ON THE EXISTENCE OF SYMPLECTIC STRUCTURES COMPATIBLE WITH LOCAL TORUS ACTIONS

TAKAHIKO YOSHIDA

1. Introduction

Let S^1 be the unit circle and $T^n := (S^1)^n$ the n-dimensional compact torus. T^n acts on the n-dimensional complex vector space $\mathbb{C}^n$ by coordinatewise complex multiplication. This action is called the *standard representation of* T^n. A T^n-action on a $2n$-dimensional manifold is said to be *locally standard* if for each point $x \in X$ there exists a coordinate neighborhood (U, ρ, φ) of X consisting of a T^n-invariant connected open neighborhood U of x, an automorphism ρ of T^n, and a ρ-equivariant diffeomorphism φ from U to some T^n-invariant open subset in $\mathbb{C}^n$. The latter means that $\varphi(u \cdot x) = \rho(u) \cdot \varphi(x)$ for $u \in T^n$ and $x \in U$. An atlas of X which consists of such coordinate neighborhoods is called a *standard atlas*, see [5, 3] for more details. A standard atlas is one of the starting point of their pioneer work [5] of Davis-Januszkiewicz and now, it plays a fundamental role in toric topology.

In [11] as a generalization of a locally standard torus action, we introduced the following notion. Let X be a compact Hausdorff space.

Definition 1.1. A *weakly standard* C^r $(0 \leq r \leq \infty)$ *atlas* of X is an atlas $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha \in \mathcal{A}}$ which satisfies the following properties

(i) for each α, φ_α^X is a homeomorphism from U_α^X to an open set of $\mathbb{C}^n$ invariant under the standard representation of T^n and

(ii) for each nonempty overlap $U_{\alpha\beta}^X := U_\alpha^X \cap U_\beta^X$, there exists an automorphism $\rho_{\alpha\beta}$ of T^n as a Lie group such that the overlap map $\varphi_{\alpha\beta}^X := \varphi_\alpha^X \circ (\varphi_\beta^X)^{-1}$ is $\rho_{\alpha\beta}$-equivariant C^r diffeomorphic with respect to the restrictions of the standard representation of T^n to $\varphi_\alpha^X(U_{\alpha\beta}^X)$ and $\varphi_\beta^X(U_{\alpha\beta}^X)$.

Two weakly standard C^r atlases $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha \in \mathcal{A}}$ and $\{(V_\beta^X, \psi_\beta^X)\}_{\beta \in \mathcal{B}}$ of X^{2n} are *equivalent* if on each nonempty overlap $U_\alpha^X \cap V_\beta^X$, there exists an automorphism ρ of T^n such that $\varphi_\alpha^X \circ (\psi_\beta^X)^{-1}$ is ρ-equivariant C^r diffeomorphic. We call an equivalence class of weakly standard C^r atlases a C^r *local* T^n*-action on* X^{2n} *modeled on the standard representation* and denote it by $\mathcal{T}$.

In the rest of this note, a C^r local T^n-action on X^{2n} modeled on the standard representation is often called a C^r local T^n-action on X^{2n}, or more simply, a local T^n-action on X if there are no confusions.

It is obvious that a standard atlas satisfies the above condition. But not all local torus actions come from local torus actions. In fact we gave an obstruction class in some cohomology set in order that a local torus action comes from a local torus action in [11]. We also defined two invariants for a local torus action and classified local torus actions topologically in terms of them. As a corollary we obtained a topological classification theorem for locally standard torus actions.

2000 *Mathematics Subject Classification.* Primary 57R15; Secondary 37J35.
Key words and phrases. Lagrangian fibrations, local torus actions.
The author is supported by Fujyukai Foundation and the 21st century COE program.

