Perfect Bott-Morse Function on Polygon Space

Takahiko Yoshida

Abstract

Let M_n be the moduli space of spatial polygons with n-edges. There is a function f_1 on M_n which associates a square of the norm of the j-th diagonal $\|a_1 + \cdots + a_{j+1}\|^2$ to $P = (a_1, ..., a_n) \in M_n$. In this paper, we show that f_1 is a perfect Bott-Morse function. We use this function to give the simple geometric computation for Betti numbers.

1 Introduction

Let M_n $(n \geq 3)$ be the moduli space of spatial polygons with n-edges defined by

$$M_n = \{P = (a_1, \cdots, a_n) \in (S^2)^n \mid a_1 + \cdots + a_n = 0\}/SO(3),$$

where $SO(3)$ acts on $(S^2)^n$ diagonally. If n is odd, the space M_n is a $(2n-6)$-dimensional smooth symplectic manifold. For $j = 1, ..., n-3$, there is a function l_j on M_n which is defined by

$$l_j(P) = \|a_1 + \cdots + a_{j+1}\| \tag{1.1}$$

i.e. the norm of the j-th diagonal for $P \in M_n$. Since the function l_j is not smooth at points $P \in {l_j}^{-1}(0)$, the Hamiltonian flow of l_j is defined only on the open dense set

$$U_j = \{P = (a_1, \cdots, a_n) \in M_n \mid a_1 + \cdots + a_{j+1} \neq 0\}. \tag{1.2}$$

In [7], Kapovich-Millson showed that each of these Hamiltonian flows induces the circle action on U_j, thus the function $l_j|_{U_j}$ is a moment map of this action. In [4], it is known that if a circle acts globally on a symplectic manifold in a Hamiltonian fashion, then a moment map of this action is a perfect Bott-Morse function. In the above case, these Hamiltonian circle actions are defined only on the open dense sets of M_n. Now we have one natural question: Can we think l_j is a perfect Bott-Morse function? Since l_j is smooth only on the open dense set, it is not a Bott-Morse function. But

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the caution why l_j is not smooth at points $P \in l_j^{-1}(0)$ is that we take the norm of the diagonal, i.e. we take a square root! Instead of l_j, considering the new function f_j on M_n $(j = 1, ..., n-3)$, which is defined by

$$f_j(P) = (l_j(P))^2 = \|a_1 + \cdots + a_{j+1}\|^2,$$

for $P \in M_n$, we can obtain the following result.

Theorem 1.1 *The function f_1, f_{n-3} are perfect Bott-Morse functions.*

In contrast to f_1 and f_{n-3}, this is false for $j = 2, ..., n-4$, as described in Remark 2.6.

If a n-dimensional torus T^n acts effectively on a $2n$-dimensional symplectic manifold (M^{2n}, ω) in a Hamiltonian fashion, (M^{2n}, ω) is equivariantly symplectomorphic to a toric variety [3]. A toric variety has been studied by many people and many important results are obtained, for example [1], [5]. In this paper, we call (M^{2n}, ω) a symplectic toric manifold. In our case, it was also shown in [7] that the Hamiltonian flows of all l_j commute each other, hence there is an effective T^n-action on $U = \cap_{j=1}^{n-3} U_j$ with $\overrightarrow{l} = (l_1, ..., l_{n-3}) : U \to \mathbb{R}^{n-3}$ as a moment map. Then the polygon space M_n has an 'almost' symplectic toric structure. As Jeffrey-Weitsman showed in [6], there is an 'almost' symplectic toric structure on the moduli space of flat connections on a Riemann surface, too. We want to know the differences between a symplectic toric and an 'almost' symplectic toric manifold. It's the first and the most important motivation.

This paper is organized as follows. In the next section, we recall Kapovich-Millson's results about the polygon space, the functions l_j for $j = 1, ..., n-3$ on M_n, and its Hamiltonian flows. Then we use these results to compute Betti numbers of M_n. Section 3 is devoted to prove Theorem 1.1.

Unless otherwise specified, all cohomologies are with rational coefficients.

