundle associated with P by the natural action of $SL_n(\mathbb{Z})$ on T^n. Note that since $SL_n(\mathbb{Z})$ is discrete, P is flat, hence this induces the flat connection to T^n_P.

Assume B is equipped with a CW complex structure. For each p-dimensional cell $e^{(p)}$ (we often say simply p-cell), let c be the barycenter of $e^{(p)}$, the map $\varphi : \overline{D^p} \to B$ denotes the characteristic map of $e^{(p)}$ from the p-dimensional closed ball $\overline{D^p}$ to B. Define the map $\widetilde{\varphi}^T : \overline{D^p} \times \pi_T^{-1}(c) \to T^n_P$ by

$$\widetilde{\varphi}^T(d,\theta) = \mathrm{Hol}^T_{\varphi\circ\gamma}(\theta)$$

for $(d,\theta) \in \overline{D^p} \times \pi_T^{-1}(c)$, where $\gamma : [0,1] \to \overline{D^p}$ is a path with $\gamma(0) = c$ and $\gamma(1) = d$ and $\mathrm{Hol}^T_{\varphi\circ\gamma} : (T^n_P)_c \to (T^n_P)_d$ is the parallel transport of T^n_P with respect to the induced connection along $\varphi \circ \gamma : [0,1] \to B$. Note that $\mathrm{Hol}^T_{\varphi\circ\gamma}$ does not depend on the choice of γ, since $\overline{D^p}$ is contractible. We shall assume that the cell decomposition of B satisfies the following conditions

(i) the restriction of $\mu : X \to B$ to each p-dimensional cell $e^{(p)}$ is a (trivial) torus bundle,
(ii) for each p-dimensional cell $e^{(p)}$, the map $\widetilde{\varphi}^T$ induces the map $\widetilde{\varphi}^X : \overline{D^p} \times \mu^{-1}(c) \to X$ such that the following diagram commutes

$$\begin{tikzcd}
\overline{D^p} \times \pi_T^{-1}(c) \arrow[rrrr, "\mathrm{id}_{\overline{D^p}} \times \nu"] \arrow[drr, "\mathrm{pr}_1"] \arrow[ddr, "\widetilde{\varphi}^T"'] & & & & \overline{D^p} \times \mu^{-1}(c) \arrow[dll, "\mathrm{pr}_1"'] \arrow[dd, "\widetilde{\varphi}^X"] \\
& & \overline{D^p} \arrow[dd, "\varphi"] & & \\
& T^n_P \arrow[rrr, "\nu"] \arrow[dr, "\pi_T"'] & & & X \arrow[dll, "\mu"] \\
& & B. & &
\end{tikzcd}$$

Remark 6.5. This assumption is achieved when each p-cell $e^{(p)}$ is included in a k-dimensional strata $\mathcal{S}^{(k)}B$ of B. Since the characteristic map φ sends the interior D^p of $\overline{D^p}$ homeomorphically to $e^{(p)}$, k is necessarily equal or greater than p.

In the rest of this section, *we shall assume the condition in Remark* 6.5. Let $B^{(p)}$ be the p-skeleton of B, and $(T^n_P)^{(p)} = \pi_T^{-1}(B^{(p)})$, $X^{(p)} = \mu^{-1}(B^{(p)})$ its inverse images by $\pi_T : T^n_P \to B$, $\mu : X \to B$, respectively. We consider the spectral sequence $\{(E_X)^{p,q}_r, d^X_r\}$ with respect to the filtration

$$S^*(X;\mathbb{Z}) \supset S^*(X, X^{(0)};\mathbb{Z}) \supset \cdots \supset S^*(X, X^{(n)};\mathbb{Z}) = 0$$

of the singular cochain complex with coefficient $\mathbb{Z}$. $\{(E_X)^{p,q}_r, d^X_r\}$ is called the *cohomology Leray spectral sequence* of the map $\mu : X \to B$. Let $c^{(p)}_\lambda$ be the barycenter of the p-cell $e_\lambda^{(p)}$. As mentioned before, we can identify the fibers $\pi_T^{-1}(c_\lambda^{(p)})$ and $\mu^{-1}(c_\lambda^{(p)})$ with tori. Then $\nu_{c_\lambda^{(p)}} : \pi_T^{-1}(c_\lambda^{(p)}) \to \mu^{-1}(c_\lambda^{(p)})$ can be thought of as a surjective homomorphism between them, and we have the exact sequence of tori

$$0 \to \ker \nu_{c_\lambda^{(p)}} \hookrightarrow \pi_T^{-1}(c_\lambda^{(p)}) \stackrel{\nu_{c_\lambda^{(p)}}}{\to} \mu^{-1}(c_\lambda^{(p)}) \to 0.$$

Since any exact sequence of tori splits, the map $\nu_{c_\lambda^{(p)}}$ induces the injective homomorphism $\nu^*_{c_\lambda^{(p)}} : H^*(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z}) \hookrightarrow H^*(\pi_T^{-1}(c_\lambda^{(p)});\mathbb{Z})$ which enable us to identify $H^q(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z})$ with its image in $H^q(\pi_T^{-1}(c_\lambda^{(p)});\mathbb{Z})$ by $\nu^*_{c_\lambda^{(p)}}$.

Let $(C^p(B;\mathcal{H}_T^q),\delta)$ be the cochain complex of the CW complex B with the Serre local system $\mathcal{H}_T^q$ of the qth cohomology with $\mathbb{Z}$-coefficient for the torus bundle $\pi_T : T_P^n \to B$. We denote by $C^p(B;\mathcal{H}_X^q)$ the subset of $C^p(B;\mathcal{H}_T^q)$ whose cochain takes a value in the image $\nu^*_{c_\lambda^{(p)}}\left(H^q(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z})\right)$ of qth cohomology $H^q(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z})$ by $\nu^*_{c_\lambda^{(p)}}$ for each p-cell $e_\lambda^{(p)}$.

Theorem 6.6. *$C^p(B;\mathcal{H}_X^q)$ is preserved by the differential δ of $C^p(B;\mathcal{H}_T^q)$, so that $C^p(B;\mathcal{H}_X^q)$ is a sub-complex of $(C^p(B;\mathcal{H}_T^q),\delta)$. We denote its cohomology by $H^p(B;\mathcal{H}_X^q)$. Then we have the isomorphisms*

$$(E_X)_1^{p,q} \cong C^p(B;\mathcal{H}_X^q), \quad (E_X)_2^{p,q} \cong H^p(B;\mathcal{H}_X^q),$$
$$(E_X)_\infty^{p,q} = F^pH^{p+q}(X;\mathbb{Z})/F^{p+1}H^{p+q}(X;\mathbb{Z}),$$

where $F^lH^k(X;\mathbb{Z})$ is the image of the map $H^k(X,X^{(l-1)};\mathbb{Z}) \to H^k(X;\mathbb{Z})$.

Proof. Let $\{(E_T)_r^{p,q}, d_r^T\}$ be the cohomology Serre spectral sequence of the torus bundle $\pi_T : T_P^n \to B$, that is, the spectral sequence with respect to the filtration

$$S^*(T_P^n;\mathbb{Z}) \supset S^*(T_P^n,(T_P^n)^{(0)};\mathbb{Z}) \supset \cdots \supset S^*(T_P^n,(T_P^n)^{(n)};\mathbb{Z}) = 0$$

of the singular cochain complex with coefficient $\mathbb{Z}$. By the excision isomorphism, the above assumption, and the Künneth formula, for E_1-terms, we have the isomorphisms

$$\begin{aligned}
(E_T)_1^{p,q} &= H^{p+q}((T_P^n)^{(p)},(T_P^n)^{(p-1)};\mathbb{Z}) \\
&\cong \sum_\lambda H^{p+q}(\pi_T^{-1}(\overline{e_\lambda^{(p)}}),\pi_T^{-1}(\overline{e_\lambda^{(p)}} - e_\lambda^{(p)});\mathbb{Z}) \\
&\cong \sum_\lambda H^{p+q}((\overline{D^p}_\lambda,\partial\overline{D^p}_\lambda)\times\pi_T^{-1}(c_\lambda^{(p)});\mathbb{Z}) \\
&\cong \sum_\lambda H^p(\overline{D^p}_\lambda,\partial\overline{D^p}_\lambda;\mathbb{Z}) \otimes H^q(\pi_T^{-1}(c_\lambda^{(p)});\mathbb{Z}),
\end{aligned}$$

$$\begin{aligned}
(E_X)_1^{p,q} &= H^{p+q}(X^{(p)},X^{(p-1)};\mathbb{Z}) \\
&\cong \sum_\lambda H^{p+q}(\mu^{-1}(\overline{e_\lambda^{(p)}}),\mu^{-1}(\overline{e_\lambda^{(p)}} - e_\lambda^{(p)});\mathbb{Z}) \\
&\cong \sum_\lambda H^{p+q}((\overline{D^p}_\lambda,\partial\overline{D^p}_\lambda)\times\mu^{-1}(c_\lambda^{(p)});\mathbb{Z}) \\
&\cong \sum_\lambda H^p(\overline{D^p}_\lambda,\partial\overline{D^p}_\lambda;\mathbb{Z}) \otimes H^q(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z}),
\end{aligned}$$

where the sum runs over all p-dimensional cells $e_\lambda^{(p)}$. On the other hand, by the injectivity of the homomorphism $\nu^*_{c_\lambda^{(p)}} : H^*(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z}) \hookrightarrow H^*(\pi_T^{-1}(c_\lambda^{(p)});\mathbb{Z})$ and the assumption (ii), the map $\nu : (T_P^n)^{(p)} \to X^{(p)}$ induces the natural injection $\nu^* : (E_X)_1^{p,q} \hookrightarrow (E_T)_1^{p,q}$ such that the following diagram commutes

$$\begin{tikzcd}
(E_X)_1^{p,q} \cong \arrow[d, hook, "\nu^*"'] & \sum_\lambda H^p(\overline{D^p}_\lambda, \partial\overline{D^p}_\lambda; \mathbb{Z}) \otimes H^q(\mu^{-1}(c_\lambda^{(p)}); \mathbb{Z}) \arrow[d, hook, "\sum_\lambda \mathrm{id}^*_{\overline{D^p}} \otimes \nu^*_{c_\lambda^{(p)}}"] \arrow[d, phantom, "\circlearrowleft", shift right=20ex] \\
(E_T)_1^{p,q} \cong & \sum_\lambda H^p(\overline{D^p}_\lambda, \partial\overline{D^p}_\lambda; \mathbb{Z}) \otimes H^q(\pi_T^{-1}(c_\lambda^{(p)}); \mathbb{Z}).
\end{tikzcd}$$

Moreover it is well known that the E_1-term $\{(E_T)_1^{p,q}, d_1^T\}$ of the Serre spectral sequence is isomorphic to the CW complex $(C^p(B; \mathcal{H}_T^q), \delta)$ with the Serre local system $\mathcal{H}_T^q$ for the torus bundle $\pi_T : T_P^n \to B$. This fact and the naturality of the maps in the spectral sequences prove Theorem 6.6. $\square$

Remark 6.7. (1) For $q = 0$, it is easy to see that $(E_X)_2^{p,0} \cong H^p(B; \mathcal{H}_X^q) \cong H^p(B; \mathbb{Z})$. Moreover $(E_X)_1^{p,q} = 0$, if q or p is greater than half the dimension of X. (2) If $n = 2$ and $\partial B \neq \emptyset$, we can take a cell decomposition of B so that all zero cells are included in ∂B. In this case, the Leray spectral sequence $\{(E_X)_r^{p,q}, d_r^X\}$ degenerates at E_2-term. In fact, $\partial B \neq \emptyset$ implies $(E_X)_2^{2,0} \cong H^2(B; \mathcal{H}_X^0) \cong H^2(B; \mathbb{Z}) = 0$, and since $e_\lambda^{(0)} \in \partial B$, the fiber $\mu^{-1}(e_\lambda^{(0)})$ of μ on $e_\lambda^{(0)}$ is diffeomorphic to the torus whose dimension is equal or less than one. Then $(E_X)_2^{0,2} \cong (E_X)_1^{0,2} \cong C^0(B; \mathcal{H}_X^2) = 0$.

