LOCALLY STANDARD TORUS FIBRATIONS

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1. Introduction

Recently topological counterparts of toric varieties are actively investigated and this research field are now called *toric topology* [3], [4], [6], [13], [14], etc. In this area, various researches have been done from the viewpoint of the theory of transformation groups and many interesting results have been obtained.

On the other hand, when we glance at not torus actions themselves, but their orbit maps, we can find a structure of certain singular torus fibrations behind them. This talk focuses on this structure. The purpose of this talk is to formulate such singular torus fibrations and to investigate their properties.

2. From torus actions to torus fibrations

2.1. **Locally standard torus actions.** Let S^1 be the unit circle in $\mathbb{C}$ and T^n the n-dimensional compact torus $(S^1)^n$. T^n acts on $\mathbb{C}^n$ by complex multiplication. This action is called the *standard T^n-action on $\mathbb{C}^n$*. Suppose that T^n acts on a $2n$-dimensional manifold X. A *standard chart* of X consists of

(i) a T^n-invariant open set $U \subset X$,
(ii) an automorphism $\rho : T^n \to T^n$, and
(iii) a ρ-equivariant diffeomorphism $\varphi : U \to V$ from U to some T^n-invariant open subset V in $\mathbb{C}^n$.

The latter means $\varphi(tx) = \rho(t)\varphi(x)$ for $t \in T^n$ and $x \in X$. The action of T^n on X is said to be *locally standard* if every point in X lies in some standard chart.

Example 2.1. An effective T^n-action on a $2n$-dimensional manifold X without nontrivial finite stabilizers are locally standard because of the slice theorem. The four-dimensional case of these actions has been studied by Orlik-Raymond in [18].

2.2. **An observation.** We observe the orbit maps of locally standard torus actions. Suppose that an $2n$-dimensional manifold X is equipped with a locally standard T^n-action. By the slice theorem the orbit space X/T^n has a structure of a manifold with corners. We denote by B the orbit space and denote by $\mu : X \to B$ the orbit map. Define the map $\mu_{\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{R}^n_+$ by

$$\mu_{\mathbb{C}^n}(z) = (|z_1|^2, \dots, |z_n|^2),$$

where $\mathbb{R}^n_+$ is the positive cone

$$\mathbb{R}^n_+ = \{\xi \in \mathbb{R}^n : \xi_i \geq 0 \text{ for } i = 1, \dots, n\}.$$

Note that $\mu_{\mathbb{C}^n}$ is naturally identified with the orbit map of the standard T^n-action on $\mathbb{C}^n$. The orbit map $\mu : X \to B$ satisfies the following condition.

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Condition A. *For each $b \in B$, there exist a coordinate neighborhood (U, φ^B) around b and a diffeomorphism $\varphi^X : \mu^{-1}(U) \to \mu_{\mathbb{C}^n}^{-1}(\varphi^B(U))$ such that the following diagram commutes*

$$\begin{tikzcd}
X \arrow[d, "\mu"] & \supset & \mu^{-1}(U) \arrow[r, "\varphi^X"] \arrow[d, "\mu"] \arrow[dr, phantom, "\circlearrowright"] & \mu_{\mathbb{C}^n}^{-1}(\varphi^B(U)) \arrow[d, "\mu_{\mathbb{C}^n}"] & \subset & \mathbb{C}^n \arrow[d, "\mu_{\mathbb{C}^n}"] \\
B & \supset & U \arrow[r, "\varphi^B"] & \varphi^B(U) & \subset & \mathbb{R}^n_+ .
\end{tikzcd}$$

Since T^n acts freely on the inverse image $\mu^{-1}(B \backslash \partial B)$ of $B \backslash \partial B$ by μ, it has a structure of a principal T^n-bundle. But once we forget this structure of the principal bundle and focus on Condition A, we can see that the restriction of μ to $\mu^{-1}(B \backslash \partial B)$ admits a structure of a fiber bundle with fiber T^n. In fact, the local identification φ^X in Condition A defines a local trivialization $\phi : \mu^{-1}(U \backslash \partial B) \to U \backslash \partial B \times T^n$ for the fiber bundle by

$$\phi(x) = \left(\mu(x), \left(\frac{\varphi_1^X(x)}{|\varphi_1^X(x)|}, \dots, \frac{\varphi_n^X(x)}{|\varphi_n^X(x)|} \right) \right), \tag{2.1}$$

where φ_i^X is the ith component of φ^X. Note that none of components of $\varphi^X(x)$ vanish for $x \in \mu^{-1}(U \backslash \partial B)$. Moreover the orbit map μ of the locally standard torus action also satisfies the following condition.

Condition B. *The transition functions with respect to local trivializations defined by* (2.1) *take values in the semidirect product $T^n \rtimes \mathrm{Aut}(T^n)$ of T^n and the group $\mathrm{Aut}(T^n)$ of automorphisms of T^n as a Lie group.*

2.3. **A definition and examples.** Inspired by this observation, we give the following definition. Let X be a closed, connected $2n$-dimensional manifold and B an n-dimensional manifold with corners. Suppose that $\mu : X \to B$ is a map, which is not necessarily the orbit map of a locally standard T^n-action.

Definition 2.2. The map $\mu : X \to B$ is called a *locally standard torus fibration* if μ satisfies Condition A and B.

Remark 2.3. In general, Condition B does not follow Condition A automatically.

In the rest of this talk, we focus on the case where all transition functions in Condition B take values in $\mathrm{Aut}(T^n)$ for simplicity.

Example 2.4 (Complete, nonsingular toric varieties)**.** A complete, nonsingular toric variety X together with the orbit map of the compact torus action is an example of locally standard torus fibrations. This can be seen as follows. X is covered by open sets U_σ each of which is determined by a top dimensional cone σ in the fan associated with X. Since each σ is nonsingular, the lattice vectors which generate σ can be taken to be a basis of the lattice and each U_σ is identified with the standard torus actions. Suppose that σ_1 and σ_2 are two top dimensional cones. Then the two bases of the lattice which generate σ_1 and σ_2 determine the automorphism of the lattice. Then the automorphism induces the automorphism of the torus which is the transition function with respect to local trivializations obtained from U_{σ_i}. For toric varieties, see [5], [8], [17].

