TWISTED TORIC STRUCTURE

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1. Introduction

By Delzant's classification theorem [3], there is a one-to-one correspondence between a symplectic toric manifold which is one of the special objects in the theory of Hamiltonian torus actions and a Delzant polytope which is a combinatorial object. Through this correspondence, various researches on the relationship between symplectic geometry, transformation groups, topology with combinatorics have been done [2, 3, 4].

On the other hand, there exists a manifold which is locally diffeomorphic to a symplectic toric manifold and whose local structures are patched together in certain sense. In this talk, as a formulation of such manifolds, we shall introduce the notion of twisted toric manifolds and generalize the weak version of Delzant's classification theorem to them. Recently, some generalizations are also considered [7, 8, 9, 10, 11]. In general, a twisted toric manifold no longer has a global torus action like that of a original symplectic toric manifold, but it has a local torus action, i.e. a torus action on a neighborhood of any point and they are patched together in certain sense. One of our motivation is to generalize the topological theory of transformation group to such a "*twisted group action*" case. The content of this talk is a part of the paper [12]. We have no time to describe the topology of twisted toric manifolds in this talk. For the topology of twisted toric manifolds, see [12].

This abstract is organized as follows. In Section 2, we recall Hamiltonian torus action and symplectic toric manifolds. Then we shall give the definition of the twisted toric manifold and some examples in Section 3. Section 4 is devoted to the classification of twisted toric manifolds.

In this talk, all manifolds are assumed to be oriented, and all maps are assumed to preserve orientations, unless otherwise stated. For simplicity, we shall consider only for the four-dimensional case, but all arguments in this talk go well for general even dimensional cases.

2. Symplectic toric manifold

A four-dimensional *symplectic manifold* (X, ω) is a smooth four-dimensional manifold equipped with a non-degenerate closed 2-form ω. Let a k-dimensional torus T^k act on X which preserves ω. In this paper, we identify T^k with $\mathbb{R}^k/\mathbb{Z}^k$. and the Lie algebra $\mathfrak{t}$ of T^k with $\mathbb{R}^k$.

Definition 2.1 ([2, 6])**.** The T^k action is called *Hamiltonian, if there is a moment map which is the map $\mu : X \to \mathbb{R}^k$ satisfying the following two conditions*

(1) $\iota(v_\xi)\omega = d\langle \mu, \xi\rangle$

(2) $\mu(\theta \cdot x) = \mu(x)$

The author is supported by Research Fellowship of the Japan Society for the Promotion of Science for Young Scientists.

for $\xi \in \mathfrak{t}$, $x \in X$, and $\theta \in T^k$, where v_ξ is the infinitesimal action, i.e. the vector field which is defined by

$$v_\xi(x) = \frac{d}{dt}\Big|_{t=0} e^{2\pi\sqrt{-1}t\xi} \cdot x,$$

and $\langle , \rangle$ is the natural inner product on $\mathbb{R}^k$. Note that a moment map of a T^k-action is determined up to additive constant.

Example 2.2. Consider the two-dimensional complex vector space $\mathbb{C}^2$ with the symplectic form $\omega_{\mathbb{C}^2} = -\frac{\sqrt{-1}}{2\pi}\sum_{i=1}^{2} dz_i \wedge d\bar{z}_i$. T^2 acts on $\mathbb{C}^2$ by $\theta \cdot z = (e^{2\pi\sqrt{-1}\theta_i} z_i)$ for $\theta = (\theta_i) \in T^2$ and $z = (z_i) \in \mathbb{C}^2$. This action is Hamiltonian and a moment map $\mu_{\mathbb{C}^n} : \mathbb{C}^2 \to \mathbb{R}^2$ is defined by $\mu_{\mathbb{C}^2}(z) = (|z_i|^2)$.

In the rest of this talk, all manifolds are assumed to be compact and connected. The following fact is fundamental for the theory of Hamiltonian T^k-actions.

Theorem 2.3 ([4])**.** *If T^k acts effectively on a four-dimensional symplectic manifold (X, ω) in a Hamiltonian fashion, then $k \leq 2$.*

In particular, in the maximal case of Theorem 2.3, i.e. $k = 2$, a compact, connected four-dimensional symplectic manifold (X, ω) equipped with an effective Hamiltonian T^2-action is called a four-dimensional *symplectic toric manifold*. Symplectic toric manifolds are classified with its moment map image.

Theorem 2.4 ([3])**.** (1) *The image $\Delta = \mu(X)$ of the moment map of a symplectic toric manifold is Delzant polytope. (For Delzant polytope, see Remark 2.5.)*
(2) *By associating the image of a moment map to a symplectic toric manifold, the set of equivariantly symplectomorphism classes of four-dimensional symplectic toric manifolds corresponds one-to-one to the set of Delzant polytopes in $\mathbb{R}^2$ up to parallel transport in $\mathbb{R}^2$.*

Remark 2.5. (1) In general, it is well known that the image of a moment map of a Hamiltonian torus action on a compact, connected symplectic manifold is the convex hull of the images of the fixed points. See [1, 5].
(2) Let $u_1, \cdots, u_d \in \mathbb{Z}^2$ and $\lambda_1, \cdots, \lambda_d \in \mathbb{R}$. A convex polytope Δ in $\mathbb{R}^2$ defined by

$$\Delta = \{\xi \in \mathbb{R}^2 \colon \langle \xi, u_i \rangle \geq \lambda_i \ (i = 1, \cdots, d)\}$$

is called *Delzant*, if Δ is compact and for each vertex $v \in \Delta$,

(i) v is defined by exactly 2-equalities $\langle v, u_{i_a} \rangle = \lambda_{i_a}$ $(a = 1, 2)$
(ii) $\{u_{i_1}, u_{i_2}\}$ in (i) spans $\mathbb{Z}^2$.