Corollary 6.8. *The Euler characteristic $\chi(X)$ is equal to the number of the 0-dimensional strata $\mathcal{S}^{(0)}B$ of B.*

Proof. Let us consider the rational coefficient cohomology Leray spectral sequence $\{(E_X)_r^{p,q}, d_r^X\}$ of the map $\mu : X \to B$. Define

$$\chi((E_X)_r) = \sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} (E_X)_r^{p,q}.$$

Since $(E_X)_1^{p,q} = C^p(B; \mathcal{H}_X^q)$,

$$\begin{aligned}
\chi((E_X)_1) &= \sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} (E_X)_1^{p,q} \\
&= \sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} C^p(B; \mathcal{H}_X^q) \\
&= \sum_{p,q} (-1)^{p+q} \sum_\lambda \dim_{\mathbb{Q}} H^q(\mu^{-1}(c_\lambda^{(p)}); \mathbb{Q}) \\
&= \sum_p \sum_\lambda (-1)^p \chi(\mu^{-1}(c_\lambda^{(p)})), \qquad (6.1)
\end{aligned}$$

where the summation $\sum_\lambda$ runs over all p-cells. From the construction of the twisted toric manifold, we have

$$\chi(\mu^{-1}(c_\lambda^{(p)})) = \begin{cases} 1 & c_\lambda^{(p)} \in \mathcal{S}^{(0)}B \\ 0 & \text{otherwise.} \end{cases}$$

By the assumption of the cell decomposition of B, $c_\lambda^{(p)} \in \mathcal{S}^{(0)}B$ if and only if $c_\lambda^{(p)}$ is the barycenter of the 0-cell in $\mathcal{S}^{(0)}B$, in particular, $p = 0$. Then (6.1) is equal to the number of $\mathcal{S}^{(0)}B$. On the other hand, it is easy to see

$$\chi((E_X)_r) = \chi((E_X)_1)$$

for any r, and since B is compact and all fibers of μ have finitely generated cohomology groups, we can obtain

$$(E_X)_\infty = (E_X)_r$$

for sufficiently large r. Moreover, from $(E_X)_\infty^{p,q} = F^pH^{p+q}(X;\mathbb{Q})/F^{p+1}H^{p+q}(X;\mathbb{Q})$, we can easily check that $\chi(X) = \chi((E_X)_\infty)$. This proves the corollary. $\square$

In the rest of this subsection, we shall calculate the cohomology groups for some examples.

Example 6.9 (Example 4.7)**.** Let us calculate the cohomology group for Example 4.7. Let $Q = [0,1]\times[0,1]$ be the square in $\mathbb{R}^2$. In this case, B is a cylinder, so B is obtained from Q by identifying each point $(0,\xi_2)$ with $(1,\xi_2)$ in Q. Then we have the natural map from Q to B which is denoted by $\varphi : Q \to B$. Q gives a cell decomposition of B as in Figure 5. The pull-back of the triangle of commutative

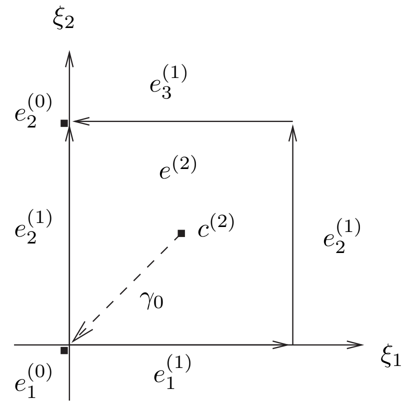


FIGURE 5. Q and cell decomposition

maps $\pi_T : T_P^2 \to B$, $\nu : T_P^2 \to X$, and $\mu : X \to B$ by φ are naturally identified with the restriction of the triangle (4.2) in Example 4.7 to Q

$$\begin{tikzcd}
Q\times T^2 \arrow[rr, "\widetilde{\varphi}^T"] \arrow[dr, "\overline{\nu}|_{Q\times T^2}"] \arrow[ddr, "\mathrm{pr}_1"'] & & T_P^2 \arrow[dr, "\nu"] \arrow[ddr, "\pi_T"] & \\
& \overline{\mu}^{-1}(Q) \arrow[rr, "\widetilde{\varphi}^X"] \arrow[d, "\overline{\mu}|_{\overline{\mu}^{-1}(Q)}"] & & X \arrow[d, "\mu"] \\
& Q \arrow[rr, "\varphi"] & & B,
\end{tikzcd}$$

where $\widetilde{\varphi}^T : Q\times T^2 \to T_P^2$ and $\widetilde{\varphi}^X : \overline{\mu}^{-1}(Q) \to X$ denote the induced fiberwisely diffeomorphisms by pull-back. Under the identifications of fibers by $\widetilde{\varphi}^T$ and $\widetilde{\varphi}^X$, the fibers $\mu^{-1}(c_\lambda^{(p)})$ on all cells $e_\lambda^{(p)}$ except for $e_2^{(1)}$ and $e^{(2)}$ are diffeomorphic to $T^2/(0\times S^1)$ and the map $\nu_{c_\lambda^{(p)}} = \nu\big|_{\pi_T^{-1}(c_\lambda^{(p)})} : \pi_T^{-1}(c_\lambda^{(p)}) \to \mu^{-1}(c_\lambda^{(p)})$ can be identified with the natural projection $T^2 \to T^2/(0\times S^1)$. Then the image of the induced injection $\nu^*_{c_\lambda^{(p)}} : H^q(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z}) \to H^q(\pi_T^{-1}(c_\lambda^{(p)});\mathbb{Z})$ can be given by

$$\nu^*_{c_\lambda^{(p)}}\left(H^q(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}\oplus 0 & q=1 \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

On the cell $c_\lambda^{(p)} = e_2^{(1)}$, $e^{(2)}$, the fiber $\mu^{-1}(c_\lambda^{(p)})$ is naturally diffeomorphic to $\pi_T^{-1}(c_\lambda^{(p)}) \cong T^2$ by $\nu_{c_\lambda^{(p)}}$.

For $q=0$, the cohomology $H^p(B;\mathcal{H}_X^0)$ of $\left(C^p(B;\mathcal{H}_X^0),\delta\right)$ is naturally identified with the cohomology $H^p(B;\mathbb{Z})$ with $\mathbb{Z}$-coefficient.

For $q=1$, the degree p cochain $u\in C^p(B;\mathcal{H}_X^1)$ takes a value as follows

$$u(e_\lambda^{(p)})\in \nu^*_{c_\lambda^{(p)}}\left(H^1(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z})\right)=\begin{cases}\mathbb{Z}\oplus 0 & p=0,\ \text{or } p=1 \text{ and } \lambda=1,3\\ \mathbb{Z}\oplus\mathbb{Z} & p=1 \text{ and } \lambda=2,\ \text{or } p=2\\ 0 & \text{otherwise.}\end{cases}$$

All differentials $\delta^p: C^p(B;\mathcal{H}_X^1)\to C^{p+1}(B;\mathcal{H}_X^1)$ vanish except for $p=0,1$, and for $p=0$, $\delta^0: C^0(B;\mathcal{H}_X^1)\to C^1(B;\mathcal{H}_X^1)$ is given as follows

$$\begin{aligned}\left(\delta^0 u\right)(e_1^{(1)}) &={}^t\rho(1)^{-1}u(e_1^{(0)})-u(e_1^{(0)}),\\ \left(\delta^0 u\right)(e_2^{(1)}) &= u(e_2^{(0)})-u(e_1^{(0)}),\\ \left(\delta^0 u\right)(e_3^{(1)}) &={}^t\rho(-1)^{-1}u(e_2^{(0)})-u(e_2^{(0)}).\end{aligned}$$

Fix the path γ_0 which starts from $c^{(2)}$ to $c_1^{(0)}$. For each $c_\lambda^{(1)}$, let γ_λ be the path which connects $c_1^{(0)}$ and $c_\lambda^{(1)}$ counterclockwisely along the boundary of Q. We identify each fiber $\pi_T^{-1}(c_\lambda^{(1)})$ with $\pi_T^{-1}(c^{(2)})$ by the parallel transport along the composition of the paths γ_0 and γ_λ with respect to the connection induced from that of P. Then $\delta^1: C^1(B;\mathcal{H}_X^1)\to C^2(B;\mathcal{H}_X^1)$ is given by

$$\left(\delta^1 u\right)(e^{(2)})=u(e_1^{(1)})+{}^t\rho(1)^{-1}u(e_2^{(1)})+{}^t\rho(1)^{-1}u(e_3^{(1)})-{}^t\rho(1)^{-1}\,{}^t\rho(-1)^{-1}u(e_2^{(1)}).$$

Then the cohomology groups are obtained by

$$H^p(B;\mathcal{H}_X^1)=\begin{cases}\mathbb{Z}/2\mathbb{Z} & p=1,2\\ 0 & \text{otherwise.}\end{cases}$$

For $q=2$, the degree p cochain $u\in C^p(B;\mathcal{H}_X^2)$ takes a value as follows

$$u(e_\lambda^{(p)})\in \nu^*_{c_\lambda^{(p)}}\left(H^1(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z})\right)=\begin{cases}\mathbb{Z} & p=1 \text{ and } \lambda=2,\ \text{or } p=2\\ 0 & \text{otherwise.}\end{cases}$$