Example 2.5 (Quasi-toric manifolds)**.** A quasi-toric manifold together with the orbit map is an example of locally quasi toric fibrations on a simple polytope. This can be seen from the construction of the canonical model. See [6, Proposition 1.8] for canonical models.

The following example does not come from torus actions.

Example 2.6. Let B be a compact, connected, and oriented surface of genus one with one boundary component and one corner points, B_1 the set of interior points $B\backslash\partial B$ of B. Let us consider the principal $SL_2(\mathbb{Z})$-bundle P on B which is

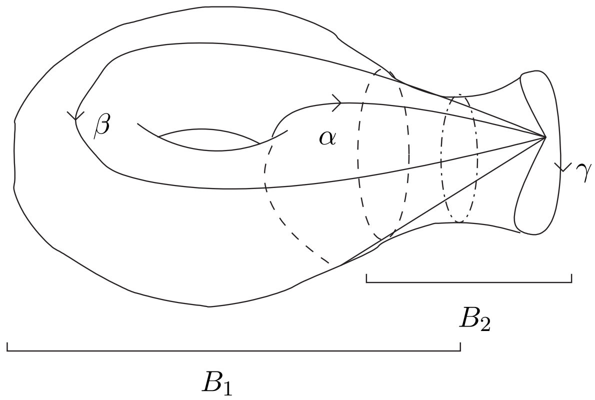

FIGURE 1. B

determined by the representation $\rho : \pi_1(B) \to SL_2(\mathbb{Z})$

$$\rho(\alpha) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\ \rho(\beta) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},\ \rho(\gamma) = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}.$$

We denote by $\pi_{\mathcal{T}} : \mathcal{T}^2 \to B$ its associated T^2-bundle by the natural action of $SL_2(\mathbb{Z})$ on T^2. Let us construct a structure of locally standard torus fibrations on a neighborhood of ∂B as follows. We define subsets $\overline{B}_2$, U_1, and U_2 of $\mathbb{R}^2_+$ by

$$\begin{aligned}
\overline{B}_2 &= \{\xi \in \mathbb{R}^2 \colon 0 \leq \xi_1 < 4,\ 0 \leq \xi_2 < 1\} \cup \{\xi \in \mathbb{R}^2 \colon 0 \leq \xi_1 < 1,\ 0 \leq \xi_2 < 4\}, \\
U_1 &= \{\xi \in \mathbb{R}^2 \colon 3 < \xi_1 < 4,\ 0 \leq \xi_2 < 1\}, \\
U_2 &= \{\xi \in \mathbb{R}^2 \colon 0 \leq \xi_1 < 1,\ 3 < \xi_2 < 4\},
\end{aligned}$$

and also define diffeomorphisms $\varphi^B : U_1 \to U_2$ and $\varphi^X : \mu_{\mathbb{C}^2}^{-1}(U_1) \to \mu_{\mathbb{C}^2}^{-1}(U_2)$ by

$$\begin{aligned}
\varphi^B(\xi) &= (\xi_2, 7 - \xi_1), \\
\varphi^X(z) &= (\frac{z_1^3 z_2}{|z_1|^3}, \sqrt{7 - |z_1|^2}\left(\frac{z_1}{|z_1|}\right)^{-1}).
\end{aligned}$$

Note that φ^B and φ^X commute with $\mu_{\mathbb{C}^2}$. We denote by X_2 the manifold which is obtained from $\mu_{\mathbb{C}^2}^{-1}(\overline{B}_2)$ by gluing $\mu_{\mathbb{C}^2}^{-1}(U_1)$ and $\mu_{\mathbb{C}^2}^{-1}(U_2)$ with φ^X and denote by B_2 the surface with one corner which is obtained from $\overline{B}_2$ by gluing U_1 and U_2 with φ^B. B_2 can be identified with a neighborhood of the boundary of B. Since φ^B and

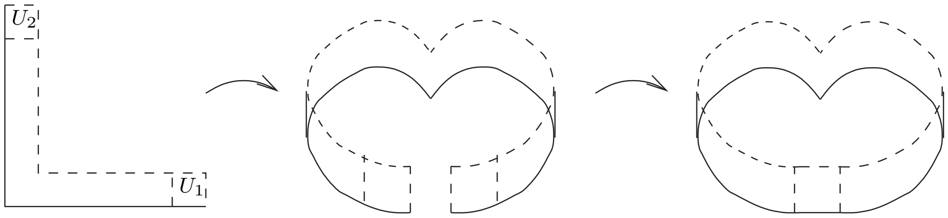

FIGURE 2. B_2

φ^X commute with $\mu_{\mathbb{C}^2}$, $\mu_{\mathbb{C}^2}$ descends to the map from X_2 to B_2. We denote it by

$\mu_2 : X_2 \to B_2$. It is easy to see that the restriction $\mu_2\big|_{\mu_2^{-1}(B_1\cap B_2)} : \mu_2^{-1}(B_1 \cap B_2) \to B_1 \cap B_2$ is a T^2-bundle with the structure group $SL_2(\mathbb{Z})$ and it is isomorphic to $\pi_{\mathcal{T}}\big|_{B_1\cap B_2} : \mathcal{T}^2\big|_{B_1\cap B_2} \to B_1 \cap B_2$. Thus we can patch $\pi_{\mathcal{T}}\big|_{B_1} : \mathcal{T}^2\big|_{B_1} \to B_1$ with $\mu_2 : X_2 \to B_2$ by this isomorphism to get the locally standard torus fibration $\mu : X \to B$ associated with $\pi : P \to B$.

Given a locally standard torus fibration $\mu : X \to B$, we can construct new locally standard torus fibrations from $\mu : X \to B$.

Example 2.7 (pullback)**.** Suppose that B_1 and B_2 are n-dimensional manifolds with corners and that $f : B_1 \to B_2$ is a map which preserves the natural stratifications induced from the structures of manifolds with corners. Let $\mu : X \to B_2$ be a locally standard torus fibration. The *pullback of* $\mu : X \to B_2$ by f, which is denoted by f^*X, is the fiber product of B_1 and X. In the similar way to the case of original fiber bundles, we can show that f^*X has a structure of a locally standard torus fibration.