2 Polygon Space

2.1 Polygon Space and Bending Flow

In this subsection, we recall the basic facts about the polygon space and the bending flow in [7].

Let $((S^2)^n, vol)$ be the 2-dimensional sphere with the $SO(3)$-invariant volume form on S^2 with the volume equal to 4π. For n-tuple of integral numbers $r = (r_1, ..., r_n) \in \mathbb{N}^n$, we give the n-products of 2-sphere $(S^2)^n$ the symplectic form $\omega = \sum_{j=1}^n p_j{}^* r_j vol$, where $p_j : (S^2)^n \to S^2$ is the j-th projection. $SO(3)$ acts diagonally on $((S^2)^n, \omega)$ with the moment map $\mu_{n,r} : (S^2)^n \to \mathfrak{su}(3)^* \cong \mathbb{R}^3$ is defined by

$$\mu_{n,r}(P) = \sum_{j=1}^{n} r_j a_j$$

for $P = (a_1, ..., a_n) \in (S^2)^n$. The moduli space $M_n(r_1, ..., r_n)$ of polygons with edge lengths $r = (r_1, ..., r_n)$ is defined by the symplectic quotient

$$M_n(r_1, ..., r_n) = \mu_{n,r}^{-1}(0)/SO(3) \tag{2.1}$$

The critical points $P = (a_1, ..., a_n)$ of $\mu_{n,r}$ is the points P such that all a_j lie on a same line in $\mathbb{R}^3$ through 0. Such points exist if and only if $r = (r_1, ..., r_n)$ satisfies the following condition

"There exists a subset $I \subset \{1, ..., n\}$ such that $\sum_{i \in I} r_i - \sum_{j \in I^c} r_j = 0$."

If the edge lengths $r = (r_1, ..., r_n)$ equal to $(1, ..., 1)$, critical points do not consist in the level set $\mu_n^{-1}(0)$ of the moment map $\mu_n = \mu_{n,(1,...,1)}$ for n odd $n \geq 3$. Hence $M_n = M_n(1, ..., 1)$ is a smooth $(2n-6)$-dimensional symplectic manifold. Moreover, in [7], M_n has a Kähler structure as follows. The tangent space at $P = (a_1, ..., a_n) \in M_n$ consists of the set of vectors $v_i \in \mathbb{R}^3$ for $i = 1, ..., n$ under the following conditions

$$\begin{cases} \cdot\ \langle a_i, v_i \rangle = 0 \text{ for } i = 1, ..., n, \\ \cdot\ v_1 + \cdots + v_n = 0, \\ \cdot\ \displaystyle\sum_{i=1}^{n} v_i \times a_i = 0, \end{cases} \tag{2.2}$$

where $\langle\ ,\ \rangle$, $\times$ are the inner product, the vector product in $\mathbb{R}^3$, respectively. Then the symplectic form ω, the complex structure J, and the Riemann metric g are defined by the formula

$$\begin{aligned} &\cdot\ \omega(u, v) = \sum_{i=1}^{n} \langle u_i \times v_i, a_i \rangle, \\ &\cdot\ J : v = (v_1, ..., v_n) \mapsto (a_1 \times v_1, ..., a_n \times v_n), \\ &\cdot\ g(u, v) = \sum_{i=1}^{n} \langle u_i, v_i \rangle \end{aligned}$$

for $u = (u_1, ..., u_n)$, $v = (v_1, ..., v_n) \in T_P M_n$.

In the following, we shall assume n is odd and $n \geq 3$, i.e. M_n is a smooth manifold.