In this case, all differentials $\delta^p: C^p(B;\mathcal{H}_X^2)\to C^{p+1}(B;\mathcal{H}_X^2)$ vanish. It is clear except for $p=1$. For $p=1$, since the holonomy $\rho(-1)$ along $e_1^{(1)}$ (resp. $\rho(1)$ along $e_3^{(1)}$) induces the identity of $H^2(\pi_T^{-1}(c_1^{(0)});\mathbb{Z})$ (resp. $H^2(\pi_T^{-1}(c_2^{(0)});\mathbb{Z})$), the differential $\delta^1: C^1\to C^2$ is given by

$$\delta^1 u(e^{(2)})=u(e_1^{(1)})+u(e_2^{(1)})+u(e_3^{(1)})-u(e_2^{(1)})=u(e_2^{(1)})-u(e_2^{(1)})=0.$$

Then the cohomology groups are obtained by

$$H^p(B;\mathcal{H}_X^2)=\begin{cases}\mathbb{Z} & p=1,2\\ 0 & \text{otherwise.}\end{cases}$$

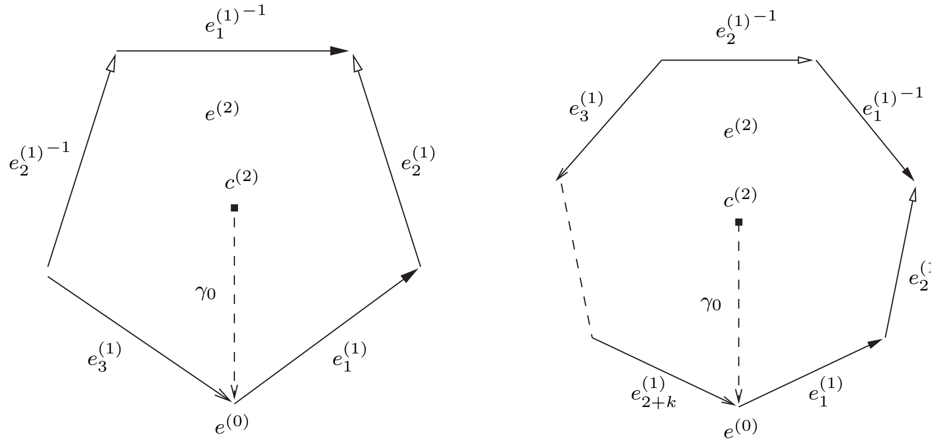
The table for the E_2-terms is in Figure 6. In particular, the Leray spectral sequence is degenerate at E^2-term in this case, and the cohomology groups of X are given by

$$H^k(X;\mathbb{Z})=\begin{cases}\mathbb{Z} & k=0,1,4\\ \mathbb{Z}/2\mathbb{Z} & k=2\\ \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & k=3\\ 0 & \text{otherwise.}\end{cases}$$

$q \backslash p$	0	1	2
2	0	$\mathbb{Z}$	$\mathbb{Z}$
1	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
0	$\mathbb{Z}$	$\mathbb{Z}$	0

FIGURE 6. the table of $(E_X^{p,q})_2$-terms for Example 4.7

Example 6.10 (Example 4.8 with $k = 0$)**.** Let us calculate the cohomology for Example 4.8 with $k = 0$. In this case, B is a compact, connected, oriented surface of genus $g \geq 0$, with boundary. (We include the case of $g = 0$, in which case, $X = S^3 \times S^1$.) Let Q_{4g+1} be the polygon with $4g + 1$ edges. Q_{4g+1} gives a cell decomposition of B with one zero-cell $e^{(0)}$, $2g + 1$ one-cells $e_1^{(1)}, \ldots, e_{2g+1}^{(1)}$, and one two-cell $e^{(2)}$ as in Figure 7. In the cell decomposition, one-cell $e_{2i-1}^{(1)}$, $e_{2i}^{(1)}$, and



(1) Q_{4g+1} for $g = 1$ and $k = 0$ (2) Q_{4g+k} for $g = 1$ and $k > 0$

FIGURE 7. polygons Q_{4g+1} for $k = 0$, Q_{4g+k} for $k > 0$, and the cell decompositions

$e_{2g+1}^{(1)}$ correspond to α_i, β_i, and γ in Figure 2, respectively. B can be obtained from Q_{4g+1} by identifying the oriented arrows corresponding to $e_i^{(1)}$ for $i = 1, \ldots, 2g$. Then we have the natural map $\varphi : Q_{4g+1} \to B$. In this case, the pull-back bundle $\varphi^* T_P^2$ is identified with the trivial bundle $Q_{4g+1} \times T^2$ and the pull-back $\varphi^* X$ of X is identified with the quotient space of $Q_{4g+1} \times T^2$ which is obtained by collapsing each fiber on the one-cell $e_{2g+1}^{(1)}$ and all vertices of Q_{4g+1} with $0 \times S^1$. In this case, all fibers $\mu^{-1}(c_\lambda^{(p)})$ except for $\mu^{-1}(c^{(0)})$ and $\mu^{-1}(c_{2g+1}^{(1)})$ are diffeomorphic to $\pi_T^{-1}(c_\lambda^{(p)}) \cong T^2$, whereas on the cell $e_\lambda^{(p)} = e^{(0)}$, $e_{2g+1}^{(1)}$, the fiber $\mu^{-1}(c_\lambda^{(p)})$ is diffeomorphic to $T^2/(0 \times S^1)$ and the map $\nu_{c_\lambda^{(p)}} : \pi_T^{-1}(c_\lambda^{(p)}) \to \mu^{-1}(c_\lambda^{(p)})$ can be identified with the natural projection $T^2 \to T^2/(0 \times S^1)$. For $e^{(0)}$ and $e^{(1)}_{2g+1}$, $\nu^*_{c^{(p)}_\lambda}\left(H^q(\mu^{-1}(c^{(p)}_\lambda);\mathbb{Z})\right)$ can be obtained by (6.2) in Example 6.9, and for the other cells, $\nu^*_{c^{(p)}_\lambda} : H^q(\mu^{-1}(c^{(p)}_\lambda);\mathbb{Z}) \to H^q(\pi_T^{-1}(c^{(p)}_\lambda);\mathbb{Z})$ is an isomorphism.

For $q = 0$, the cohomology $H^p(B;\mathcal{H}^0_X)$ of $\left(C^p(B;\mathcal{H}^0_X),\delta\right)$ is naturally identified with the cohomology $H^p(B;\mathbb{Z})$ with $\mathbb{Z}$-coefficient.

For $q = 1$, the degree p cochain $u \in C^p(B;\mathcal{H}^1_X)$ takes values as follows

$$u(e^{(p)}_\lambda) \in \nu^*_{c^{(p)}_\lambda}\left(H^q(\mu^{-1}(c^{(p)}_\lambda);\mathbb{Z})\right) = \begin{cases} \mathbb{Z}\oplus 0 & p=0 \text{ or } p=1 \text{ and } \lambda = 2g+1 \\ \mathbb{Z}\oplus\mathbb{Z} & p=1 \text{ and } \lambda = 1,\dots,2g \text{ or } p=2 \\ 0 & \text{otherwise.} \end{cases}$$

All differentials $\delta^p : C^p(B;\mathcal{H}^1_X) \to C^{p+1}(B;\mathcal{H}^1_X)$ vanish except for $p = 0, 1$, and in this case of $p = 0$, δ^0 is given as follows

$$\begin{aligned} \left(\delta^0 u\right)(e^{(1)}_{2i-1}) &= {}^t\rho(\alpha_i)^{-1}u(e^{(0)}) - u(e^{(0)}) \text{ for } i = 1,\dots,g, \\ \left(\delta^0 u\right)(e^{(1)}_{2i}) &= {}^t\rho(\beta_i)^{-1}u(e^{(0)}) - u(e^{(0)}) \text{ for } i = 1,\dots,g, \\ \left(\delta^0 u\right)(e^{(1)}_{2g+1}) &= {}^t\rho(\gamma)^{-1}u(e^{(0)}) - u(e^{(0)}) = 0. \end{aligned}$$

We identify each fiber $\pi_T^{-1}(c^{(1)}_\lambda)$ with $\pi_T^{-1}(c^{(0)})$ as in Example 6.9. Then δ^1 is given by

$$\begin{aligned} &\left(\delta^1 u\right)(e^{(2)}) \\ &= u(e^{(1)}_1) + {}^t\rho(\alpha_1)^{-1}u(e^{(1)}_2) - {}^t\rho(\alpha_1\beta_1\alpha_1^{-1})^{-1}u(e^{(1)}_1) - {}^t\rho([\alpha_1\beta_1])^{-1}u(e^{(1)}_2) \\ &+\cdots \\ &+ {}^t\rho(\prod_{i=1}^{g-1}[\alpha_i\beta_i])^{-1}u(e^{(1)}_{2g-1}) + {}^t\rho(\prod_{i=1}^{g-1}[\alpha_i\beta_i]\alpha_g)^{-1}u(e^{(1)}_{2g}) \\ &- {}^t\rho(\prod_{i=1}^{g-1}[\alpha_i\beta_i]\alpha_g\beta_g\alpha_g^{-1})^{-1}u(e^{(1)}_{2g-1}) - {}^t\rho(\prod_{i=1}^{g}[\alpha_i\beta_i])^{-1}u(e^{(1)}_{2g}) \\ &+ {}^t\rho(\prod_{i=1}^{g}[\alpha_i\beta_i])^{-1}u(e^{(1)}_{2g+1}) \\ &= \left(1 - {}^t\rho(\beta_1)^{-1}\right)u(e^{(1)}_1) + \left({}^t\rho(\alpha_1)^{-1} - 1\right)u(e^{(1)}_2) \\ &+\cdots \\ &= \left(1 - {}^t\rho(\beta_g)^{-1}\right)u(e^{(1)}_{2g-1}) + \left({}^t\rho(\alpha_g)^{-1} - 1\right)u(e^{(1)}_{2g}) + u(e^{(1)}_{2g+1}). \end{aligned}$$

Then the cohomology groups are calculated by

$$H^p(B;\mathcal{H}^1_X) = \begin{cases} \mathbb{Z} & p = 0,2 \\ \mathbb{Z}^{\oplus 4g} & p = 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $a_i = b_i = 0$. In the other case,

$$H^p(B;\mathcal{H}^1_X) = \begin{cases} \Lambda & p = 1 \\ \mathbb{Z}/(a_i,b_j)\mathbb{Z} & p = 2 \\ 0 & \text{otherwise,} \end{cases}$$

where (a_i, b_j) is the greatest common measure of all a_i and b_j which are not equal to 0 and

$$\Lambda = \left\{ (u_1, v_1, \ldots, u_g, v_g) \in \mathbb{Z}^{\oplus 2g} \colon \sum_{i=1}^{g} (b_i u_i - a_i v_i) = 0 \right\}$$
$$\oplus\, \mathbb{Z}^{\oplus 2g} / \left\{ (-a_1 u, -b_1 u, \ldots, -a_g u, -b_g u) \colon u \in \mathbb{Z} \right\}.$$