Example 2.8 (blowing up)**.** First we shall explain the blowing up of the local model $\mu : \mathbb{C}^n \to \mathbb{R}^n_+$ at the origin of $\mathbb{C}^n$. For a positive real number $\varepsilon > 0$, we denote by $\mathbb{C}^n(\varepsilon)$ the quotient space of $\{(z,w) \in \mathbb{C}^n \times \mathbb{C} \colon \|z\|^2 - |w|^2 = \varepsilon\}$ by the circle action which is defined by

$$t \cdot (z,w) := (tz_1, \dots, tz_n, t^{-1}w).$$

We define the map $\mu_{\mathbb{C}^n(\varepsilon)} : \mathbb{C}^n(\varepsilon) \to \mathbb{R}^n$ by

$$\mu_{\mathbb{C}^n(\varepsilon)}([z,w]) = (|z_1|^2, \dots, |z_n|^2)$$

for $[z,w] \in \mathbb{C}^n(\varepsilon)$. The image of $\mu_{\mathbb{C}^n(\varepsilon)}$ is

$$\mathbb{R}^n_\epsilon := \{\xi \in \mathbb{R}^n_+ \colon \sum_{i=1}^n \xi_i \geq \varepsilon\}.$$

We define the map ψ_ϵ from the subset $\mathbb{C}^n \backslash \overline{D}_\varepsilon(0)$ of $\mathbb{C}^n$ obtained by removing the closed disc $\overline{D}_\varepsilon(0)$ with center 0 and radius $\varepsilon^{1/2}$ to $\{[z,w] \in \mathbb{C}^n(\epsilon) \colon w \neq 0\}$ by

$$\psi_\epsilon(z) = \left[z, (\|z\|^2 - \varepsilon)^{1/2}\right]$$

for $z \in \mathbb{C}^n \backslash \overline{D}_\varepsilon(0)$. It is easy to see that ψ_ϵ is a diffeomorphism and satisfies

$$\mu_{\mathbb{C}^n}(z) = \mu_{\mathbb{C}^n(\varepsilon)} \circ \psi_\varepsilon(z)$$

for $z \in \mathbb{C}^n \backslash \overline{D}_\varepsilon(0)$. This is a smooth part of the symplectic blowing up by Guillemin-Sternberg in [11].

Let $\mu : X \to B$ be a locally standard torus fibration on an n-dimensional manifold B with corners. Suppose that B has the non-empty zero-dimensional stratum and that b is a point in 0-stratum. Then the fiber $\mu^{-1}(b)$ of μ on b consists of one point which is denoted by x. By definition of locally standard torus fibrations a neighborhood of x can be identified with a neighborhood of the origin of the map $\mu_{\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{R}^n_+$. With this identification, we remove a closed disc $\overline{D}_\varepsilon(0)$ for a sufficiently small $\epsilon > 0$ from the neighborhood of x and glue $\mu_{\mathbb{C}^n(\epsilon)} : \mathbb{C}^n(\epsilon) \to \mathbb{R}^n_\epsilon$ by ψ_ε. Then the obtained map is a locally standard torus fibration which we call the *blowing up of* $\mu : X \to B$ *at* x.

Example 2.9 (gluing)**.** Suppose that $\mu_1 : X_1 \to B_1$ and $\mu_2 : X_2 \to B_2$ are locally standard torus fibrations on n-dimensional manifolds. Let U_i be an open set in the n-dimensional stratum of B_i. If necessary, by taking U_i sufficiently small, we may assume that the restriction of μ_i to the inverse image $\mu_i^{-1}(\overline{U}_i)$ of the closure of U_i is identified with the restriction of $\mu_{\mathbb{C}^n}$ to the inverse image $\mu_{\mathbb{C}^n}^{-1}(D)$ of a closed disc D

in the interior of $\mathbb{R}^n_+$. In particular they are identified each other. Then, by gluing $X_1\backslash\mu_1^{-1}(U_1)$ and $X_2\backslash\mu_2^{-1}(U_2)$ by the identification of $\mu_1^{-1}(\partial\overline{U}_1)$ with $\mu_2^{-1}(\partial\overline{U}_2)$, we can obtain a new locally standard torus fibration on the connected sum $B_1\#B_2$.

3. The topological classification

In [6] Davis-Januszkiewicz give a topological classification for quasi-toric manifolds. We generalize this classification to locally standard torus fibrations. To do this, we recall some stuffs which are necessary to describing the classification. We assume that $\partial B \neq \emptyset$.

Proposition 3.1. *For any locally standard torus fibration $\mu : X \to B$, there exist a fiber bundle $\pi_{\mathcal{T}} : \mathcal{T}^n_\mu \to B$ of B with fiber T^n and a surjective continuous map $\nu : \mathcal{T}^n_\mu \to X$ which satisfy the following condition: For each coordinate neighborhood (U, φ^B) in Condition A, there exists a local trivialization $\varphi^T : \pi_{\mathcal{T}}^{-1}(U) \to U \times T^n$ such that the following diagram commutes*

$$\begin{tikzcd}
\pi_{\mathcal{T}}^{-1}(U) \arrow[rr, "\varphi^B\times \mathrm{id}_{T^n}\circ\varphi^T"] \arrow[dr, "\nu"] \arrow[ddr, "\pi_{\mathcal{T}}"'] & & \varphi^B(U)\times T^n \arrow[dr, "\nu_{\mathbb{C}^n}"] \arrow[dd, "\mathrm{pr}_1"] & \\
& \mu^{-1}(U) \arrow[rr, "\varphi^X"] \arrow[d, "\mu"] & & \mu_{\mathbb{C}^n}^{-1}(\varphi^B(U)) \arrow[dl, "\mu_{\mathbb{C}^n}"] \\
& U \arrow[r, "\varphi^B"] & \varphi^B(U), &
\end{tikzcd}$$

where $\nu_{\mathbb{C}^n}$ is the map $\nu_{\mathbb{C}^n} : \mathbb{R}^n_+ \times T^n \to \mathbb{C}^n$ which is defined by

$$\nu_{\mathbb{C}^n}(\xi, t) = (t_1\sqrt{\xi_1}, \ldots, t_n\sqrt{\xi_n}) \tag{3.1}$$

for $(\xi, t) \in \mathbb{R}^n_+ \times T^n$.