In [7], Kapovich-Millson discovered a symplectic toric structure on an open dense set of M_n as follows. For $j = 1, ..., n-3$, the function l_j on M_n is defined in (1.1). Since l_j is not smooth at points $P \in {l_j}^{-1}(0)$, the Hamiltonian flow φ_j^t of l_j is defined only on the open dense set U_j (for the definition of U_j, see (1.2)). The flow φ_j^t induces the circle action on U_j with the following geometric description. If $P \in U_j$, the j-th diagonal $a_1 + \cdots + a_{j+1}$ divides the polygon P into two pieces. The first piece has $a_1, ..., a_{j+1}$, and $a_1 + \cdots a_{j+1}$ as edges, and vectors $a_{j+2}, ..., a_n$, and $a_1 + \cdots a_{j+1}$ are the edges of the second piece. φ_j^t keeps the second piece and rotates the first piece around the j-th diagonal with the angular velocity equal to 2π. In [7], these flows φ_j^t are called bending flows. It is easy to see that bending flows commute each other, then there is a symplectic toric structure on $U = \cap_{j=1}^{n-3} U_j$. But these flows do not extend globally on M_n.

2.2 Bott-Morse Theory on M_n

In this subsection, we give a short review of Bott-Morse theory. For more details, see [1], [9]. Let M be a compact manifold. A smooth function $f : M \to \mathbb{R}$ is called a Bott-Morse function if its critical point set is a finitely disjoint union of connected submanifolds called critical manifolds, and the Hessian of f is nondegenerate (fibrewisely) on the normal bundle of critical manifolds. An index $\lambda(C)$ of the critical manifold C is the dimension of the negative eigenspace of the Hessian of f on the normal bundle of C.

For a Bott-Morse function f, the famous Morse inequalities holds as follows. For a manifold M, let $P_t(M)$ be the Poincaré series defined by

$$P_t(M) = \sum_{j \geq 0} t^j \cdot \dim \mathrm{H}^j(M; \mathbb{Q}).$$

Then Morse inequalities say that

$$\sum_C t^{\lambda(C)} P_t(C) - P_t(M) = (1+t)R(t),$$

where $R(t)$ is a series with non negative integer coefficients and the sum runs over all critical manifolds. A Bott-Morse function f is called perfect if Morse inequalities are equalities, that is

$$P_t(M) = \sum_C t^{\lambda(C)} P_t(C). \tag{2.3}$$

In our case, to see the meaning of Theorem 1.1 concretely, let us compute Morse equalities (2.3).

Lemma 2.1 *f_1 has the following three kinds of critical manifolds.*
(i) *The S^2 bundle E_{n-2} on M_{n-2} defined by*

$$E_{n-2} = \mu_{n-2}^{-1}(0) \times_{SO(3)} S^2 \longrightarrow M_{n-2}.$$

The index of E_{n-2} is 0.
(ii) *The polygon space $M_{n-1}(2, 1, ..., 1)$ defined in (2.1). The index of $M_{n-1}(2, 1, ..., 1)$ is* 2.
(iii) *Polygons $P = (a_1, ..., a_n)$ with $a_k = \pm(a_1 + a_2)$ for $k = 3, ..., n$. The all indices of these polygons are $n - 3$.*

Proof. The S^2 bundle E_{n-2} corresponds to the critical manifold for the minimum value. Since 0 is the minimum value of f_1, the level set of 0

$$f_1^{-1}(0) = \{P = (a_1, ..., a_n) \in (S^2)^n \mid a_1 + \cdots + a_n = 0,\ a_1 + a_2 = 0\}/SO(3)$$

is the critical manifold. It is easy to see that $f_1^{-1}(0)$ is diffeomorphic to E_{n-2}. This fibre bundle has the section $s : M_{n-2} \to E_{n-2}$ which is defined by

$$s(P) = [a_3, (a_3, ..., a_n)]$$

for $P = (a_3, ..., a_n) \in M_{n-2}$. Hence the Euler class vanishes and by Leray-Hirsh's Theorem (see [2]), the following equality holds

$$\mathrm{H}^*(E_{n-2}) = \oplus \mathrm{H}^*(M_{n-2}) \otimes \mathrm{H}^*(S^2).$$

Since 0 is the minimum value, the index of $f_1^{-1}(0)$ is 0.

Similarly the polygon space $M_{n-1}(2, 1, ..., 1)$ corresponds to the critical manifold for the maximum value 4. The level set

$$f_1^{-1}(4) = \{P = (a_1, ..., a_n) \in (S^2)^n \mid a_1 + \cdots + a_n = 0,\ a_1 = a_2\}/SO(3)$$

is naturally identified with $M_{n-2}(2, 1, ..., 1)$. The index of $f_1^{-1}(4)$ is equal to the codimension of $f_1^{-1}(4)$, which is 2.