For $q = 2$, the degree p cochain $u \in C^p(B; \mathcal{H}^2_X)$ takes values as follows

$$u(e^{(p)}_\lambda) \in \nu^*_{c^{(p)}_\lambda} \left(H^q(\mu^{-1}(c^{(p)}_\lambda); \mathbb{Z}) \right) = \begin{cases} \mathbb{Z} & p = 1 \text{ and } \lambda = 1, \ldots, 2g \text{ or } p = 2 \\ 0 & \text{otherwise.} \end{cases}$$

All differentials $\delta^p : C^p(B; \mathcal{H}^2_X) \to C^{p+1}(B; \mathcal{H}^2_X)$ vanish. It is clear except for $p = 1$. In the case of $p = 1$, since all holonomies along $e^{(1)}_\lambda$ induce the identity of $H^2(\pi_T^{-1}(c^{(0)}); \mathbb{Z})$, the differential δ^1 is given by

$$\begin{aligned} \delta^1 u(e^{(2)}) =& u(e^{(1)}_1) + u(e^{(1)}_2) - u(e^{(1)}_1) - u(e^{(1)}_2) \\ & + \cdots \\ & + u(e^{(1)}_{2g-1}) + u(e^{(1)}_{2g}) - u(e^{(1)}_{2g-1}) - u(e^{(1)}_{2g}) + u(e^{(1)}_{2g+1}) = 0. \end{aligned}$$

Then the cohomology groups are obtained by

$$H^p(B; \mathcal{H}^2_X) = \begin{cases} \mathbb{Z}^{\oplus 2g} & p = 1 \\ \mathbb{Z} & p = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The table for the E_2-terms is in Figure 8. In particular, the Leray spectral sequence

(1) $a_i = b_j = 0$

$q \backslash p$	0	1	2
2	0	$\mathbb{Z}^{2g}$	$\mathbb{Z}$
1	$\mathbb{Z}$	$\mathbb{Z}^{4g}$	$\mathbb{Z}$
0	$\mathbb{Z}$	$\mathbb{Z}^{2g}$	0

(2) the other case

$q \backslash p$	0	1	2
2	0	$\mathbb{Z}^{2g}$	$\mathbb{Z}$
1	0	Λ	$\mathbb{Z}/(a_i, b_j)\mathbb{Z}$
0	$\mathbb{Z}$	$\mathbb{Z}^{2g}$	0

FIGURE 8. the table of $(E^{p,q}_X)_2$-terms for $k = 0$ in Example 4.8

is degenerate at E^2-term, too, and the cohomology groups of X are given by

$$H^k(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 4 \\ \mathbb{Z}^{\oplus 2g+1} & k = 1, 3 \\ \mathbb{Z}^{\oplus 4g} & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

for all $a_i = b_i = 0$. In the other case,

$$H^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0,4 \\ \mathbb{Z}^{\oplus 2g} & k = 1 \\ \Lambda & k = 2 \\ \mathbb{Z}^{\oplus 2g} \oplus \mathbb{Z}/(a_i, b_j)\mathbb{Z} & k = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Example 6.11 (Example 4.8 with $k = 2$)**.** In the case of $k = 2$ in Example 4.8, we must replace Q_{4g+1} in Example 6.10 with the polygon Q_{4g+2} with $4g+2$ edges $e_1^{(1)}$, $e_2^{(1)}$, $(e_1^{(1)})^{-1}$, $(e_2^{(1)})^{-1}$, …, $e_{2g-1}^{(1)}$, $e_{2g}^{(1)}$, $(e_{2g-1}^{(1)})^{-1}$, $(e_{2g}^{(1)})^{-1}$, $e_{2g+1}^{(1)}$, and $e_{2g+2}^{(1)}$. $e_{2i-1}^{(1)}$ and $e_{2i}^{(1)}$ correspond to α_i, β_i in Figure 2, respectively, and $e_{2g+1}^{(1)}$ and $e_{2g+2}^{(1)}$ correspond to the edge arcs γ_1 and γ_2 in Figure 2, respectively. See also Figure 7. As before, this gives B a cell decomposition with two zero-cells $e_1^{(0)}$ and $e_2^{(0)}$, $2g+2$ one-cells $e_1^{(1)}$, …, $e_{2g+2}^{(1)}$, and one two-cell $e^{(2)}$. By the same way in Example 6.10, we have the natural map $\varphi : Q_{4g+2} \to B$. In this case, the pull-back bundle $\varphi^* T_P^2$ is also identified with the trivial bundle $Q_{4g+2} \times T^2$ but the pull-back $\varphi^* X$ of X is identified with its quotient space which is obtained from $Q_{4g+2} \times T^2$ by collapsing each fiber on the one-cell $e_{2g+1}^{(1)}$ with $S^1 \times 0$, on the one-cell $e_{2g+2}^{(1)}$ with $0 \times S^1$, and on all vertices of Q_{4g+2} with T^2. The fibers $\mu^{-1}(c_\lambda^{(p)})$ are diffeomorphic as follows

$$\mu^{-1}(c_\lambda^{(p)}) \cong \begin{cases} \text{one point} & p = 0 \\ T^2/S^1 \times 0 & p = 1 \text{ and } \lambda = 2g+1 \\ T^2/0 \times S^1 & p = 1 \text{ and } \lambda = 2g+2 \\ \pi_T^{-1}(c_\lambda^{(p)}) \cong T^2 & \text{otherwise.} \end{cases}$$

For $e_\lambda^{(0)}$, $e_{2g+1}^{(1)}$, and $e_{2g+2}^{(1)}$, $\nu^*_{c_\lambda^{(p)}}\left(H^p(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z})\right)$ is identified as follows

$$\nu^*_{c_\lambda^{(0)}}\left(H^q(\mu^{-1}(c_\lambda^{(0)});\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu^*_{c_{2g+1}^{(1)}}\left(H^q(\mu^{-1}(c_{2g+1}^{(1)});\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 \oplus \mathbb{Z} & q = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu^*_{c_{2g+2}^{(1)}}\left(H^q(\mu^{-1}(c_{2g+2}^{(1)});\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z} \oplus 0 & q = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and $\nu^*_{c_\lambda^{(p)}} : H^q(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z}) \to H^q(\pi_T^{-1}(c_\lambda^{(p)});\mathbb{Z})$ is isomorphic for the other cells. The similar calculus as in Example 6.10 gives the table of the E_2-terms in Figure 9. The Leray spectral sequence is degenerate at E^2-term, and the cohomology groups of X are given by

$$H^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0,4 \\ \mathbb{Z}^{\oplus 2g} & k = 1,3 \\ \mathbb{Z}^{\oplus 4g} & k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Example 6.12 (Example 4.8 with $k = 3$)**.** In the case of $k = 3$ in Example 4.8, we take $X_\Delta = \mathbb{C}P^2$ as a symplectic toric manifold with the triangle as its Delzant polytope. In this case, we change Q_{4g+1} in Example 6.10 with the polygon Q_{4g+3}

with $4g+3$ edges $e_1^{(1)}$, $e_2^{(1)}$, $(e_1^{(1)})^{-1}$, $(e_2^{(1)})^{-1}$, …, $e_{2g-1}^{(1)}$, $e_{2g}^{(1)}$, $(e_{2g-1}^{(1)})^{-1}$, $(e_{2g}^{(1)})^{-1}$, $e_{2g+1}^{(1)}$, …, $e_{2g+3}^{(1)}$. $e_{2i-1}^{(1)}$ and $e_{2i}^{(1)}$ correspond to α_i, β_i in Figure 2, respectively, and $e_{2g+1}^{(1)}$, …, $e_{2g+3}^{(1)}$ correspond to the edge arcs γ_1, …, γ_3 in Figure 2, respectively. See also Figure 7. This gives B a cell decomposition with three zero-cells $e_1^{(0)}$, …, $e_3^{(0)}$, $2g+3$ one-cells $e_1^{(1)}$, …, $e_{2g+3}^{(1)}$, and one two-cell $e^{(2)}$. In this case, the pull-back bundle $\varphi^* T_P^2$ is also identified with the trivial bundle $Q_{4g+3} \times T^2$ but the pull-back $\varphi^* X$ of X is identified with its quotient space which is obtained from $Q_{4g+3} \times T^2$ by collapsing each fiber on the one-cell $e_{2g+1}^{(1)}$ with $S^1 \times 0$, on the one-cell $e_{2g+2}^{(1)}$ with $0 \times S^1$, on the one-cell $e_{2g+3}^{(1)}$ with the circle in T^2 generated by $(-1,-1)$ which we denote by $-\mathrm{diag}(S^1)$, and on all vertices of Q_{4g+3} with T^2. The fibers $\mu^{-1}(c_\lambda^{(p)})$ are diffeomorphic as follows

$$\mu^{-1}(c_\lambda^{(p)}) \cong \begin{cases} \text{one point} & p=0 \\ T^2/S^1 \times 0 & p=1 \text{ and } \lambda = 2g+1 \\ T^2/0 \times S^1 & p=1 \text{ and } \lambda = 2g+2 \\ T^2/-\mathrm{diag}(S^1) & P=1 \text{ and } \lambda = 2g+3 \\ \pi_T^{-1}(c_\lambda^{(p)}) \cong T^2 & \text{otherwise.} \end{cases}$$

For $e_\lambda^{(0)}$, $e_{2g+1}^{(1)}$, …, $e_{2g+3}^{(1)}$, $\nu^*_{c_\lambda^{(p)}}\left(H^p(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z})\right)$ is identified as follows

$$\nu^*_{c_\lambda^{(0)}}\left(H^q(\mu^{-1}(c_\lambda^{(0)});\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu^*_{c_{2g+1}^{(1)}}\left(H^q(\mu^{-1}(c_{2g+1}^{(1)});\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q=0 \\ 0 \oplus \mathbb{Z} & q=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu^*_{c_{2g+2}^{(1)}}\left(H^q(\mu^{-1}(c_{2g+2}^{(1)});\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z} \oplus 0 & q=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu^*_{c_{2g+3}^{(1)}}\left(H^q(\mu^{-1}(c_{2g+3}^{(1)});\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q=0 \\ \mathrm{offdiag}(\mathbb{Z}) & q=1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{offdiag}(\mathbb{Z})$ is the sub-lattice of $\mathbb{Z} \oplus \mathbb{Z}$ which is generated by $(1,-1)$. For other cells, $\nu^*_{c_\lambda^{(p)}} : H^p(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z}) \to H^p(\pi_T^{-1}(c_\lambda^{(p)});\mathbb{Z})$ is isomorphic. The similar calculus as in Example 6.10 gives the table of the E_2-terms in Figure 9. Then the Leray spectral sequence is degenerate at E^2-term, and the cohomology groups of X are given by

$$H^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0,4 \\ \mathbb{Z}^{\oplus 2g} & k=1,3 \\ \mathbb{Z}^{\oplus 4g+1} & k=2 \\ 0 & \text{otherwise.} \end{cases}$$