A sketch of the proof. The fiber bundle $\pi_{\mathcal{T}} : \mathcal{T}^n_\mu \to B$ in Proposition 3.1 is an extension of the restriction $\mu\big|_{\mu^{-1}(B\backslash\partial B)} : \mu^{-1}(B\backslash\partial B) \to B\backslash\partial B$ to the whole B. Since the automorphism group of T^n is discrete, such a fiber bundle exists. Next we describe the map $\nu : \mathcal{T}^n \to B$. On each open set U_α of B in Condition A, we denote by $\nu_\alpha : \varphi^B_\alpha(U_\alpha)\times T^n \to \mu_{\mathbb{C}^n}^{-1}(\varphi^B_\alpha(U_\alpha))$ the restriction of the map $\nu_{\mathbb{C}^n}$ defined by (3.1) to $\varphi^B_\alpha(U_\alpha)\times T^n$. On each nonempty intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we have the following diagram

$$\begin{tikzcd}
\varphi^B_\alpha(U_{\alpha\beta})\times T^n \arrow[d, "\nu_\alpha"'] & \pi_{\mathcal{T}}^{-1}(U_{\alpha\beta}) \arrow[l, "\varphi^B_\alpha\times\mathrm{id}_{T^n}\circ\varphi^T_\alpha"'] \arrow[r, "\varphi^B_\beta\times\mathrm{id}_{T^n}\circ\varphi^T_\beta"] & \varphi^B_\beta(U_{\alpha\beta})\times T^n \arrow[d, "\nu_\beta"] \\
\mu_{\mathbb{C}^n}^{-1}(\varphi^B_\alpha(U_{\alpha\beta})) & \mu^{-1}(U_{\alpha\beta}) \arrow[l, "\varphi^X_\alpha"'] \arrow[r, "\varphi^X_\beta"] & \mu_{\mathbb{C}^n}^{-1}(\varphi^B_\beta(U_{\alpha\beta})).
\end{tikzcd}$$

We claim that this diagram commutes for all intersections. In fact, this is true for $(\xi, t) \in \varphi^B_\alpha(U_{\alpha\beta}\backslash\partial B)\times T^n$ by the construction of the local trivializations (2.1). Since $\varphi^B_\alpha(U_{\alpha\beta}\backslash\partial B)$ is dense in $\varphi^B_\alpha(U_{\alpha\beta})$ and since $\nu_{\mathbb{C}^n}$ is continuous, this diagram must commute. Then we can glue the local maps $\{\nu_\alpha\}$ together to get the map $\nu : \mathcal{T}^n_\mu \to X$ which satisfies the desired condition. □

The structure group of the fiber bundle $\pi_{\mathcal{T}} : \mathcal{T}^n_\mu \to B$ is contained in $\mathrm{Aut}(T^n)$. We call it a *structure group of* $\mu : X \to B$. Let G be the structure group of $\mu : X \to B$. We denote by $\pi_{P_\mu} : P_\mu \to B$ the principal G-bundle on B associated with the T^n-bundle $\pi_{\mathcal{T}_\mu} : \mathcal{T}^n_\mu \to B$ and call it the *underlying principal G-bundle of* μ.

Let B be an n-dimensional manifold with corners. We do not assume that B is a base space of a locally standard torus fibration. Since B is a manifold with corners, B is equipped with a natural stratification. We denote by $\mathcal{S}^{(k)}B$ the k-dimensional stratum of B. That is, $\mathcal{S}^{(k)}B$ consists of those points which have exactly k nonzero components in the local coordinate φ^B in Condition A. Let G be a subgroup of the group of automorphisms of T^n and $\pi_P : P \to B$ a principal G-bundle on B. Since the differential of any automorphism of T^n at the unit element preserves the integral lattice Λ of T^n, there is a natural homomorphism $\rho : G \to \mathrm{GL}(\Lambda)$. We denote by $\pi_\Lambda : \Lambda_P \to B$ the Λ-bundle associated with P by ρ. Let $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(n-1)}B$ be a rank one sub-lattice bundle of the restriction of Λ_P to the codimension one stratum $\mathcal{S}^{(n-1)}B$. For each k and any point $b \in \mathcal{S}^{(k)}B$, let U be an open neighborhood of b in B on which P is trivialize and $\varphi^P : \pi_P^{-1}(U) \cong U \times G$ a local trivialization of P. Then φ^P induces the local trivialization of $\pi_\Lambda : \Lambda_P \to B$ which is denoted by $\varphi^\Lambda : \pi_\Lambda^{-1}(U) \cong U \times \Lambda$. If necessary, by shrinking U, we can assume that the intersection $U \cap \mathcal{S}^{(n-1)}B$ of U with the codimension one stratum $\mathcal{S}^{(n-1)}B$ has exactly $n-k$ connected components, say, $(U \cap \mathcal{S}^{(n-1)}B)_1, \ldots, (U \cap \mathcal{S}^{(n-1)}B)_{n-k}$. Since Λ is discrete, there exist $n-k$ rank one sub-lattices $L_1, \ldots, L_{n-k}$ in Λ such that φ^Λ sends the inverse image of each connected component $(U \cap \mathcal{S}^{(n-1)}B)_a$ by $\pi_{\mathcal{L}}$ identically to $(U \cap \mathcal{S}^{(n-1)}B)_a \times L_a$.

$$\begin{array}{ccc}
\pi_\Lambda^{-1}(U) & \overset{\varphi^\Lambda}{\cong} & U \times \Lambda \\
\cup & & \cup \\
\pi_\Lambda^{-1}\left((U \cap \mathcal{S}^{(n-1)}B)_a\right) & \cong & (U \cap \mathcal{S}^{(n-1)}B)_a \times \Lambda \\
\cup & & \cup \\
\pi_{\mathcal{L}}^{-1}\left((U \cap \mathcal{S}^{(n-1)}B)_a\right) & \cong & (U \cap \mathcal{S}^{(n-1)}B)_a \times L_a
\end{array}$$

Definition 3.2. $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(n-1)}B$ is said to be *unimodular*, if for each k and any point $b \in \mathcal{S}^{(k)}B$, the sub-lattice $L_1 + \cdots + L_{n-k}$ generated by $L_1, \ldots, L_{n-k}$ is a rank $n-k$ direct summand of Λ. (In [6], such a sub-lattice is called an $(n-k)$*-dimensional unimodular subspace* of Λ.)