One of the important class of manifolds equipped with local torus actions is the class of locally toric Lagrangian fibrations (for the definition, see Definition 2.1). It is a natural generalization of a moment map of a symplectic toric manifold. We will see that the total space of a locally toric Lagrangian fibration admits a C^∞ local torus action in Proposition 2.2. The purpose of this talk is to give a necessary and sufficient condition in order that a manifold with a local torus action becomes a locally toric Lagrangian fibration. For a local T^n-action $(X, \mathcal{T})$ we defined the orbit space and the orbit map which are denoted by B_X and $\mu_X \colon X \to B_X$, respectively, in [11]. First we see that if X admits a symplectic form ω so that $\mu_X \colon (X, \omega) \to B_X$ is a locally toric Lagrangian fibration, then B_X admits a rigid structure called an integral affine structure. An integral affine structure is a generalization of a Delzant polytope. As a corollary we see that the structure group of T^*B_X is reduced to $\mathrm{GL}_n(\mathbb{Z})$. Let $\pi_{T_X} \colon T_X \to B_X$ be the associated T^n-bundle of the frame bundle of T^*B_X by the natural $\mathrm{GL}_n(\mathbb{Z})$-action on T^n. Then we also see that T_X is equipped with a symplectic structure ω_{T_X} so that $\pi_{T_X} \colon (T_X, \omega_{T_X}) \to B_X$ is a nonsingular Lagrangian fibration.

From an integral affine structure we construct a new locally toric Lagrangian fibration on B_X which is locally isomorphic to $(X, \mathcal{T})$ as C^∞ local torus actions. By the standard argument we obtain a Čech one cocycle of B_X with values in the sheaf of germs of sections of $\pi_{T_X} \colon T_X \to B_X$. Then a necessary and sufficient condition is as follows (Theorem 4.4).

Theorem 1 ([11])**.** *For a local T^n-action $(X, \mathcal{T})$ there exists a symplectic structure ω on X such that $\mu_X \colon (X, \omega) \to B_X$ is a locally toric Lagrangian fibration if and only if B_X admits an integral affine structure and the above Čech one cocycle takes values in the sheaf of germs of Lagrangian sections of $\pi_{T_X} \colon (T_X, \omega_{T_X}) \to B_X$.*

Next we suppose that a local T^n-action $(X, \mathcal{T})$ satisfies the condition in Theorem 1. Hence X admits a symplectic structure ω so that $\mu_X \colon (X, \omega) \to B_X$ is a locally toric Lagrangian fibration. The above Čech one cocycle defines a cohomology class in the first Čech cohomology of B_X with coefficient in the sheaf of germs of Lagrangian sections of $\pi_{T_X} \colon (T_X, \omega_{T_X}) \to B_X$. We call it a *Lagrangian class* of $\mu_X \colon (X, \omega) \to B_X$. Then we obtain the following classification theorem for locally toric Lagrangian fibrations (Theorem 5.1).

Theorem 2 ([2])**.** *Locally toric Lagrangian fibrations are classified by the integral affine structures and the Lagrangian classes up to fiber-preserving symplectomorphisms.*

This result has already been obtained by Boucetta-Molino [2]. We will prove this theorem by refining the method used to prove the topological classification theorem for local torus actions.

This note is organized as follows. In the next section we define a locally toric Lagrangian fibration and show that the total space of a locally toric Lagrangian fibration admits a C^∞ local torus action. In Section 3 we recall the orbit space and the orbit map of a local torus action. In Section 4 we give a necessary and sufficient condition in order that a manifold with a local torus action becomes a locally toric Lagrangian fibration. Finally Section 5 is devoted to the classification theorem for locally toric Lagrangian fibrations.

1.1. **Conventions.** We denote by $\mathrm{Aut}(T^n)$ the group of automorphisms of T^n. $\mathrm{Aut}(T^n)$ can be identified with $\mathrm{GL}_n(\mathbb{Z})$ because of the decomposition $T^n = (S^1)^n$. We often identify an automorphism of T^n with an element of $\mathrm{GL}_n(\mathbb{Z})$ by this isomorphism. In this note we work in the C^∞ category unless otherwise stated.