Example 6.13 (Example 4.8 with $k=4$)**.** In the case of $k=4$ in Example 4.8, we take $X_\Delta = S^2 \times S^2$ as a symplectic toric manifold with the square as its Delzant polytope. In this case, we change Q_{4g+1} in Example 6.10 with the polygon Q_{4g+4} with $4g+4$ edges $e_1^{(1)}$, $e_2^{(1)}$, $(e_1^{(1)})^{-1}$, $(e_2^{(1)})^{-1}$, …, $e_{2g-1}^{(1)}$, $e_{2g}^{(1)}$, $(e_{2g-1}^{(1)})^{-1}$, $(e_{2g}^{(1)})^{-1}$, $e_{2g+1}^{(1)}$, …, $e_{2g+4}^{(1)}$. $e_{2i-1}^{(1)}$ and $e_{2i}^{(1)}$ correspond to α_i, β_i in Figure 2, respectively, and

$e^{(1)}_{2g+1}$, …, $e^{(1)}_{2g+4}$ correspond to the edge arcs γ_1, …, γ_4 in Figure 2, respectively. See also Figure 7. This gives B a cell decomposition with four zero-cells $e^{(0)}_1$, …, $e^{(0)}_4$, $2g+4$ one-cells $e^{(1)}_1$, …, $e^{(1)}_{2g+4}$, and one two-cell $e^{(2)}$. In this case, the pull-back bundle $\varphi^* T^2_P$ is also identified with the trivial bundle $Q_{4g+4} \times T^2$ but the pull-back $\varphi^* X$ of X is identified with its quotient space which is obtained from $Q_{4g+4} \times T^2$ by collapsing each fiber on the one-cell $e^{(1)}_{2g+1}$ with $S^1 \times 0$, on the one-cell $e^{(1)}_{2g+2}$ with $0 \times S^1$, on the one-cell $e^{(1)}_{2g+3}$ with $-S^1 \times 0$, on the one-cell $e^{(1)}_{2g+4}$ with $0 \times -S^1$, and on all vertices of Q_{4g+4} with T^2, where $-S^1 \times 0$ is the circle generated by $(-1, 0)$ etc. The fibers $\mu^{-1}(c^{(p)}_\lambda)$ are diffeomorphic as follows

$$\mu^{-1}(c^{(p)}_\lambda) \cong \begin{cases} \text{one point} & p = 0 \\ T^2/S^1 \times 0 & p = 1 \text{ and } \lambda = 2g+1 \\ T^2/0 \times S^1 & p = 1 \text{ and } \lambda = 2g+2 \\ T^2/-S^1 \times 0 & P = 1 \text{ and } \lambda = 2g+3 \\ T^2/0 \times -S^1 & P = 1 \text{ and } \lambda = 2g+4 \\ \pi_T^{-1}(c^{(p)}_\lambda) \cong T^2 & \text{otherwise.} \end{cases}$$

For $e^{(0)}_\lambda$, $e^{(1)}_{2g+1}$, …, $e^{(1)}_{2g+4}$, $\nu^*_{c^{(p)}_\lambda}\left(H^p(\mu^{-1}(c^{(p)}_\lambda);\mathbb{Z})\right)$ is identified as follows

$$\nu^*_{c^{(0)}_\lambda}\left(H^q(\mu^{-1}(c^{(0)}_\lambda);\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu^*_{c^{(1)}_\lambda}\left(H^q(\mu^{-1}(c^{(1)}_\lambda);\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 \oplus \mathbb{Z} & q = 1 \\ 0 & \text{otherwise,} \end{cases} \qquad (\lambda = 2g+1, 2g+3)$$

$$\nu^*_{c^{(1)}_\lambda}\left(H^q(\mu^{-1}(c^{(1)}_\lambda);\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z} \oplus 0 & q = 1 \\ 0 & \text{otherwise} \end{cases} \qquad (\lambda = 2g+2, 2g+4),$$

and $\nu^*_{c^{(p)}_\lambda} : H^p(\mu^{-1}(c^{(p)}_\lambda);\mathbb{Z}) \to H^p(\pi_T^{-1}(c^{(p)}_\lambda);\mathbb{Z})$ is an isomorphism for the other cells. The similar calculus as in Example 6.10 gives the table of the E_2-terms in Figure 9. Then the Leray spectral sequence is degenerate at E^2-term, and the

q \ p	0	1	2
2	0	$\mathbb{Z}^{2g}$	$\mathbb{Z}$
1	0	$\mathbb{Z}^{4g+k-2}$	0
0	$\mathbb{Z}$	$\mathbb{Z}^{2g}$	0

FIGURE 9. $(E^{p,q}_X)_2$-terms for $k = 2, 3, 4$ in Example 4.8

cohomology groups of X are given by

$$H^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 4 \\ \mathbb{Z}^{\oplus 2g} & k = 1, 3 \\ \mathbb{Z}^{\oplus 4g+2} & k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Example 6.14 (Example 4.9)**.** Let us calculate the cohomology groups for Example 4.9. In this case, we take the pentagon Q_5 with edges $e_1^{(1)}$, $e_2^{(1)}$, $(e_1^{(1)})^{-1}$, $(e_2^{(1)})^{-1}$, and $e_3^{(1)}$. $e_1^{(1)}$, $e_2^{(1)}$, and $e_3^{(1)}$ correspond to α, β, and the edge arc γ in Figure 2, respectively. See also Figure 7. This gives B a cell decomposition with one zero-cell $e^{(0)}$, three one-cells $e_1^{(1)}, \ldots, e_3^{(1)}$, and one two-cell $e^{(2)}$. In this case, the pull-back bundle $\varphi^* T_P^2$ is also identified with the trivial bundle $Q_5 \times T^2$ but the pull-back $\varphi^* X$ of X is identified with its quotient space which is obtained from $Q_5 \times T^2$ by collapsing each fiber on the one-cell $e_3^{(1)}$ with $0 \times S^1$, and on all vertices of Q_5 with T^2. The fibers $\mu^{-1}(c_\lambda^{(p)})$ are diffeomorphic as follows

$$\mu^{-1}(c_\lambda^{(p)}) \cong \begin{cases} \text{one point} & p = 0 \\ T^2/0 \times S^1 & p = 1 \text{ and } \lambda = 3 \\ \pi_T^{-1}(c_\lambda^{(p)}) \cong T^2 & \text{otherwise.} \end{cases}$$

For $e^{(0)}$ and $e_3^{(1)}$, $\nu^*_{c_\lambda^{(p)}}\left(H^p(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z})\right)$ is identified as follows

$$\nu^*_{c^{(0)}}\left(H^q(\mu^{-1}(c^{(0)});\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu^*_{c_3^{(1)}}\left(H^q(\mu^{-1}(c_3^{(1)});\mathbb{Z})\right) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z} \oplus 0 & q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

and $\nu^*_{c_\lambda^{(p)}} : H^p(\mu^{-1}(c_\lambda^{(p)});\mathbb{Z}) \to H^p(\pi_T^{-1}(c_\lambda^{(p)});\mathbb{Z})$ is an isomorphism for the other cells. The similar calculus as in Example 6.10 gives the table of the E_2-terms in Figure 10. Then the Leray spectral sequence is degenerate at E^2-term, and the

$q \backslash p$	0	1	2
2	0	$\mathbb{Z}^2$	$\mathbb{Z}$
1	0	$\mathbb{Z}^3$	0
0	$\mathbb{Z}$	$\mathbb{Z}^2$	0

FIGURE 10. $(E_X^{p,q})_2$-terms for Example 4.9

cohomology groups of X are given by

$$H^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0,4 \\ \mathbb{Z}^{\oplus 2} & k=1,3 \\ \mathbb{Z}^{\oplus 3} & k=2 \\ 0 & \text{otherwise.} \end{cases}$$

6.3. **Signatures.** In this subsection, we shall give the method of computing the signature for a four-dimensional case by using the Novikov additivity. Let B be a surface with at least one corner, and X a twisted toric manifold associated with a principal $SL_2(\mathbb{Z})$-bundle P on B. For simplicity, assume that B has only one boundary component. We divide B into two parts B_1 and B_2, where B_2 is the closed neighborhood of the boundary ∂B such that ∂B is a deformation retract of B_2 and B_1 is the closure $B_1 = \overline{B \backslash B_2}$ of the remainder. We set $X_i = \mu^{-1}(B_i)$ for $i = 1, 2$, and denote by $\sigma(X_i)$ and $\sigma(X)$ the signature of X_i and X, respectively. The Novikov additivity says that

$$\sigma(X) = \sigma(X_1) + \sigma(X_2). \tag{6.3}$$

First let us compute the signature $\sigma(X_1)$ of X_1. We notice that X_1 is the associated T^2-bundle for P. When the genus of B is equal to zero, B_1 is contractible. In this case, the signature $\sigma(X_1)$ is zero.

When the genus of B is greater than zero, we give B_1 a trinion decomposition $B_1 = \cup_{i=1}^k (B_1)_i$, where each $(B_1)_i$ is a trinion, that is, a surface obtained from S^2 by removing three distinct open discs. Let $(X_1)_i = \mu^{-1}((B_1)_i)$ for $i = 1, \dots, k$. From the Novikov additivity, we have

$$\sigma(X_1) = \sum_{i=1}^k \sigma((X_1)_i), \tag{6.4}$$

and each $\sigma((X_1)_i)$ can be computed as follows. We take the oriented boundary loops γ_1, γ_2, and γ_3 of $(B_1)_i$ as in Figure 11 which represent generators of $\pi_1((B_1)_i)$ with $[\gamma_1]\cdot[\gamma_2]\cdot[\gamma_3] = 1$. Let $\rho : \pi_1((B_1)_i) \to SL_2(\mathbb{Z})(= Sp(2;\mathbb{Z}))$ be the representation which determines T^2-bundle $(X_1)_i$ on $(B_1)_i$. We set $C_j = \rho([\gamma_j])$ for $j = 1, 2, 3$. For C_1 and C_2, define the vector space V_{C_1,C_2} and the bilinear form $\langle\ ,\ \rangle_{C_1,C_2}$ on

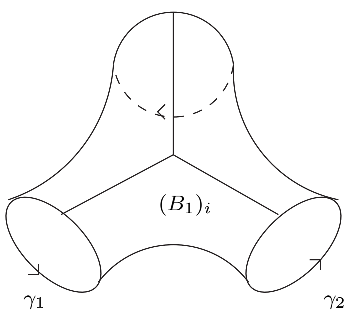


FIGURE 11. $(B_1)_i$ and γ_j

V_{C_1,C_2} by

$$V_{C_1,C_2} = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \colon (C_1^{-1} - I)x + (C_2 - I)y = 0\},$$
$$\langle (x,y),(x',y')\rangle_{C_1,C_2} =^t (x+y)J(I - C_2)y'$$

for $(x,y),\ (x',y') \in V_{C_1,C_2}$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It is easy to show that $\langle\ ,\ \rangle_{C_1,C_2}$ is symmetric and we denote the signature of $\langle\ ,\ \rangle_{C_1,C_2}$ by $\tau_1(C_1,C_2)$.