Are there another critical manifolds? In $f_1^{-1}((0, 4))$, the critical points of $f_1|_{f_1^{-1}((0,4))}$ coincide with those of $l_1|_{f_1^{-1}((0,4))}$. Since $l_1|_{U_1}$ is a moment map, critical points of $l_1|_{f_1^{-1}((0,4))}$ are equal to the fixed points of the bending flow of l_1. It is easy to see that the fixed points in $f_1^{-1}((0, 4))$ correspond to the degenerate polygons $P = (a_1, ..., a_n)$ with $a_k = \pm(a_1 + a_2)$ for $k \geq 2$. Let I^+, I^- be the subset in $\{3, 4, ..., n\}$ defined by

$$I^+ = \{i \mid a_i = a_1 + a_2\},$$
$$I^- = \{j \mid a_j = -(a_1 + a_2)\}.$$

Since polygons are closed, i.e. $a_1 + \cdots + a_n = 0$, the cardinaries of I^+, I^- are equal to $\frac{n-3}{2}$, $\frac{n-1}{2}$, respectively. Hence there are exactly $\binom{n-2}{\frac{n-3}{2}}$ such polygons.

Now we find out these indices. Let $P = (a_1, ..., a_n)$ be the polygon corresponding to the fixed point. Then the bending flow φ_1^t of l_1 induces the infinitesimal action $d_P\varphi_1^t$ on the tangent space $T_P M_n$ at P. By [9], the index of P is equal to twice the numbers of negative weights of this action. To see the index of these fixed points, it is sufficient to identify the index of the following polygons $P = (a_1, ..., a_n)$, where $a_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, $a_2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$, $a_i = (1, 0, 0)$ for $i \in I^+$, and $a_j = (-1, 0, 0)$ for $j \in I^-$. From (2.2), all vectors v_i for $i \in I^+$ and any $\frac{n-3}{2}$ vectors in v_j for $j \in I^-$ form the basis of $T_P M_n$ as a complex vector space. In these coordinates

v_k, $d_P\varphi_1^t$ acts diagonally on $T_PM_n \cong \mathbb{C}^{n-3}$ for $t \in S^1$, and $d_P\varphi_1^t$ sends v_k to $e^{\mp ti}v_k$, where the sign $-$ or $+$ is taken according to $k \in I^+$ or $k \in I^-$, respectively. Then the number of the negative weights are equal to $\frac{n-3}{2}$. Thus the index of P is $n-3$. □

Corollary 2.2 *The critical manifold $f_1^{-1}(0) \cong E_{n-2} \subset M_n$ is a coisotropic submanifold of codimension equal to 2. The symplectic form ω vanishes along the fibre of E_{n-2}. For the special case of $n=5$, $f_1^{-1}(0) \cong S^2$ is a Lagrangian submanifold.*

From the above argument, we can compute Morse inequalities (2.3).

Corollary 2.3 *The Poincaré series of M_n satisfies the following relation.*

$$P_t(M_n) = (1+t^2)P_t(M_{n-2}) + \binom{n-2}{\frac{n-3}{2}} t^{n-3} + t^2 P_t(M_{n-1}(2,1,...,1)).$$

Example 2.4 (1) For $n=5$, we have

$$P_t(M_5) = (1+t^2)P_t(M_3) + 3t^2 + t^2 P_t(M_4(2,1,...,1)).$$

But it is easy to see that $M_3 = \{\text{1-point}\}$ and $M_4(2,1,...,1)$ is biholomorphic to $\mathbb{C}P^1$. Then we have

$$\begin{aligned} P_t(M_5) &= 1 + t^2 + 3t^2 + t^2(1+t^2) \\ &= 1 + 5t^2 + t^4. \end{aligned}$$