2. Locally toric Lagrangian fibrations

Let $\omega_{\mathbb{C}^n} := \frac{1}{2\pi\sqrt{-1}}\sum_{k=1}^n dz_k \wedge d\overline{z}_k$ be the standard symplectic structure on $\mathbb{C}^n$. The standard representation of T^n preserves $\omega_{\mathbb{C}^n}$ and the map $\mu_{\mathbb{C}^n}\colon \mathbb{C}^n \to \mathbb{R}^n$ defined by

$$\mu_{\mathbb{C}^n}(z) = (|z_1|^2, \dots, |z_n|^2) \tag{2.1}$$

for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ is a moment map of the standard representation of T^n. Notice that the image of $\mu_{\mathbb{C}^n}$ is an n-dimensional standard positive cone

$$\mathbb{R}^n_+ := \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \colon \xi_i \geq 0 \ i = 1, \dots, n\}.$$

Let (X, ω) be a $2n$-dimensional symplectic manifold and B an n-dimensional manifold with corners.

Definition 2.1 ([8])**.** A map $\mu\colon (X,\omega) \to B$ is called a *locally toric Lagrangian fibration* if there exists a system $\{(U_\alpha, \varphi^B_\alpha)\}$ of coordinate neighborhoods of B modeled on $\mathbb{R}^n_+$, and for each α there exists a symplectomorphism $\varphi^X_\alpha\colon (\mu^{-1}(U_\alpha), \omega) \to (\mu^{-1}_{\mathbb{C}^n}(\varphi^B_\alpha(U_\alpha)), \omega_{\mathbb{C}^n})$ such that $\mu_{\mathbb{C}^n} \circ \varphi^X_\alpha = \varphi^B_\alpha \circ \mu$.

A locally toric Lagrangian fibration is a natural generalization of a moment map of a symplectic toric manifold. In the case of $\partial B = \emptyset$, it is a nonsingular Lagrangian fibration. Conversely, by the Arnold-Liouville theorem [1], a nonsingular Lagrangian fibration with closed connected fibers on a closed manifold is also such an example.

The following proposition shows that the total space of a locally toric Lagrangian fibration admits a local torus action.

Proposition 2.2. *Let $\mu\colon (X,\omega) \to B$ be a locally toric Lagrangian fibration on an n-dimensional base B and $\{(U_\alpha, \varphi^B_\alpha, \varphi^X_\alpha)\}$ a system of local identifications of μ with $\mu_{\mathbb{C}^n}$. Then, on each connected component of a nonempty overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$, there exists an element $\rho_{\alpha\beta} \in \mathrm{Aut}(T^n)$ such that $\varphi^X_\alpha \circ (\varphi^X_\beta)^{-1}$ is $\rho_{\alpha\beta}$-equivariant.*

for the proof, see [11].

3. The orbit structures of local torus actions

In this section we recall the orbit space and the orbit map of a local torus action. Let $(X, \mathcal{T})$ be a $2n$-dimensional manifold equipped with a local T^n-action and $\{(U^X_\alpha, \varphi^X_\alpha)\}_{\alpha\in\mathcal{A}} \in \mathcal{T}$ the maximal weakly standard atlas of X. We define the orbit space B_X of X by

$$B_X := \coprod_\alpha \left(\varphi^X_\alpha(U^X_\alpha)/T^n\right) / \sim_{orb},$$

where $b_\alpha \in \varphi^X_\alpha(U^X_\alpha)/T^n \sim_{orb} b_\beta \in \varphi^X_\beta(U^X_\beta)/T^n$ if and only if $b_\alpha \in \varphi^X_\alpha(U^X_{\alpha\beta})/T^n$, $b_\beta \in \varphi^X_\beta(U^X_{\alpha\beta})/T^n$ and a homeomorphism induced by the overlap map $\varphi^X_{\alpha\beta}$ sends b_β to b_α.