Theorem 6.15 ([5, 14])**.** $\sigma((X_1)_i) = \tau_1(C_1,C_2)$.

We should notice that our orientation of $(X_1)_i$ is different from that in [5, 14]. From (6.4) and Theorem 6.15, we can compute $\sigma(X_1)$.

Remark 6.16. Meyer shows in [14] that τ_1 defines the cocycle $\tau_1 : Sp(2;\mathbb{Z}) \times Sp(2;\mathbb{Z}) \to \mathbb{Z}$ of $Sp(2;\mathbb{Z})$ which is called Meyer's signature cocycle. The author was taught Meyer's signature cocycle by Endo [5].

Next, we shall compute $\sigma(X_2)$. For X_2, we can show the following lemma.

Lemma 6.17. *$\mu^{-1}(\partial B)$ is a deformation retract of X_2.*

Proof. Let $h : B_2 \times I \to B_2$ be a deformation retraction with $h(\cdot,0) = \mathrm{id}_{B_2}$ and $h(\cdot,1) \in \partial B$. Define the map $\widetilde{h} : X_2 \times I \to X_2$ of h by

$$\widetilde{h}(x,s) = \nu \circ \mathrm{Hol}_{\gamma_{\mu(x),s}}(x)$$

for $(x,s) \in X_2 \times I$, where $\gamma_{\mu(x),s}$ is the path $\gamma_{\mu(x),s} : I \to B_2$ which is defined by $\gamma_{\mu(x),s}(t) = h(\mu(x), st)$ for $t \in I$ and $\mathrm{Hol}_{\gamma_{\mu(x),s}}$ is the parallel transport $\mathrm{Hol}_{\gamma_{\mu(x),s}} : \pi_T^{-1}(\mu(x)) \to \pi_T^{-1}(\gamma_{\mu(x),s}(1))$ of T_P^2 along $\gamma_{\mu(x),s}$ with respect to the connection induced from that of P. Then $\widetilde{h}$ is a deformation retraction. $\square$

Assume that B has k corner points. Then the one-dimensional strata $\mathcal{S}^{(1)}B$ has exactly k connected components $(\mathcal{S}^{(1)}B)_1, \ldots, (\mathcal{S}^{(1)}B)_k$, and $\mu^{-1}(\partial B) = \cup_{i=1}^k \mu^{-1}(\overline{(\mathcal{S}^{(1)}B)_i})$, where $\overline{(\mathcal{S}^{(1)}B)_i}$ is the closure of $(\mathcal{S}^{(1)}B)_i$. It is easy to see from the similar construction of the twisted toric structure on a neighborhood of ∂B as in Example 4.9 and Theorem 5.8 that each $\mu^{-1}(\overline{(\mathcal{S}^{(1)}B)_i})$ is homeomorphic to the two-dimensional sphere S^2 if $k \geq 2$, and is homeomorphic to S^2 with one self-intersection at north and south points if $k = 1$. If $\overline{(\mathcal{S}^{(1)}B)_i} \cap \overline{(\mathcal{S}^{(1)}B)_j} \neq \emptyset$ for $i \neq j$, then they have two intersections if $k = 2$, and have one intersection if $k > 2$. In all cases, the intersections are transversal since by definition, a neighborhood of each intersection in X is identified with that of the intersection of $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$ in $\mathbb{C}^2$. Then $\mu^{-1}(\partial B)$ looks like a necklace of k spheres and the homology group of X_2 is given by

$$H_p(X_2;\mathbb{Z}) = H_p(\mu^{-1}(\partial B);\mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0,1 \\ \mathbb{Z}^{\oplus k} & p = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover the homology classes $[\mu^{-1}(\overline{(\mathcal{S}^{(1)}B)_1})], \ldots, [\mu^{-1}(\overline{(\mathcal{S}^{(1)}B)_k})] \in H_2(X_2;\mathbb{Z})$ represented by $\mu^{-1}(\overline{(\mathcal{S}^{(1)}B)_i})$ for $i = 1, \ldots, k$ are generators of $H_2(X_2;\mathbb{Z})$. We set $S_i^2 = \mu^{-1}(\overline{(\mathcal{S}^{(1)}B)_i})$ for $i = 1, \ldots, k$. From the above fact, we can obtain the following proposition.

Proposition 6.18. *For $i \neq j$, the intersection number $[S_i^2] \cdot [S_j^2]$ of $[S_i^2]$ and $[S_j^2]$ is given as follows*

$$[S_i^2] \cdot [S_j^2] = \begin{cases} 0 & S_i^2 \cap S_j^2 = \emptyset \\ 1 & S_i^2 \cap S_j^2 \neq \emptyset \text{ and } k > 2 \\ 2 & S_i^2 \cap S_j^2 \neq \emptyset \text{ and } k = 2. \end{cases}$$

Assume that $k > 1$. Let us compute the self-intersection number of $[S_i^2]$. We can take a contractible neighborhood U of $\overline{(\mathcal{S}^{(1)}B)_i}$ in B so that $U \cap \mathcal{S}^{(1)}B$ has exactly two connected components $(U \cap \mathcal{S}^{(1)}B)_1$ and $(U \cap \mathcal{S}^{(1)}B)_2$ except for $(\mathcal{S}^{(1)}B)_i$. We may assume that $(U \cap \mathcal{S}^{(1)}B)_1$ and $(U \cap \mathcal{S}^{(1)}B)_2$ are located in Figure 12. Let

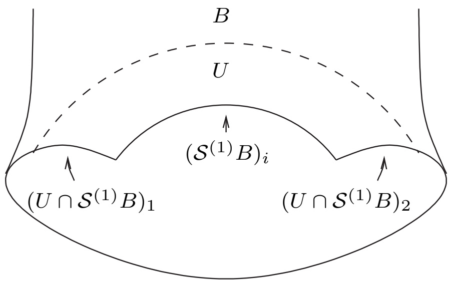


FIGURE 12. $(\mathcal{S}^{(1)}B)_i$ and $(U \cap \mathcal{S}^{(1)}B)_a$

$\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(1)}B$ be the characteristic bundle of X. Since $\mathcal{L}$ is primitive, we can take the local trivialization $\varphi^{\mathbb{Z}} : \pi_{\mathbb{Z}}^{-1}(U) \cong U \times \mathbb{Z}^2$ of $\pi_{\mathbb{Z}} : \mathbb{Z}_P^2 \to B$ so that $\varphi^{\mathbb{Z}}$ also gives the trivializations $\pi_{\mathcal{L}}^{-1}((\mathcal{S}^{(1)}B)_i) \cong (\mathcal{S}^{(1)}B)_i \times L$ and $\pi_{\mathcal{L}}^{-1}((U \cap \mathcal{S}^{(1)}B)_a) \cong (U \cap \mathcal{S}^{(1)}B)_a \times L_a$ on $(\mathcal{S}^{(1)}B)_i$ and $(U \cap \mathcal{S}^{(1)}B)_a$ for $a = 1, 2$, where L and L_a are rank one sub-lattices of $\mathbb{Z}^2$. We take the generators u, u_a of L and L_a such that both of the determinants of (u_1, u) and (u, u_2) are equal to one, where (u_1, u) (resp. (u, u_2)) denotes the matrix given by arranging the column vectors u_1 and u (resp. u and u_2) in this order.

Proposition 6.19. *The self-intersection number $[S_i^2] \cdot [S_i^2]$ is equal to the negative determinant $-\det(u_1, u_2)$ of (u_1, u_2).*

Remark 6.20. The determinant of (u_1, u_2) does not depend on the choice of the local trivialization $\varphi^{\mathbb{Z}}$ since the structure group of the bundle is $SL_2(\mathbb{Z})$.

Proof. Since the self-intersection number of S_i^2 is equal to the Euler number $\int_{S_i^2} e(\mathcal{N}_{S_i^2})$ of the normal bundle $\mathcal{N}_{S_i^2}$ of S_i^2 in X, for example, see [3], we identify $\mathcal{N}_{S_i^2}$. If necessary, by replacing the local trivialization, we may assume that $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then u_2 must be of the form $u_2 = \begin{pmatrix} -1 \\ m \end{pmatrix}$ for some $m \in \mathbb{Z}$ since $\det(u, u_2) = 1$. The primitive vector u_2 defines the Hamiltonian S^1-action on $(\mathbb{C}^2, \omega_{\mathbb{C}^2})$ by

$$t \cdot z = (e^{-2\pi\sqrt{-1}t} z_1, e^{2\pi\sqrt{-1}mt} z_2)$$

with the moment map

$$\mu_{u_2}(z) = -|z_1|^2 + m|z_2|^2.$$

For $\varepsilon < 0$, $\overline{\mathbb{C}^2}_{\mu_{u_2} \geq \varepsilon}$ denotes the cut space which is obtained from $\mathbb{C}^2$ by the symplectic cutting for this circle action. More precisely, let $\Phi : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{R}$ be the map defined by

$$\Phi(z, w) = -|z_1|^2 + m|z_2|^2 - |w|^2 - \varepsilon$$

for $(z, w) \in \mathbb{C}^2 \times \mathbb{C}$. By the construction of the symplectic cutting, $\overline{\mathbb{C}^2}_{\mu_{u_2} \geq \varepsilon} = \Phi^{-1}(0)/S^1$, where the S^1-action on $\Phi^{-1}(0)$ is

$$t \cdot (z, w) = (t \cdot z, e^{-2\pi\sqrt{-1}t} w).$$

Then it is easy to see from the similar construction of the twisted toric structure on a neighborhood of ∂B as in Example 4.9 and Theorem 5.8 that the the neighborhood of S^2_i in X is orientation preserving diffeomorphic to that of

$$\{(z,w) \in \Phi^{-1}(0) \colon z_2 = 0\}/S^1$$

in $\overline{\mathbb{C}^2}_{\mu_{u_2} \geq \varepsilon}$ for some $\varepsilon < 0$. By this diffeomorphism, we identify S^2_i with