Remark 3.3. Definition 3.2 does not depend on the choice of a neighborhood U and a local trivialization φ^P because the condition for a sub-lattice to be unimodular is invariant by an automorphism of Λ.

For $i = 1, 2$, let $(P_i, \mathcal{L}_i)$ be a pair of a principal G-bundle $\pi_{P_i} : P_i \to B$ and a unimodular rank one sub-lattice bundle $\pi_{\mathcal{L}_i} : \mathcal{L}_i \to \mathcal{S}^{(n-1)}B$ of the restriction of the associated Λ-bundle $\pi_{\Lambda_i} : \Lambda_{P_i} \to B$ of P_i by ρ to the codimension one stratum $\mathcal{S}^{(n-1)}B$ of B.

Definition 3.4. They are isomorphic, if there exists a bundle isomorphism $\psi^P : P_1 \to P_2$ such that its associated lattice bundle isomorphism $\psi^\Lambda : \Lambda_{P_1} \to \Lambda_{P_2}$ sends $\mathcal{L}_1$ isomorphically to $\mathcal{L}_2$.

Let $\mu : X \to B$ be a locally standard torus fibration with structure group G and $\pi_{P_\mu} : P_\mu \to B$ its underlying principal G-bundle. As we showed in [20], the map μ determines the unique unimodular rank one sub-lattice bundle $\pi_{\mathcal{L}_\mu} : \mathcal{L}_\mu \to \mathcal{S}^{(n-1)}B$ of the restriction of the associated Λ-bundle $\pi_\Lambda : \Lambda_{P_\mu} \to B$ of P_μ by ρ to the codimension one stratum $\mathcal{S}^{(n-1)}B$. We call $\mathcal{L}_\mu$ a *characteristic bundle of* μ. This is a generalization of a characteristic function of quasi-toric manifolds, or torus manifolds.

In [6, Section 1.5], Davis-Januszkiewicz constructed the canonical model of a quasi-toric manifold from the based polytope and the characteristic function. The similar construction for locally standard torus manifolds has been done by Masuda-Panov in [15, Section 3.2]. In the case of locally standard torus fibrations, we can construct the canonical model of a locally standard torus fibration from the pair of the underlying principal G-bundle and the characteristic bundle in the following way. Let $\mu : X \to B$ be a locally standard torus fibration and $(P_\mu, \mathcal{L}_\mu)$ the pair of the underlying principal G-bundle and the characteristic bundle of μ. For any point $b \in B$, there is a unique stratum which contains b. Suppose b is contained in the k-dimensional stratum $\mathcal{S}^{(k)}B$. By definition of $\mathcal{L}_\mu$, if we take a sufficiently small open neighborhood U of b and a local trivialization $\varphi^P : \pi_{P_\mu}^{-1}(U) \to U \times G$ of P_μ, there exist $n-k$ rank one sub-lattices $L_1, \ldots, L_{n-k}$ of Λ which are determined by the local trivialization $\varphi^P : \pi_{P_\mu}^{-1}(U) \to U \times G$ of P_μ and $\mathcal{L}_\mu$. Since $\mathcal{L}_\mu$ is unimodular, $L_1, \ldots, L_{n-k}$ generate the $(n-k)$-dimensional sub-torus of T^n which is denoted by T_{b,φ^P}. Note that T_{b,φ^P} depends on the choice of $\varphi^P : \pi_{P_\mu}^{-1}(U) \to U \times G$. Let $\pi_{\mathcal{T}_\mu} : \mathcal{T}_\mu^n \to B$ be the T^n-bundle associated with P by the natural action of G on T^n. The local trivialization $\varphi^P : \pi_{P_\mu}^{-1}(U) \to U \times G$ of P_μ induces the local trivialization $\varphi^T : \pi_{\mathcal{T}_\mu}^{-1}(U) \to U \times T^n$ of $\mathcal{T}_\mu^n$ on U. Let $z_i \in \pi_{\mathcal{T}_\mu}^{-1}(U)$ with $(b, t_i) = \varphi^T(z_i)$ for $i = 1$, 2. z_1 and z_2 are said to be fiberwisely equivalent, or $z_1 \sim_{\mathcal{L}_\mu} z_2$, if $t_2^{-1}t_1 \in T_{b,\varphi^P}$. Note that $\sim_{\mathcal{L}_\mu}$ does not depend on the choice of $\varphi^P : \pi_{P_\mu}^{-1}(U) \to U \times G$, but depends only on $\mathcal{L}_\mu$ since the structure group G of P_μ is a subgroup of the automorphism group of T^n. We define an equivalent relation of $\mathcal{T}_\mu^n$ as follows. Two elements z_1 and z_2 of $\mathcal{T}_\mu^n$ are equivalent, or $z_1 \sim z_2$, if and only if $\pi_T(z_1) = \pi_T(z_2)$ and $z_1 \sim_{\mathcal{L}_\mu} z_2$. We denote the quotient space $\mathcal{T}_\mu^n / \sim$ by $X_{(P_\mu, \mathcal{L}_\mu)}$. The projection $\pi_T : \mathcal{T}_\mu^n \to B$ descends to the projection $\mu_{(P_\mu,\mathcal{L}_\mu)} : X_{(P_\mu,\mathcal{L}_\mu)} \to B$. Since $\mathcal{L}_\mu$ is unimodular and B is a manifold with corners, we can show that the space $X_{(P_\mu,\mathcal{L}_\mu)}$ is a closed, connected, topological $2n$-dimensional manifold and the continuous map $\mu_{(P_\mu,\mathcal{L}_\mu)} : X_{(P_\mu,\mathcal{L}_\mu)} \to B$ admits a structure of a topological locally standard torus fibration by the similar way to that of Davis-Januszkiewicz [6, Section 1.5], or Masuda-Panov [15, Section 3.2]. The following is a straightforward generalization of the result [6, Proposition 1.8] by Davis-Januszkiewicz and the result [15, Lemma 3.6] by Masuda-Panov.