Definition 3.1. Let B be a Hausdorff space. A *structure of an n-dimensional C^0 manifold with corners* on B is a system of coordinate neighborhoods onto open subsets of $\mathbb{R}^n_+$ so that overlap maps are homeomorphisms which preserve stratifications induced from the natural stratification of $\mathbb{R}^n_+$. See [4, Section 6] for a C^0 manifold with corners.

Proposition 3.2. *B_X is endowed with a structure of an n-dimensional C^0 manifold with corners.*

Proof. A structure of an n-dimensional C^0 manifold with corners on B_X is constructed as follows. We put $U_\alpha^B := \varphi_\alpha^X(U_\alpha^X)/T^n$. The restriction of $\mu_{\mathbb{C}^n}$ to $\varphi_\alpha^X(U_\alpha^X)$ induces the homeomorphism from U_α^B to the open subset $\mu_{\mathbb{C}^n}(\varphi_\alpha^X(U_\alpha^X))$ of $\mathbb{R}_+^n$, which we denote by φ_α^B. By the construction, on each overlap $U_{\alpha\beta}^B := U_\alpha^B \cap U_\beta^B$, the overlap map $\varphi_{\alpha\beta}^B := \varphi_\alpha^B \circ (\varphi_\beta^B)^{-1} \colon \mu_{\mathbb{C}^n}(\varphi_\beta^X(U_{\alpha\beta}^X)) \to \mu_{\mathbb{C}^n}(\varphi_\alpha^X(U_{\alpha\beta}^X))$ preserves the natural stratifications of $\mu_{\mathbb{C}^n}(\varphi_\alpha^X(U_{\alpha\beta}^X))$ and $\mu_{\mathbb{C}^n}(\varphi_\beta^X(U_{\alpha\beta}^X))$. Thus, $\{(U_\alpha^B, \varphi_\alpha^B)\}_{\alpha\in\mathcal{A}}$ is the desired atlas. $\square$

By the construction of B_X, the map $\coprod_\alpha \pi \circ \varphi_\alpha^X \colon \coprod_\alpha U_\alpha^X \to \coprod_\alpha \left(\varphi_\alpha^X(U_\alpha^X)/T^n\right)$ induces a map from X to B_X. We call it an *orbit map* of the local T^n-action $\mathcal{T}$ on X and denote it by $\mu_X \colon X \to B_X$.

Remark 3.3. The atlas $\{(U_\alpha^B, \varphi_\alpha^B)\}_{\alpha\in\mathcal{A}}$ of B_X constructed in the proof of Proposition 3.2 has following property; for each α, $U_\alpha^X = \mu_X^{-1}(U_\alpha^B)$, $\varphi_\alpha^X(U_\alpha^X) = \mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha^B))$ and the following diagram commutes

$$\begin{array}{ccccccc}
X & \supset & \mu_X^{-1}(U_\alpha^B) & \xrightarrow{\varphi_\alpha^X} & \mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha^B)) & \subset & \mathbb{C}^n \\
\downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_{\mathbb{C}^n}} & & \downarrow{\scriptstyle \mu_{\mathbb{C}^n}} \\
B_X & \supset & U_\alpha^B & \xrightarrow{\varphi_\alpha^B} & \varphi_\alpha^B(U_\alpha^B) & \subset & \mathbb{R}_+^n.
\end{array}$$

4. A necessary and sufficient condition

In this section we give a necessary and sufficient condition in order that a manifold with a local torus action becomes a locally toric Lagrangian fibration. Let $(X, \mathcal{T})$ be a $2n$-dimensional manifold equipped with a local T^n-action $\mathcal{T}$.