(2) For $n=7$, we have

$$P_t(M_7) = (1+t^2)P_t(M_5) + 10t^4 + t^2 P_t(M_6(2,1,...,1)).$$

We must calculate the Poincaré series $P_t(M_6(2,1,...,1))$ of $M_6(2,1,...,1)$. Consider the function f on $M_6(2,1,...,1)$ defined by

$$f(P) = \|2a_1 + a_2\|$$

for $P \in M_6(2,1,...,1)$. In this case, f is smooth totally on $M_6(2,1,...,1)$ and its Hamiltonian flow induces the bending flow on $M_6(2,1,...,1)$ as in the case of M_n (see [7]). Thus f is a moment map of the bending flow, that is a perfect Bott-Morse function. Morse inequalities (2.3) for f hold

$$P_t(M_6(2,1,...,1)) = P_t(M_5) + 4t^4 + t^2 P_t(M_5(3,1,...,1)).$$

Repeating the same argument for $M_5(3,1,...,1)$, we have

$$P_t(M_5(3,1,...,1)) = 1 + t^2 + t^4.$$

Then we have the following formula

$$\begin{aligned} P_t(M_6(2,1,...,1)) &= 1+5t^2+t^4+4t^4+t^2(1+t^2+t^4) \\ &= 1+6t^2+6t^4+t^6. \end{aligned}$$

$$\begin{aligned} P_t(M_7) &= (1+t^2)(1+5t^2+t^4)+10t^4 \\ &\quad +t^2(1+6t^2+6t^4+t^6)) \\ &= 1+7t^2+22t^4+7t^6+t^8. \ \square \end{aligned}$$

For generally n, to calculate $P_t(M_n)$, we need to know the Poincaré series of the polygon space $M_n(n-2k,1,...,1)$ for $k=1,...,\frac{n-2}{2}$. Let f_k be the function on $M_n(n-2k,1,...,1)$ for $k=1,...,\frac{n-2}{2}$, which is defined by

$$f_k(P) = \|(n-2k)a_1 + a_2\|$$

for $P \in M_n(n-2k,1,...,1)$. Using the same argument for the function f_k on $M_n(n-2k,1,...,1)$, we obtain

$$\begin{aligned} P_t(M_n(n-2k,1,...,1)) &= P_t(M_{n-1}(n-1-2k,1,...,1)) \\ &\quad +\binom{n-2}{k-1}t^{2n-2k-4} \\ &\quad +t^2P_t(M_{n-1}(n+1-2k,1,...,1) \end{aligned}$$

for $k=2,...,\frac{n-2}{2}$, and

$$\begin{aligned} P_t(M_n(n-2,1,...,1)) &= P_t(M_{n-1}(n-3,1,...,1)) \\ &\quad +t^{2n-6}P_t(M_3(n-2,1,n-2)) \end{aligned}$$

for $k=1$. From the above formulae, we can calculate the Poincaré series $P_t(M_n(n-2k,1,...,1))$ of $M_n(n-2k,1,...,1)$ by the induction.

Lemma 2.5 *For $k=1,...,\frac{n-2}{2}$, the Poincaré series $P_t(M_n(n-2k,1,...,1))$ is given by the formula*

$$\begin{aligned} &P_t(M_n(n-2k,1,...,1)) \\ &\quad = \sum_{j=0}^{k-1}\{1+\binom{n-1}{1}+\cdots+\binom{n-1}{j}\}(t^{2j}+t^{2n-6-2j}) \\ &\qquad +\{1+\binom{n-1}{1}+\cdots+\binom{n-1}{k-1}\}\sum_{j=k}^{n-3-k}t^{2j}. \end{aligned}$$

From Corollary 2.3 and Lemma 2.5, we can get the Poincaré series of M_n.