Lemma 3.5. *Let $\mu : X \to B$ be a locally standard torus fibration with structure group G and $(P_\mu, \mathcal{L}_\mu)$ the pair of the underlying principal G-bundle and the characteristic bundle of $\mu : X \to B$. Then there is a homeomorphism $\psi_\mu : X_{(P_\mu,\mathcal{L}_\mu)} \to X$ such that such that the diagram*

$$\begin{tikzcd}
X_{(P_\mu,\mathcal{L}_\mu)} \arrow[rr, "\psi_\mu"] \arrow[dr, "\mu_{(P_\mu,\mathcal{L}_\mu)}"'] & \arrow[d, phantom, "\circlearrowleft"] & X \arrow[dl, "\mu"] \\
& B &
\end{tikzcd}$$

is commutative and the restriction of ψ_μ to $\mu_{(P_\mu,\mathcal{L}_\mu)}^{-1}(B \backslash \partial B)$ is a bundle isomorphism.

In fact, by proposition 3.1, there is a map $\nu : \mathcal{T}_\mu^n \to X$. Then it is easy to see from the construction of $X_{(P_\mu,\mathcal{L}_\mu)}$ that the map ν descends to the required fiberwisely homeomorphism. We call the map ψ_μ in Lemma 3.5 a fiberwisely homeomorphism. More precisely, let $\mu_i : X_i \to B$ be a locally standard torus fibration with structure group G for $i = 1, 2$.

Definition 3.6. They are *fiberwisely homeomorphic* if there exists a homeomorphism $\psi^X : X_1 \to X_2$ such that the diagram

$$\begin{tikzcd} X_1 \arrow[rr, "\psi^X"] \arrow[dr, "\mu_1"'] & \arrow[d, phantom, "\circlearrowleft"] & X_2 \arrow[dl, "\mu_2"] \\ & B, & \end{tikzcd}$$

is commutative and the restriction of ψ^X to $\mu_1^{-1}(B \backslash \partial B)$ is a bundle isomorphism.

As topological spaces, locally standard torus fibrations are classified by the pair of the underlying principal G-bundle together with the characteristic bundle.

Theorem 3.7 ([21])**.** *By associating the pair $(P_\mu, \mathcal{L}_\mu)$ to a locally standard torus fibration $\mu : X \to B$ with structure group G, the set of fiberwisely homeomorphism classes of locally standard torus fibrations corresponds one-to-one to the set of isomorphism classes of a pair of a principal G-bundle $\pi_P : P \to B$ together with a unimodular rank one sub-lattice bundle $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(n-1)}B$ of the restriction of the associated Λ-bundle $\pi_\Lambda : \Lambda_P \to B$ of P by ρ to the codimension one stratum $\mathcal{S}^{(n-1)}B$ of B.*

Remark 3.8 (Smoothness)**.** We constructed the canonical model of a locally standard torus fibration from the pair of the underlying principal G-bundle together with the characteristic bundle topologically. We can equip the canonical model with a smooth structure if the base manifold B has a coordinate neighborhood system $\{(U_\alpha, \varphi_\alpha^B)\}_{\alpha \in A}$ which satisfies the following condition: On a nonempty overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$, suppose that the coordinate changing map $\varphi_{\beta\alpha}^B := \varphi_\beta^B|_{U_{\alpha\beta}} \circ (\varphi_\alpha^B|_{U_{\alpha\beta}})^{-1}$ sends $\{\xi \in \varphi_\alpha^B(U_{\alpha\beta})\colon \xi_i = 0\}$ to $\{\zeta \in \varphi_\beta^B(U_{\alpha\beta})\colon \zeta_j = 0\}$ for some i, j. Then the jth component $\varphi_{\beta\alpha_j}^B$ of $\varphi_{\beta\alpha}^B$ satisfies

$$\varphi_{\beta\alpha_j}^B(\xi) = \xi_i$$

on a sufficiently small neighborhood of $\{\xi \in \varphi_\alpha(U_{\alpha\beta})\colon \xi_i = 0\}$ of $\varphi_\alpha(U_{\alpha\beta})$. Note that convex polytopes and surfaces have such coordinate neighborhood systems. For more details, see [20].

4. Topology

4.1. **Fundamental groups.** Let $\mu : X \to B$ be a locally standard torus fibration. Take a point b in the interior of B and a point x in the fiber $\mu^{-1}(b)$ as base points of B and X, respectively. Comparing the fundamental group of $\mathcal{T}_\mu^n$ with that of X by using the homomorphism induced from $\nu : \mathcal{T}_\mu^n \to X$, we have the following result.

Theorem 4.1 ([20], [21])**.** *Suppose that B has the nonempty zero-dimensional stratum. Then the map μ induces the isomorphism $\mu_* : \pi_1(X, x) \cong \pi_1(B, b)$ of fundamental groups.*

4.2. **Cohomology groups.** Suppose that B is equipped with a CW complex structure so that each p-cell $e^{(p)}$ is contained in some stratum $\mathcal{S}^{(k)}B$ of B. Let $B^{(p)}$ be the p-skeleton and $X^{(p)} = \mu^{-1}(B^{(p)})$ its inverse image by μ. We consider the cohomology Leray spectral sequence of the map $\mu : X \to B$, that is, the spectral sequence $\{(E_X)_r^{p,q}, d_r^X\}$ associated with the exact couple

$$\begin{tikzcd} \bigoplus_{p,q} A^{p,q} \arrow[rr, "i"] & & \bigoplus_{p,q} A^{p,q} \arrow[dl, "j"] \\ & \bigoplus_{p,q} E^{p,q}, \arrow[ul, "k"] & \end{tikzcd}$$

where

$$A^{p,q} := H^{p+q}(X, X^{(p-1)}; \mathbb{Z}),\ E^{p,q} := H^{p+q}(X^{(p)}, X^{(p-1)}; \mathbb{Z})$$

and i, j are the maps

$$i : A^{p,q} \to A^{p-1,q+1},\ j : A^{p,q} \to E^{p,q}$$

induced from inclusions $(X, X^{(p-2)}) \subset (X, X^{(p-1)})$ and $(X^{(p)}, X^{(p-1)}) \subset (X, X^{(p-1)})$, respectively, and k is the map

$$k : E^{p,q} \to A^{p+1,q}$$

which is the connecting homomorphism of the exact sequence of the triple $(X, X^{(p)}, X^{(p-1)})$. We denote the barycenter of a p-cell $e^{(p)}$ by $c^{(p)}$ and the restriction of the map $\nu : \mathcal{T}_\mu^n \to X$ in Proposition 3.1 to the fiber $\pi_{\mathcal{T}}^{-1}(c^{(p)})$ by