Definition 4.1. Let $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}} \in \mathcal{T}$ be a weakly standard atlas of X and $\{(U_\alpha^B, \varphi_\alpha^B)\}_{\alpha\in\mathcal{A}}$ the atlas of B_X induced by $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$. For each connected component of a nonempty overlap $U_{\alpha\beta}^X \neq \emptyset$, let $\rho_{\alpha\beta} \in \mathrm{Aut}(T^n)$ be the automorphism in (ii) of Definition 1.1 with respect to $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$. We call $\{(U_\alpha^B, \varphi_\alpha^B)\}_{\alpha\in\mathcal{A}}$ an *integral affine structure* compatible with $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$ if on each connected component of a nonempty overlap $U_{\alpha\beta}^B \neq \emptyset$, the overlap map $\varphi_{\alpha\beta}^B \colon \varphi_\beta^B(U_{\alpha\beta}^B) \to \varphi_\alpha^B(U_{\alpha\beta}^B)$ is of the form

$$\varphi_{\alpha\beta}^B(\xi) = \rho_{\alpha\beta}^{-T}(\xi) + c_{\alpha\beta}, \tag{4.1}$$

for some constant $c_{\alpha\beta} \in \mathbb{R}^n$.

Lemma 4.2. *If there exists a symplectic structure ω on X and there also exists a weakly standard atlas $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}} \in \mathcal{T}$ of X such that on each U_α^X, φ_α^X preserves symplectic forms, namely, $\omega = {\varphi_\alpha^X}^* \omega_{\mathbb{C}^n}$, then the atlas $\{(U_\alpha^B, \varphi_\alpha^B)\}_{\alpha\in\mathcal{A}}$ of B_X induced by $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$ is an integral affine structure compatible with $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$. In particular, B_X becomes a smooth manifold with corners.*

See [11] for the proof.

Let $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}} \in \mathcal{T}$ be a weakly standard atlas of X. Suppose that the induced atlas $\{(U_\alpha^B, \varphi_\alpha^B)\}_{\alpha\in\mathcal{A}}$ of B_X is an integral affine structure compatible with $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}} \in \mathcal{T}$. By (4.1) the structure group of the cotangent bundle T^*B_X is reduced to $\mathrm{GL}_n(\mathbb{Z})$. Let $\pi_{P_X} \colon P_X \to B_X$ be the the frame bundle of T^*B_X with structure group $\mathrm{GL}_n(\mathbb{Z})$. Let Λ be the lattice of integral elements of the Lie algebra $\mathfrak{t}$ of T^n, namely, $\Lambda := \{t \in \mathfrak{t} \colon \exp(t) = 1\}$. We denote by $\pi_{\Lambda_X} \colon \Lambda_X \to B_X$ and $\pi_{T_X} \colon T_X \to B_X$ the associated Λ-bundle and T^n-bundle by the natural action of $\mathrm{GL}_n(\mathbb{Z})$ on Λ and T^n, respectively. Then we have the following exact sequence of associated fiber bundles of P_X

$$0 \longrightarrow \Lambda_X \longrightarrow T^*B_X \longrightarrow T_X \longrightarrow 0.$$

As is well-known, T^*B_X is equipped with the standard symplectic structure $\omega_{T^*B_X}$. Since the natural fiberwise action of Λ_X on T^*B_X preserves $\omega_{T^*B_X}$, $\omega_{T^*B_X}$ descends to a symplectic structure on T_X, which is denoted by ω_{T_X}, so that $\pi_{T_X}: (T_X, \omega_{T_X}) \to B_X$ is a nonsingular Lagrangian fibration.