Corollary 2.6 *The Poincaré series $P_t(M_n)$ of M_n satisfies the following formula*

$$
\begin{aligned}
P_t(M_n) &= 1+\{1+\binom{n-1}{1}\}t^2 \\
&\vdots \\
&+\{1+\binom{n-1}{1}+\binom{n-1}{2}+\cdots+\binom{n-1}{\frac{n-5}{2}}\}t^{n-5} \\
&+\{1+\binom{n-1}{1}+\binom{n-1}{2}+\cdots+\binom{n-1}{\frac{n-3}{2}}\}t^{n-3} \\
&+\{1+\binom{n-1}{1}+\binom{n-1}{2}+\cdots+\binom{n-1}{\frac{n-5}{2}}\}t^{n-1} \\
&\vdots \\
&+\{1+\binom{n-1}{1}\}t^{2n-8}+t^{2n-6}.
\end{aligned}
$$

Remark 2.7 (1) The Poincaré series of the polygon space M_n for odd $n \geq 3$ were first calculated by Kirwan in [8].
(2) In contrast to f_1, f_{n-3}, the function f_j is not a Bott-Morse function for $j = 2, ..., n-4$. In fact, the level set of the minimum value 0 is identified with the bundle $\mu_{j+1}^{-1}(0) \times_{SO(3)} \mu_{n-j-1}^{-1}(0)$. Since either $j+1$ or $n-j-1$ is even, $f_j^{-1}(0)$ has a singular point.

If M_n is a symplectic toric manifold, all f_j are perfect Bott-Morse functions, and all critical manifolds are symplectic submanifolds. Corollary 2.2 and the above observation represent the difference between a symplectic toric and an 'almost' symplectic toric manifolds. □

3 The proof of Theorem

The proof of Theorem 1.1 consists of 2 steps. First we shall prove that f_1, f_{n-3} are Bott-Morse functions in Lemma 3.1, then that they are perfect in Lemma 3.4. Though we prove Theorem 1.1 only for f_1, the proof for f_{n-3} is similar to the case of f_1.

Lemma 3.1 *f_1 is a Bott-Morse function.*

Since $l_1|_{U_1} : U_1 \to \mathbb{R}$ is a moment map, $l_1|_{U_1}$ is a Bott-Morse function. So is $f_1|_{U_1} : U_1 \to \mathbb{R}$. Then it is sufficient to show that the Hessian of f_1 is nondegenerate fibrewisely on the normal bundle of $f_1^{-1}(0)$. The following Lemma describes the behavior of f_1 on the neighborhood of $f_1^{-1}(0)$.

Lemma 3.2 *There is a diffeomorphism from a neighborhood of $f_1^{-1}(0) \subset M_n$ to that of* $(\mathrm{graph}(-id) \times \mu_{n-2}^{-1}(0))/SO(3) \subset (S^2 \times S^2 \times \mu_{n-2}^{-1}(0))/SO(3)$, *where $SO(3)$ acts diagonally on $S^2 \times S^2 \times \mu_{n-2}^{-1}(0)$, and* graph$(-id)$ *is the graph of the involution $-id : S^2 \to S^2$, associating the antipodal point $-a$ to $a \in S^2$. Under this diffeomorphism, f_1 is identified with the function $(S^2 \times S^2 \times \mu_{n-2}^{-1}(0))/SO(3) \to \mathbb{R}$ by taking the norm $\|a_1 + a_2\|^2$ to $[a_1, a_2, P] \in (S^2 \times S^2 \times \mu_{n-2}^{-1}(0))/SO(3)$.*

We define the function $\widetilde{f_1} : \mu_n^{-1}(0) \to \mathbb{R}$ by

$$\widetilde{f_1}(P) = \|a_1 + a_2\|^2$$

for $P \in \mu_n^{-1}(0)$. To prove Lemma 3.2, we need the following Lemma.

Lemma 3.3 *The map $\psi : \mu_n^{-1}(0) \to S^2 \times S^2$ taking the first two factors of $\mu_n^{-1}(0) \subset (S^2)^n$ is a submersion at points in $\widetilde{f_1}^{-1}(0)$, i.e. the differential of ψ at $P \in \widetilde{f_1}^{-1}(0)$ is surjective.*