$$\nu_{c^{(p)}} : \pi_{\mathcal{T}}^{-1}(c^{(p)}) \to \mu^{-1}(c^{(p)}).$$

Let $(C^p(B; \mathcal{H}_T^q), \delta)$ be the cochain complex of the CW complex B with the Serre local system $\mathcal{H}_T^q$ of the qth cohomology with $\mathbb{Z}$-coefficient for the fiber bundle $\pi_{\mathcal{T}} : \mathcal{T}_\mu^n \to B$. We denote by $C^p(B; \mathcal{H}_X^q)$ the subset of $C^p(B; \mathcal{H}_T^q)$ whose cochain takes a value in the image $\nu^*_{c_\lambda^{(p)}} \left(H^q(\mu^{-1}(c_\lambda^{(p)}); \mathbb{Z}) \right)$ of $H^q(\mu^{-1}(c_\lambda^{(p)}); \mathbb{Z})$ by $\nu^*_{c_\lambda^{(p)}}$ for each p-cell $e_\lambda^{(p)}$.

Theorem 4.2 ([20], [21]). *The subset $C^p(B; \mathcal{H}_X^q)$ is preserved by the differential δ of $C^p(B; \mathcal{H}_T^q)$, that is, $C^p(B; \mathcal{H}_X^q)$ is a sub-complex of $(C^p(B; \mathcal{H}_T^q), \delta)$. We denote its cohomology by $H^p(B; \mathcal{H}_X^q)$. Then we have the isomorphisms*

$$\begin{aligned} &(E_X)_1^{p,q} \cong C^p(B; \mathcal{H}_X^q), \quad (E_X)_2^{p,q} \cong H^p(B; \mathcal{H}_X^q), \\ &(E_X)_\infty^{p,q} = F^p H^{p+q}(X; \mathbb{Z}) / F^{p+1} H^{p+q}(X; \mathbb{Z}), \end{aligned}$$

where $F^l H^k(X; \mathbb{Z})$ is the image of the map $H^k(X, X^{(l-1)}; \mathbb{Z}) \to H^k(X; \mathbb{Z})$.

For the spectral sequence of cohomology groups, see [12].

Remark 4.3. (1) For $q = 0$, it is easy to see that $(E_X)_2^{p,0} \cong H^p(B; \mathcal{H}_X^q) \cong H^p(B; \mathbb{Z})$. Moreover $(E_X)_1^{p,q} = 0$, if q or p is greater than half the dimension of X.
(2) If $n = 2$ and $\partial B \neq \emptyset$, we can take a cell decomposition of B so that all zero cells are included in ∂B. In this case, the Leray spectral sequence $\{(E_X)_r^{p,q}, d_r^X\}$ degenerates at E_2-term. In fact, $\partial B \neq \emptyset$ implies $(E_X)_2^{2,0} \cong H^2(B; \mathcal{H}_X^0) \cong H^2(B; \mathbb{Z}) = 0$, and since $e_\lambda^{(0)} \in \partial B$, the fiber $\mu^{-1}(e_\lambda^{(0)})$ of μ on $e_\lambda^{(0)}$ is diffeomorphic to the torus whose dimension is equal or less than one. Then $(E_X)_2^{0,2} \cong (E_X)_1^{0,2} \cong C^0(B; \mathcal{H}_X^2) = 0$.

Corollary 4.4 ([20], [21]). *The Euler characteristic $\chi(X)$ is equal to the number of elements in the zero-dimensional stratum $\mathcal{S}^{(0)}B$ of B.*

4.3. **K-groups.** By replacing the cohomology functor $H^*(\ )$ by the K-functor $K^*(\ )$ in the cohomology Leray spectral sequence of the map $\mu : X \to B$, the similar method is available for computing K-groups. We also denote such a spectral sequence by the same notation $\{(E_X)_r^{p,q}, d_r^X\}$ and call it an Atiyah-Hirzebruch type spectral sequence for the map $\mu : X \to B$. For K-theory, see [1] and for the spectral sequence of K-theory, see [2]. Let $(C^p(B; \mathcal{K}_{\mathcal{T}}^q), \delta)$ be the cochain complex of the CW complex B with the local system $\mathcal{K}_{\mathcal{T}}^q$ with respect to the qth K-group of the fiber of the fiber bundle $\pi_{\mathcal{T}} : \mathcal{T}_\mu^n \to B$. We denote by $C^p(B; \mathcal{K}_X^q)$ the subset of $C^p(B; \mathcal{K}_{\mathcal{T}}^q)$ whose cochain takes a value in the image $\nu_{c^{(p)}}^* \left(K^q(\mu^{-1}(c^{(p)}))\right)$ of $K^q(\mu^{-1}(c^{(p)}))$ by $\nu_{c^{(p)}}^*$ for each p-cell $e^{(p)}$.

Theorem 4.5 ([21])**.** *The subset $C^p(B; \mathcal{K}_X^q)$ is preserved by the differential δ of $C^p(B; \mathcal{K}_{\mathcal{T}}^q)$, that is, $C^p(B; \mathcal{K}_X^q)$ is a sub-complex of $(C^p(B; \mathcal{K}_{\mathcal{T}}^q), \delta)$. We denote by $H^p(B; \mathcal{K}_X^q)$ its cohomology. Then we have the isomorphisms*

$$(E_X)_1^{p,q} \cong C^p(B; \mathcal{K}_X^q), \quad (E_X)_2^{p,q} \cong H^p(B; \mathcal{K}_X^q),$$
$$(E_X)_\infty^{p,q} = F^pK^{p+q}(X)/F^{p+1}K^{p+q}(X),$$

where $F^pK^*(X)$ is the image of the map $K^*(X, X^{(p-1)}) \to K^*(X)$.