$$\{(z,w) \in \Phi^{-1}(0) \colon z_2 = 0\}/S^1.$$

Let $\pi_1 : \Phi^{-1}(0) \to \overline{\mathbb{C}^2}_{\mu_{u_2} \geq \varepsilon}$ and $\pi_2 : \{(z,w) \in \Phi^{-1}(0) \colon z_2 = 0\} \to S^2_i$ be the natural projections. It is easy to see from the symplectic cutting construction that

$$\pi_1^* T\overline{\mathbb{C}^2}_{\mu_{u_2} \geq \varepsilon} \oplus \underline{\mathrm{Lie}(S^1) \otimes_{\mathbb{R}} \mathbb{C}} = T(\mathbb{C}^2 \times \mathbb{C})\Big|_{\Phi^{-1}(0)}$$

$$\pi_2^* TS^2_i \oplus \underline{\mathrm{Lie}(S^1) \otimes_{\mathbb{R}} \mathbb{C}} = T(\mathbb{C} \times \{0\} \times \mathbb{C})\Big|_{\{(z,w) \in \Phi^{-1}(0) \colon z_2 = 0\}},$$

where $\underline{\mathrm{Lie}(S^1) \otimes_{\mathbb{R}} \mathbb{C}}$ is the trivial complex line bundle. By the above identification, the normal bundle $\mathcal{N}_{S^2_i}$ of S^2_i in X is isomorphic to the associated complex line bundle

$$\{(z,w) \in \Phi^{-1}(0) \colon z_2 = 0\} \times_{S^1} \mathbb{C} \to S^2_i$$

of $\pi_2 : \{(z,w) \in \Phi^{-1}(0) \colon z_2 = 0\} \to S^2_i$ with respect to the irreducible S^1-representation of weight m. This is the complex line bundle $\mathcal{O}(-m)$ on $\mathbb{C}P^1$. Since the Euler class is equal to the first Chern class for a complex line bundle, the Euler number $\displaystyle\int_{S^2_i} e(\mathcal{N}_{S^2_i})$ is equal to $-m$. □

From Proposition 6.18 and Proposition 6.19, we can compute the signature $\sigma(X_2)$ case-by-case for $k > 1$. In the case of $k = 1$, by blowing up the fiber on the corner point, which consists of the one point, of B, we can reduce to the case of $k > 1$ as in the following example.

Example 6.21. Let us compute the signature of the twisted toric manifold X in Example 4.9. Recall that B is a surface of genus one with one corner. As described above, we divide B into two part B_1 and B_2. We give B_1 the trinion decomposition as in Figure 13. Then the easy computation shows that the value $\tau_1(\rho([\alpha^{-1}]), \rho([\gamma^{-1}]))$ of the Meyer cocycle vanishes. This implies the signature $\sigma(X_1)$ of X_1 is zero.

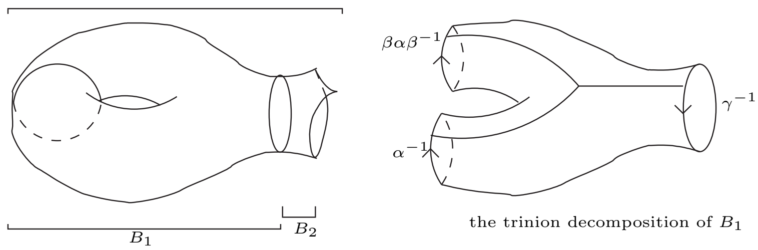


FIGURE 13. B, B_i, and the trinion decomposition of B_1

Next we consider X_2. In general, the fiber of $\mu : X \to B$ on the corner point consists of only one point which we denote by x_0, and by definition, the neighborhood of x_0 is orientation preserving diffeomorphic to that of the origin of $\mathbb{C}^2$. Then we can blow up X_2 at x_0 and denote by $\widetilde{X_2}$ its blow-up. $\widetilde{X_2}$ is a twisted toric manifold on the surface $\widetilde{B_2}$ with two corners. For a blowing up of a symplectic toric manifold, see [7, 15]. Let $(\mathcal{S}^{(1)}\widetilde{B_2})_1$ and $(\mathcal{S}^{(1)}\widetilde{B_2})_2$ denote connected components of the codimension one strata of $\widetilde{B_2}$ as in Figure 14. Since the inverse image

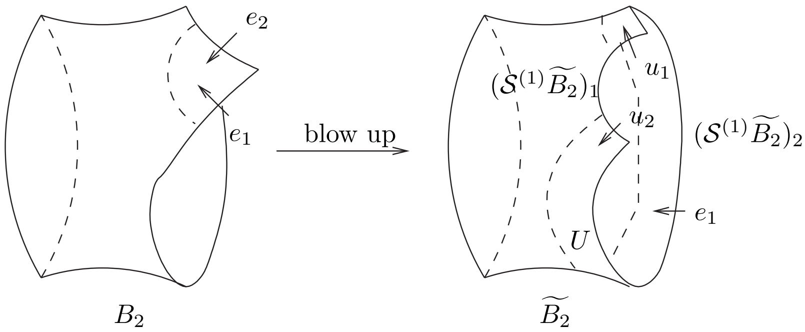


Figure 14. B_2 and $\widetilde{B_2}$

$S_1^2 = \mu^{-1}(\overline{(\mathcal{S}^{(1)}\widetilde{B_2})_1})$ of $(\mathcal{S}^{(1)}\widetilde{B_2})_1$ is an exceptional divisor, its self-intersection number $[S_1^2]\cdot[S_1^2]$ is equal to -1.

We compute the self-intersection number $[S_2^2]\cdot[S_2^2]$ of $S_2^2 = \mu^{-1}(\overline{(\mathcal{S}^{(1)}\widetilde{B_2})_2})$. By the construction of X_2 in Example 4.9, we can take u_1, u_2 in Proposition 6.19 of the forms $u_1 = \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, hence $[S_2^2]\cdot[S_2^2] = -\det(u_1, u_2) = -5$. The above computation and Proposition 6.18 for $k = 2$ show that the intersection matrix of $\widetilde{X_2}$ is $\begin{pmatrix} -1 & 2 \\ 2 & -5 \end{pmatrix}$ and the signature $\sigma(\widetilde{X_2})$ is equal to -2. Since $\widetilde{X_2}$ is the blow-up of X_2 at x_0, the signature of X_2 is $\sigma(X_2) = \sigma(\widetilde{X_2}) + 1 = -1$. Then the computations of $\sigma(X_1)$ and $\sigma(X_2)$ together with (6.3) shows that $\sigma(X) = -1$.

Appendix A. Smoothness of cut spaces

For $u_1, \dots, u_d \in \mathbb{Z}^n$, define the Hamiltonian T^d-action on $(T^*T^n \times \mathbb{C}^d, \omega_{T^*T^n} \oplus \omega_{\mathbb{C}^d})$ by

$$t \cdot (\xi, \theta, z) = (\xi, \theta + \sum_{i=1}^{d} t_i u_i, (e^{-2\pi\sqrt{-1}t_i} z_i)).$$

for $t = (t_1, \dots, t_d) \in T^d$ and $(\xi, \theta, z) \in T^*T^n \times \mathbb{C}^d$ with the moment map $\Phi : T^*T^n \times \mathbb{C}^d \to \mathbb{R}^d$

$$\Phi(\xi, \theta, z) = (\langle u_i, \xi\rangle - |z_i|^2).$$

For $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$, we define the simultaneous cut space $\overline{(T^*T^n)}_{\{\mu_{u_i} \geq \lambda_i\}_{i=1,\dots,d}}$ by

$$\overline{(T^*T^n)}_{\{\mu_{u_i} \geq \lambda_i\}_{i=1,\dots,d}} = \Phi^{-1}(\lambda)/T^d.$$

Let us investigate the smoothness of $\overline{(T^*T^n)}_{\{\mu_{u_i} \geq \lambda_i\}_{i=1,\dots,d}}$.

Lemma A.1. *For a linear independent tuple $\{u_1, \dots, u_k\}$ of k vectors in $\mathbb{Z}^n$, the following conditions are equivalent.*

(i) *$\{u_1, \dots, u_k\}$ is primitive*

(ii) *if $(t_1, \dots, t_k) \in \mathbb{R}^k$ satisfies $\sum_{i=1}^{k} t_i u_i \in \mathbb{Z}^n$, then $(t_1, \dots, t_k) \in \mathbb{Z}^k$*

Proof. Let $A = (u_1, \dots, u_k) \in M_{n\times k}(\mathbb{Z})$, and $\varphi_A : \mathbb{R}^k \to \mathbb{R}^n$ be the linear map defined by A. Then

$$\begin{aligned}\text{(ii)} &\iff \varphi_A^{-1}(\mathbb{Z}^n) = \mathbb{Z}^k \\ &\iff \mathrm{Im}\varphi_A \cap \mathbb{Z}^n = \varphi_A(\mathbb{Z}^k)\end{aligned}$$

since φ_A is injective. (The condition $\mathrm{Im}\varphi_A \cap \mathbb{Z}^n \supset \varphi_A(\mathbb{Z}^k)$ is trivial since $A \in M_{n\times k}(\mathbb{Z})$.)

On the other hand, if necessary, by replacing a basis of $\mathbb{Z}^n$, A can be of the form

$$A = \begin{array}{c c} & k \\ \begin{array}{c} k \\ \\ n-k \end{array} & \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_k \\ & \mathbf{0} & \end{pmatrix} \end{array},$$

and

$$\begin{aligned} \mathrm{Im}\varphi_A &= \overbrace{\mathbb{R} \oplus \cdots \oplus \mathbb{R}}^{k} \oplus \{0\} \oplus \cdots \oplus \{0\} \subset \mathbb{R}^n \\ \mathrm{Im}\varphi_A \cap \mathbb{Z}^n &= \overbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}^{k} \oplus \{0\} \oplus \cdots \oplus \{0\} \subset \mathbb{Z}^n \\ \varphi_A(\mathbb{Z}^k) &= d_1\mathbb{Z} \oplus \cdots \oplus d_k\mathbb{Z} \oplus \{0\} \oplus \cdots \oplus \{0\} \subset \mathbb{Z}^n. \end{aligned}$$

(Use the fundamental divisor theory.) Then

$$\begin{aligned} \therefore\ \ \mathrm{Im}\varphi_A \cap \mathbb{Z}^n \subset \varphi_A(\mathbb{Z}^k) &\iff d_i = \pm 1\ (i = 1, \dots, k) \\ &\iff \{u_1, \dots, u_k\}\text{: primitive.} \end{aligned}$$