Proof. From the definition of μ_n, it is clear that

$$\begin{aligned} d_P\mu_n &= d_{(a_1,a_2)}\mu_2 + d_{(a_3,...,a_n)}\mu_{n-2} \\ &: T_P(S^2)^n = T_{(a_1,a_2)}(S^2)^2 \times T_{(a_3,...,a_n)}(S^2)^{n-2} \longrightarrow \mathbb{R}^3 \end{aligned}$$

for $P = (a_1, ..., a_n) \in (S^2)^n$. If P is in $\widetilde{f_1}^{-1}(0)$, $d_{(a_3,...,a_n)}\mu_{n-2}$ is surjective. Then, for arbitrary $(\xi_1, \xi_2) \in T_{(a_1,a_2)}(S^2)^2$, there exists $(\xi_3, ..., \xi_n) \in T_{(a_3,...,a_n)}(S^2)^{n-2}$ such that $d_{(a_3,...,a_n)}\mu_{n-2}(\xi_3, ..., \xi_n) = -d_{(a_1,a_2)}\mu_2(\xi_1, \xi_2)$. This implies $(\xi_1, ..., \xi_n) \in \ker d_P\mu_n = T_P\mu_n^{-1}(0)$. □

Proof of Lemma 3.2. It is clear that there is a natural $SO(3)$ equivariant diffeomorphism from $\widetilde{f_1}^{-1}(0)$ to graph$(-id) \times \mu_{n-2}^{-1}(0)$. We want to extend this diffeomorphism $SO(3)$ equivariantly to a neighborhood of $\widetilde{f_1}^{-1}(0)$ and that of graph$(-id) \times \mu_{n-2}^{-1}(0)$. But from Lemma 3.3, it is easy to see that the map ψ induces the isomorphism between the normal bundle of $\widetilde{f_1}^{-1}(0) \subset \mu_n^{-1}(0)$ and that of graph$(-id) \times \mu_{n-2}^{-1}(0) \subset S^2 \times S^2 \times \mu_{n-2}^{-1}(0)$. Then using the equivariant tubular neighborhood Theorem [1], we can obtain the diffeomorphism we need in this Lemma. □

Proof of Lemma 3.1. Locally f_1 is identified with the function $f : S^2 \times S^2 \to \mathbb{R}$ defined by $f(a_1, a_2) = \|a_1 + a_2\|^2$. But it is easy to see that the Hessian of f is nondegenerate on the normal bundle of graph$(-id) \subset S^2 \times S^2$. This proves Lemma 3.1. □

It remains to show that f_1 is perfect.

Lemma 3.4 *f_1 is perfect.*

Proof. Let $\phi = V_0 \subset V_1 \subset V_2 \subset V_3 = M_n$ be the filtration of M_n by the open sets V_j, which are defined by

$$V_0 = f_1^{-1}((-\frac{1}{2}, -\frac{1}{2})),\ V_1 = f_1^{-1}((-\frac{1}{2}, \frac{1}{2})),\ V_2 = f_1^{-1}((-\frac{1}{2}, 2)),$$
$$\text{and } V_3 = f_1^{-1}((-\frac{1}{2}, 5)).$$

This filtration induces the filtration of the cochain complex of M_n

$$\{0\} = C^*(M_n, V_3) \overset{\pi}{\hookrightarrow} C^*(M_n, V_2) \overset{\pi}{\hookrightarrow} C^*(M_n, V_1) \overset{\pi}{\hookrightarrow} C^*(M_n, V_0) = C^*(M_n).$$

Consider the spectral sequence of this filtered complex. As described in [2], there is a short exact sequence

$$0 \to \oplus_j C^*(M_n, V_j) \xrightarrow{\pi} \oplus_j C^*(M_n, V_j) \to \oplus_j C^*(V_j, V_{j-1}) \to 0.$$

This leads to an exact couple

$$\begin{array}{ccc} \oplus_j \mathrm{H}^*(M_n, V_j) & \to & \oplus_j \mathrm{H}^*(M_n, V_j) \\ \nwarrow & & \swarrow \\ & \oplus_j \mathrm{H}^*(V_j, V_{j-1}), & \end{array}$$

whose derived couples abut to $\mathrm{H}^*(M_n)$. On the other hand, using the Morse lemma and the Thom isomorphism theorem, the following isomorphism holds

$$\mathrm{H}^*(V_j, V_{j-1}) \cong \mathrm{H}^{*-\lambda(C_j)}(C_j),$$

where C_j is the critical manifold contained in $V_j \backslash V_{j-1}$. This implies that the spectral sequence of this filtered complex has $\oplus_j \mathrm{H}^*(C_j)$ as E_1-term.