Example 4.6. Let us compute K-groups of X in Example 2.6. We give B the following cell decomposition. One-cells $e_1^{(1)}$, $e_2^{(1)}$, and $e_3^{(1)}$ correspond to α, β, and

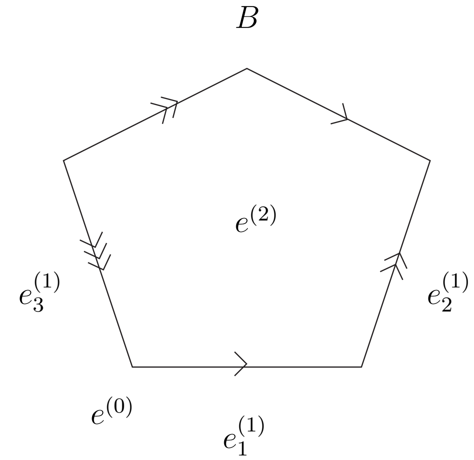

FIGURE 3. A cell decomposition of B

the edge arc γ in Figure 1, respectively.

For even q, since a fiber of $\pi_{\mathcal{T}} : \mathcal{T}_\mu^n \to B$ is T^2, by [9] or [21, Lemma A.1], the K-group of its fiber is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and all homomorphisms between them which are induced from parallel transports are identity. Also by [21], images of $\nu_{c_\lambda^{(p)}}^* : K^q(\mu^{-1}(c_\lambda^{(p)})) \to K^q(\pi_{\mathcal{T}}^{-1}(c_\lambda^{(p)}))$ are computed as follows

$$\nu_{c_\lambda^{(p)}}^* \left(K^q(\mu^{-1}(c_\lambda^{(p)}))\right) = \begin{cases} \mathbb{Z} \oplus 0 & p = 0, \text{ or } p = 1 \text{ and } \lambda = 3 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{otherwise.} \end{cases}$$

A similar computation in [20, Example 6.14] shows that

$$H^p(B; \mathcal{K}_X^q) = \begin{cases} \mathbb{Z} & p = 0, 2 \\ \mathbb{Z}^{\oplus 4} & p = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For odd q, by [21, Lemma A.3], we have an isomorphism

$$H^p(B; \mathcal{K}_X^q) \cong H^p(B; \mathcal{H}_X^q).$$

$H^p(B; \mathcal{H}_X^q)$ is computed in [20, Example 6.14] and the result is given as follows

$$H^p(B; \mathcal{K}_X^q) = \begin{cases} \mathbb{Z}^{\oplus 3} & p = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The table of E^2-terms is in Figure 4. In particular, the spectral sequence is

$$\begin{array}{c|ccc}
q & & & \\
 & \mathbb{Z} & \mathbb{Z}^{\oplus 4} & \mathbb{Z} \\
 & 0 & \mathbb{Z}^{\oplus 3} & 0 \\
 & \mathbb{Z} & \mathbb{Z}^{\oplus 4} & \mathbb{Z} \\
\hline
 & 0 & \mathbb{Z}^{\oplus 3} & 0 \quad\quad p \\
 & \mathbb{Z} & \mathbb{Z}^{\oplus 4} & \mathbb{Z}
\end{array}$$

FIGURE 4. the table of $(E_X)_2^{p,q}$-terms

degenerate at E^2-term, and K-groups of X are given by

$$K^k(X) = \begin{cases} \mathbb{Z}^{\oplus 5} & k : \text{even} \\ \mathbb{Z}^{\oplus 4} & k : \text{odd.} \end{cases}$$

5. COMPATIBLE SYMPLECTIC FORMS

Suppose that $\mu : X \to B$ is a locally standard torus fibration and ω is a symplectic form on X. ω is said to be *compatible with* $\mu : X \to B$ if for any $b \in B$, a coordinate neighborhood (U, φ^B) and a diffeomorphism $\varphi^X : \mu^{-1}(U) \to \mu_{\mathbb{C}^n}^{-1}(\varphi^B(U))$ in Condition A can be taken so that φ^X is a symplectomorphism with respect to ω and $\omega_{\mathbb{C}^n}$, where $\omega_{\mathbb{C}^n}$ is the symplectic form on $\mathbb{C}^n$ which is defined by

$$\omega_{\mathbb{C}^n} = -\frac{\sqrt{-1}}{2\pi} \sum_{i=1}^n dz_i \wedge d\overline{z}_i.$$

The map $\mu : X \to B$ with a compatible symplectic form is a *singular Lagrangian fibration with non-degenerate elliptic singularities.* For more details, see [22, Section 4.1], [19, Section 4]. We give the necessary and sufficient condition when $\mu : X \to B$ admits a compatible symplectic form.

Definition 5.1. An *integral affine structure on B compatible with $\mu : X \to B$* is a coordinate neighborhood system $\{(U_\alpha, \varphi_\alpha^B)\}$ of B such that each $(U_\alpha, \varphi_\alpha^B)$ satisfies Condition A and for each non-empty intersection $U_{\alpha\beta} \neq \emptyset$, the coordinate transition function is of the form

$$\varphi_{\beta\alpha}^B(\xi) := \left(\varphi_\beta^B\big|_{U_{\alpha\beta}}\right) \circ \left(\varphi_\alpha^B\big|_{U_{\alpha\beta}}\right)^{-1}(\xi) = g_{\beta\alpha}(\xi)^{-T}\xi + c_{\alpha\beta},$$

where $g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}_n(\mathbb{Z})$ is the transition function of $\pi_{\mathcal{T}} : \mathcal{T}^n_\mu \to B$ on $U_{\alpha\beta}$ with respect to local coordinates φ^T in Proposition 3.1 and $c_{\alpha\beta}$ is a constant. Here we identify $\mathrm{Aut}(T^n)$ with $\mathrm{GL}_n(\mathbb{Z})$ by using the decomposition $T^n \cong (S^1)^n$.

Theorem 5.2 ([21]). *Let $\mu : X \to B$ be a locally standard torus fibration. $\mu : X \to B$ has a compatible symplectic form if and only if B admits an integral affine structure compatible with $\mu : X \to B$.*

Remark 5.3. For nonsingular Lagrangian fibrations, this theorem is well known. For example, see [7], [19], [16]. Recently this result are extended to the case where some degeneracy of a symplectic form is allowed by Gay-Symington [10].

Corollary 5.4 ([21]). *If B admits an integral affine structure compatible with $\mu : X \to B$, then the characteristic bundle $\pi_{\mathcal{L}_\mu} : \mathcal{L}_\mu \to \mathcal{S}^{(n-1)}B$ of $\mu : X \to B$ has a section which generates $\mathcal{L}$ fiberwisely.*

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