□

Assume $\Phi^{-1}(\lambda) \neq \emptyset$. For each $(\xi, \theta, z) \in \Phi^{-1}(\lambda)$, we set

$$I_{(\xi,\theta,z)} = \{i \in \{1, \dots, d\} | z_i = 0\}.$$

Lemma A.2. (1) *The following conditions are equivalent.*

(i) *λ is a regular value of Φ.*
(ii) *For arbitrary $(\xi, \theta, z) \in \Phi^{-1}(\lambda)$, $\{u_i\}_{i \in I_{(\xi,\theta,z)}}$ is linear independent.*

(2) *Under the condition of* (1)*, the following conditions are equivalent.*

(i) *T^k-action on $\Phi^{-1}(0)$ is free.*
(ii) *For arbitrary $(\xi, \theta, z) \in \Phi^{-1}(\lambda)$, $\{u_i\}_{i \in I_{(\xi,\theta,z)}}$ is primitive.*

Proof. Φ can be decomposed into two maps $\mu_{\mathbb{C}^d} : \mathbb{C}^d \to \mathbb{R}^d$ in Example 2.2 and $\Phi_1 : T^*T^n \to \mathbb{R}^d$ which is defined by

$$\Phi_1(\xi, \theta) = {}^tB\xi,$$

where $B = (u_1, \dots, u_d) \in M_{n\times d}(\mathbb{Z})$. Let us pay attention to $\mu_{\mathbb{C}^d}$. Since $\mu_{\mathbb{C}^d}$ is the moment map of T^d-action on $\mathbb{C}^d$, we have $\mathrm{Im}(d(\mu_{\mathbb{C}^d})_z)^\perp = \mathrm{Lie}(T^d_z)$, where T^d_z is the stabilizer of z of T^d-action on $\mathbb{C}^d$. Since T^d_z can be written by

$$T^d_z = \{t \in T^d \colon i \notin I_{(\xi,\theta,z)} \Rightarrow t_i = 0\},$$

we have

$$\mathrm{Im}(d(\mu_{\mathbb{C}^d})_z)^\perp = \{(\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d \colon i \notin I_{(\xi,\theta,z)} \Rightarrow \lambda_i = 0\}. \tag{A.1}$$

On the other hand, $d(\Phi_1)_{(\xi,\theta)} : T_{(\xi,\theta)}(T^*T^n) \cong T_\xi\mathbb{R}^n \oplus T_\theta T^n \to \mathbb{R}^d$ is

$$d(\Phi_1)_{(\xi,\theta)}(v_{\mathbb{R}^n}, v_{T^n}) = {}^tBv_{\mathbb{R}^n}. \tag{A.2}$$

From (A.1) and (A.2), we have

$$(\xi,\theta,z) \text{ is a regular point of } \Phi \Leftrightarrow \{u_i\}_{i\in I_{(\xi,\theta,z)}} \text{are linear independent.}$$

This proves (1).

(2) is obtained from the fact that the stabilizer $T^d_{(\xi,\theta,z)}$ of (ξ,θ,z) of T^d-action on $T^*T^n \times \mathbb{C}^d$ is written by

$$T^d_{(\xi,\theta,z)} = \{t \in T^d \colon \sum_{i\in I_{(\xi,\theta,z)}} t_i u_i \in \mathbb{Z}^n\}$$

and Lemma A.1. □

Theorem A.3. *$\overline{(T^*T^n)}_{\{\mu_{u_i}\geq\lambda_i\}_{i=1,\dots,d}}$ is smooth, if and only if for each $[\xi,\theta,z] \in \overline{(T^*T^n)}_{\{\mu_{u_i}\geq\lambda_i\}_{i=1,\dots,d}}$, $\{u_i\}_{i\in I_{(\xi,\theta,z)}}$ is primitive.*

Proof. This is a direct consequence of Lemma A.2. □

By Remark 2.7, $\overline{(T^*T^n)}_{\{\mu_{u_i}\geq\lambda_i\}_{i=1,\dots,d}}$ is equipped with the Hamiltonian T^n-action which is induced from the natural T^n-action on T^*T^n.

Theorem A.4. *Under the condition in Theorem A.3, in the particular case where $d \leq n$ and $\lambda_1 = \cdots = \lambda_d = 0$, there exists an automorphism ρ of T^n such that $\overline{(T^*T^n)}_{\{\mu_{u_i}\geq 0\}_{i=1,\dots,d}}$ is ρ-equivariantly symplectomorphic to the T^n-action on $(\mathbb{C}^d \times T^*T^{n-d}, \omega_{\mathbb{C}^d} \oplus \omega_{T^*T^{n-d}})$ that is the direct product of the T^d-action on $\mathbb{C}^d$ in Example 2.2 and the T^{n-d}-action on T^*T^{n-d} in Example 2.3.*

Proof. For primitive $u_1, \dots, u_d$, there exists an element $\rho \in \mathrm{GL}_n(\mathbb{Z})$ such that ρ maps u_i to the ith fundamental vector e_i for $i = 1, \dots, d$. Then, ρ induces the symplectomorphism $\widetilde{\varphi} = {}^t\rho^{-1} \times \rho$ of $(T^*T^n = \mathbb{R}^n \times T^n, \omega_{T^*T^n})$. The source T^*T^n is equipped with d commutative Hamiltonian circle actions defined by u_1, …, u_d with moment maps μ_{u_1}, …, μ_{u_d}, and on the target T^*T^n, e_1, …, e_d define d commutative Hamiltonian circle actions with moment maps μ_{e_1}, …, μ_{e_d}, respectively. Since for each i, $\widetilde{\varphi}$ is equivariant with respect to the Hamiltonian circle action defined by u_i on the source and that defined by e_i on the target, $\widetilde{\varphi}$ descends to the ρ-equivariantly symplectomorphism φ from the simultaneous cut space $\overline{(T^*T^n)}_{\{\mu_{u_i}\geq 0\}_{i=1,\dots,d}}$ to $\overline{(T^*T^n)}_{\{\mu_{e_i}\geq 0\}_{i=1,\dots,d}}$. But by Example 2.8, $\overline{(T^*T^n)}_{\{\mu_{e_i}\geq 0\}_{i=1,\dots,d}}$ is equivariantly symplectomorphic to $\mathbb{C}^d \times T^*T^{n-d}$. This proves Theorem A.4. □

Appendix B. Smoothness of induced maps between cut spaces

Let $u, v \in \mathbb{Z}^n$ be two vectors each of which is primitive, $\rho \in SL_n(\mathbb{Z})$ with $\rho(u) = \pm v$, and $\varphi : U \to V$ an orientation preserving diffeomorphism between two open sets U, $V \subset \mathbb{R}^n$which satisfies

$$\varphi\left(\{\xi \in U \colon \langle u, \xi\rangle \geq 0\}\right) = \{\eta \in V \colon \langle v, \eta\rangle \geq 0\}.$$

In this Appendix, we shall show the following lemma.

Lemma B.1. *If φ satisfies the condition*

$$\langle u, \xi\rangle = \langle v, \varphi(\xi)\rangle$$

on a sufficiently small neighborhood of $\{\xi \in U \colon \langle u, \xi\rangle = 0\}$ in $\{\xi \in U \colon \langle u, \xi\rangle \geq 0\}$, the map $\varphi \times \rho : U \times T^n \to V \times T^n$ descends to an orientation preserving diffeomorphism between cut spaces $\overline{(U \times T^n)}_{\mu_u\geq 0}$ and $\overline{(V \times T^n)}_{\mu_v\geq 0}$.

Proof. Let $\Phi_u : U \times T^n \times \mathbb{C} \to \mathbb{R}$ and $\Phi_v : V \times T^n \times \mathbb{C} \to \mathbb{R}$ be the maps which are defined by

$$\Phi_u(\xi, \theta, z) = \langle u, \xi \rangle - |z|^2, \quad \Phi_v(\eta, \tau, w) = \langle v, \eta \rangle - |w|^2.$$

Define the diffeomorphism $\overline{\psi} : \Phi_u^{-1}(0) \to \Phi_v^{-1}(0)$ by

$$\overline{\psi}(\xi, \theta, z) = \left(\varphi(\xi), \rho(\theta), \sqrt{\frac{\langle v, \varphi(\xi) \rangle}{\langle u, \xi \rangle}} \overline{z}^{\rho} \right),$$

where

$$\overline{z}^{\rho} = \begin{cases} z & \text{if } \rho(u) = v \\ \overline{z} & \text{if } \rho(u) = -v. \end{cases}$$

Note that $\overline{\psi}$ is well-defined from the assumption of the lemma. $\overline{\psi}$ is equivariant with respect to the free circle actions

$$t \cdot (\xi, \theta, z) = (\xi, \theta + tu, e^{-2\pi\sqrt{-1}t} z)$$

on $\Phi_u^{-1}(0)$ and

$$t \cdot (\eta, \tau, w) = (\eta, \tau + t\rho(u), \overline{e^{-2\pi\sqrt{-1}t}}^{\rho} w)$$

on $\Phi_v^{-1}(0)$. Then $\overline{\psi}$ descends to the orientation preserving diffeomorphism $\psi : \Phi_u^{-1}(0)/S^1 \to \Phi_v^{-1}(0)/S^1$ since φ and ρ preserve orientations. $\square$

By using Lemma B.1 repeatedly, we can show the version of the lemma for the case of simultaneous cut spaces. Let $\{u_1, \ldots, u_d\}$, $\{v_1, \ldots, v_d\} \subset \mathbb{Z}^n$ be the primitive tuples of vectors, $\rho \in SL_n(\mathbb{Z})$ with $\rho(u_i) = \pm v_i$ for $i = 1, \ldots, d$, and $\varphi : U \to V$ an orientation preserving diffeomorphism between two open sets U, $V \subset \mathbb{R}^n$which satisfies

$$\varphi\left(\{\xi \in U\colon \langle u_i, \xi \rangle \geq 0 \ i = 1, \ldots, d\}\right) = \{\eta \in V\colon \langle v_i, \eta \rangle \geq 0 \ i = 1, \ldots, d\}.$$

Lemma B.2. *If φ satisfies the condition*

$$\langle u_i, \xi \rangle = \langle v_i, \varphi(\xi) \rangle$$

on a sufficiently small neighborhood of $\{\xi \in U\colon \langle u_i, \xi \rangle = 0, \ \langle u_j, \xi \rangle \geq 0 \ j \neq i\}$ in $\{\xi \in U\colon \langle u_j, \xi \rangle \geq 0 \ j = 1, \ldots, d\}$ for each $i = 1, \ldots, d$, the map $\varphi \times \rho : U \times T^n \to V \times T^n$ descends to the orientation preserving diffeomorphism between simultaneous cut spaces $\overline{(U \times T^n)}_{\{\mu_{u_i} \geq 0\}_{i=1,\ldots,d}}$ and $\overline{(V \times T^n)}_{\{\mu_{v_i} \geq 0\}_{i=1,\ldots,d}}$.

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