Since B_X is a manifold with corners it is equipped with a natural stratification. Let $\mathcal{S}^{(k)}B_X$ be the k-dimensional part of B_X with respect to the natural stratification, namely, $\mathcal{S}^{(k)}B_X$ consists of those points which have exactly k nonzero components in a local coordinate. For any point b of B_X, let $(U_\alpha^B, \varphi_\alpha^B)$ be a coordinate neighborhood in the integral affine structure which contains b. Suppose that b lies in $\mathcal{S}^{(k)}B_X$. Then the stabilizer of the T^n-action on $\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(b))$ is an $(n-k)$-dimensional subtorus and by Lemma 4.2 it defines a unique $(n-k)$-dimensional subtorus of the fiber $\pi_{T_X}^{-1}(b)$ of $\pi_{T_X}: T_X \to B_X$ at b which is denoted by Z_b. Notice that a fiber of $\pi_{T_X}: T_X \to B_X$ admits a group structure since its structure group is $\mathrm{GL}_n(\mathbb{Z})$. We define the equivalence relation $\sim_{can}$ on T_X by $t \sim_{can} t'$ if and only if $\pi_{T_X}(t) = \pi_{T_X}(t')$ and $t't^{-1} \in Z_{\pi_{T_X}(t)}$, and denote the quotient space of $\sim_{can}$ by X_{can}. By the construction of X_{can} the bundle projection π_{T_X} descends to the projection $\mu_{can}: X_{can} \to B_X$.

Lemma 4.3 ([11])**.** *X_{can} is a $2n$-dimensional smooth manifold. Moreover, ω_{T_X} induces a symplectic structure ω_{can} on X_{can} so that $\mu_{can}: (X_{can}, \omega_{can}) \to B_X$ is a locally toric Lagrangian fibration.*

Roughly speaking, the proof is as follows. The integral affine structure defines a Hamiltonian action of some subtorus of T^n on each $\pi_{T_X}^{-1}(U_\alpha^B)$. (X_{can}, ω_{can}) can be obtained from (T_X, ω_{T_X}) by the symplectic cutting technique with respect to these Hamiltonian torus actions. For more details, see [11]. $\mu_{can}: (X_{can}, \omega_{can}) \to B_X$ is called a canonical model for the integral affine structure on B_X.

By the construction of $\mu_{can}: (X_{can}, \omega_{can}) \to B_X$, it is locally isomorphic, as local torus actions, to the orbit map $\mu_X: X \to B_X$, namely, on each U_α^B there is a equivariantly diffeomorphism $h_\alpha: \mu_X^{-1}(U_\alpha^B) \to \mu_{can}^{-1}(U_\alpha^B)$ with respect to the T^n-actions which covers the identity on U_α^B. Note that on $\mu_X^{-1}(U_\alpha^B)$ and on $\mu_{can}^{-1}(U_\alpha^B)$ there are T^n-actions under the identifications with $\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha^B))$. On each nonempty overlap $U_{\alpha\beta}$ the equation

$$h_\alpha \circ h_\beta^{-1}(x) = \theta_{\alpha\beta}(b)x$$

for $b \in U_{\alpha\beta}^B$ and $x \in \mu_{can}^{-1}(b)$ determines a local section $\theta_{\alpha\beta}$ of $\pi_{T_X}: T_X \to B_X$ on $U_{\alpha\beta}^B$. Then a necessary and sufficient condition in order that $\mu_X: X \to B_X$ becomes a locally toric Lagrangian fibration is given as follows.

Theorem 4.4. *Let $(X, \mathcal{T})$ be a $2n$-dimensional manifold equipped with a local T^n-action $\mathcal{T}$. There exists a symplectic structure ω on X and there also exists a weakly standard atlas $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}} \in \mathcal{T}$ of X such that on each U_α^X, $\omega = {\varphi_\alpha^X}^* \omega_{\mathbb{C}^n}$ if and only if the atlas $\{(U_\alpha^B, \varphi_\alpha^B)\}_{\alpha\in\mathcal{A}}$ of B_X induced by $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$ is an integral affine structure compatible with $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$ and on each nonempty overlap $U_{\alpha\beta}^B$, $\theta_{\alpha\beta}$ is a Lagrangian section, namely, $\theta_{\alpha\beta}^*\omega_{T_X}$ vanishes.*

For nonsingular Lagrangian fibrations, this result is obtained by Duistermaat [6]. See also [10], [9]. Recently, in [7] Gay-Symington showed the similar result for near-symplectic four-manifolds.