To prove Lemma 3.4, we need to show that all the differentials d_r of this spectral sequence are vanished. Since there are only three non zero terms in the cochain filtration, only the first two differentials d_1 and d_2 can possibly be zero. By the definition, the first differential d_1 is the direct sum of maps $\mathrm{H}^*(C_j) \to \mathrm{H}^*(C_{j+1})$ induced by the upper part of the following diagram.

$$\begin{array}{ccccc} & & \mathrm{H}^k(C_j) & & \\ & & \searrow & & \\ & & \mathrm{H}^{k+\lambda(C_j)}(V_j, V_{j-1}) & & \\ & & \downarrow & & \\ \mathrm{H}^{k+\lambda(C_j)+1}(M_n, V_{j+1}) & \to & \mathrm{H}^{k+\lambda(C_j)+1}(M_n, V_j) & \to & \mathrm{H}^{k+\lambda(C_j)+1}(V_{j+1}, V_j) \\ \downarrow & & & & \searrow \\ \mathrm{H}^{k+\lambda(C_j)+1}(V_{j+2}, V_{j+1}) & & & & \mathrm{H}^{k+\lambda(C_j)+1-\lambda(C_{j+1})}(C_{j+1}) \\ \searrow & & & & \\ \mathrm{H}^{k+\lambda(C_j)+1-\lambda(C_{j+2})}(C_{j+2}) & & & & \end{array}$$

Here all maps are induced by inclusions, except the diagonal arrows, which are Thom isomorphisms.

An element of $\mathrm{H}^*(C_j)$ is in the kernel of the first differential if only if it maps to $0 \in \mathrm{H}^*(V_{j+1}, V_j)$. By the exactness of the middle arrows, it comes from some element of $\mathrm{H}^*(M_n, V_{j+1})$, and its image in $\mathrm{H}^*(C_{j+2})$ is the value of the second differential.

Since we showed in Lemma 2.1 that all the indices $\lambda(C_j)$ are even, from the above diagram, d_1 and d_2 send the odd (resp. even) dimensional cohomology classes to the even (resp. odd) dimensional cohomology classes. But in [8], it has shown that the odd degree cohomologies H^{odd} of the polygon spaces M_n and $M_{n-1}(2, 1, ..., 1)$ vanish. This implies that d_1 and d_2 are vanishing. □

References

[1] M. Audin: *The topology of torus actions on symplectic manifold*, Progress in Mathematics 93, Birkhäuser 1991.

[2] R. Bott, L. W. Tu: *Differential forms in algebraic topology*, Graduate Texts in Mathematics 82, Springer-Verlag 1982.

[3] T. Delzant: *Hamiltoniens periodiques et images convexes de l'application moment*, Bull. Soc. Math. France 116 (1988) N 3, 315–339.

[4] T. Frankel: *Fixed points on Kähler manifolds*, Ann. Math. 70 (1959), 1–8.

[5] W. Fulton: *Introduction to toric varieties*, Ann. of Math. Stud. Vol. 131, Princeton Univ. Press, Princeton, 1993

[6] L. Jeffrey, J. Weitsman: *Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde Dimension formula*, Commun. Math. Phys. 150 (1992), 593–630.

[7] M. Kapovich, J. J. Millson: *The symplectic geometry of polygons in Euclidean space*, J. Diff. Geom. 44 (1996), 479–513.

[8] F. C. Kirwan: *Cohomology of quotients in algebraic and symplectic geometry*, Mathematical Notes 31 Princeton 1984.

[9] D.McDuff, D.Salamon: *Introduction to symplectic topology*, Oxford University Press, 1994.

Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo
153-8914, Japan
e-mail: takahiko@ms.u-tokyo.ac.jp