5. Classification of locally toric Lagrangian fibrations

Suppose that $(X, \mathcal{T})$ be a $2n$-dimensional manifold equipped with a local T^n-action $\mathcal{T}$ satisfies the condition in Theorem 4.4. Then X is equipped with a symplectic structure ω so that $\mu_X: (X, \omega) \to B_X$ is a locally toric Lagrangian fibration. In this case, by Theorem4.4, the local sections $\theta_{\alpha\beta}$ define a Čech cohomology class $\lambda(X) \in H^1(B_X; \mathscr{S}_{T_X}^{Lag})$ of B_X with values in the sheaf $\mathscr{S}_{T_X}^{Lag}$ of germs of Lagrangian sections of $\pi_{T_X} : (T_X, \omega_{T_X}) \to B_X$. $\lambda(X)$ is called a *Lagrangian class* for $\mu_X : (X, \omega) \to B_X$. Now we state the classification theorem for locally toric Lagrangian fibrations.

Theorem 5.1 ([2]). *Locally toric Lagrangian fibrations are classified by integral affine structures on the bases and $\lambda(X)$ up to fiber-preserving symplectomorphisms.*

The idea of the proof is as follows. If for two locally toric Lagrangian fibrations $\mu_{X_1} : (X_1, \omega_1) \to B_{X_1}$ and $\mu_{X_2} : (X_2, \omega_2) \to B_{X_2}$, there is a diffeomorphism $f : B_{X_1} \to B_{X_2}$ which preserves the integral affine structures, then the canonical models are same under the identification f. By definition, $\lambda(X_i)$ measures the difference between $\mu_{X_i} : (X_i, \omega_i) \to B_{X_i}$ and its canonical model. So if $\lambda(X_1) = f^*\lambda(X_2)$, then their differences are same. This implies that they are fiber-preserving symplectomorphic. For more details, see [11].

References

1. V. I. Arnold, *Mathematical methods of classical mechanics*, second ed., Graduate texts in Mathematics, vol. 60, Springer-Verlag, New York, 1989.
2. M. Boucetta and Molino P., *Géométrie globale des systèmes hamiltoniens complètement intègrables : fibrations lagrangiennes singulières et coordonnées action-angle à singularités*, C. R. Acad. Sci. Paris Sér. I Math. **308** (1989), no. 13, 421–424.
3. V. Buchstaber and T. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, vol. 24, Amer. Math. Soc., Providence, RI, 2002.
4. M. Davis, *Group generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2) **117** (1983), no. 2, 293–324.
5. M. Davis and T. Januszkiewicz, *Convex polytopes, coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451.
6. J. J. Duistermaat, *On global action-angle coordinates*, Comm. Pure Appl. Math. **33** (1980), no. 6, 687–706.
7. D. T. Gay and M. Symington, *Toric structures on near-symplectic 4-manifolds*, arXiv:math.SG/0609753, 2006.
8. M. Hamilton, *Quantization of toric manifolds via real polizations*, The talk in International Conference on Toric Topology, 2006.
9. K. N. Mishachev, *The classification of lagrangian bundles over surface*, Differential Geom. Appl. **6** (1996), no. 4, 301–320.
10. M. Symington, *Four dimensions from two in symplectic topology*, Topology and geometry of manifolds (Athens, GA, 2001), Proc. Sympos. Pure Math., vol. 71, Amer. Math. Soc., Providence, RI, 2003, pp. 153–208.
11. T. Yoshida, *Local torus actions modeled on the standard represenation*, preprint, 2007.

Graduate School of Mathematical Sciences, The University of Tokyo,
8-1 Komaba 3-chome, Meguro-ku, Tokyo, 153-8914, Japan
E-mail address: takahiko@ms.u-tokyo.ac.jp