For more details, see [3].
(3) Theorem 2.4 says the symplectic toric manifold X_Δ associated with a Delzant polytope $\Delta \subset \mathbb{R}^2$ is recovered from Δ. As a topological set, X_Δ is obtained as the quotient space $X_\Delta = \Delta \times T^2/\sim$ of the trivial T^2-bundle on Δ by the equivalent relation $\sim$ which is defined as follows. Two elements (ξ, θ) and (ξ', θ') are equivalent, or $(\xi, \theta) \sim (\xi', \theta')$, if and only if $\xi' = \xi$ and

$$\begin{cases} \theta' = \theta & \text{if } \xi \text{ is in the interior of } \Delta \\ \theta' - \theta \in S^1_{u_i} & \text{if } \xi \text{ is in an edge } \{\langle \xi, u_i \rangle = \lambda_i\} \text{ of } \Delta \\ \xi \text{ is a vertex of } \Delta. & \end{cases}$$

where $S^1_{u_i}$ is the sub-circle of T^2 generated by u_i.
(4) A $2n$-dimensional symplectic toric manifold is locally identified with the Hamiltonian T^2-action on $\mathbb{C}^2$ in Example 2.2. More precisely, for any $\xi \in \Delta$, there exist

(i) an automorphism $\rho \in SL_2(\mathbb{Z})$ of T^2
(ii) an open neighborhood U of ξ in Δ which is sent to an open set ${}^t\rho^{-1}(U)$ in the first quadrant $\mathbb{R}^2_{\geq 0} = \{\xi \in \mathbb{R}^2 \colon \xi_i \geq 0 \ i = 1, 2\}$ by ${}^t\rho^{-1}$
(iii) a symplectomorphism $\varphi : \mu^{-1}(U) \to \mu_{\mathbb{C}^2}^{-1}({}^t\rho^{-1}(U))$ which satisfies $\varphi(\theta \cdot x) = \rho(\theta) \cdot \varphi(x)$ for $x \in \mu^{-1}(U)$ and $\theta \in T^2$

such that the following diagram commutes

$$\begin{array}{ccccccccc}
X \supset & & \mu^{-1}(U) & \overset{\varphi}{\cong} & \mu_{\mathbb{C}^2}^{-1}({}^t\rho^{-1}(U)) & & \subset \mathbb{C}^2 \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & \circlearrowright & \downarrow{\scriptstyle \mu_{\mathbb{C}^2}} & & \downarrow{\scriptstyle \mu_{\mathbb{C}^2}} \\
\Delta \supset & & U & \overset{{}^t\rho^{-1}}{\cong} & {}^t\rho^{-1}(U) & & \subset \mathbb{R}^2_{\geq 0}.
\end{array}$$

3. TWISTED TORIC MANIFOLD

By the topological construction, a four-dimensional symplectic toric manifold X_Δ is obtained from the trivial T^2-bundle on the Delzant polytope Δ by collapsing each fiber on the edge $\{\xi \in \Delta \colon \langle u_i, \xi \rangle = \lambda_i\}$ by the circle subgroup $S^1_{u_i}$ generated by u_i. By replacing the trivial T^2-bundle on Δ to a fiber bundle with fiber T^2 on a surface with corners, we can obtain the notion of four-dimensional twisted toric manifolds.

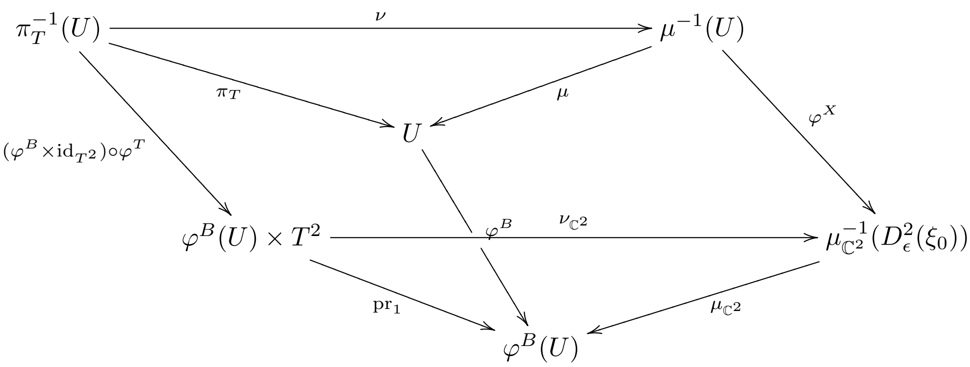
Let B be a surface with corners, $\pi_P : P \to B$ a principal $SL_2(\mathbb{Z})$-bundle, and the associated T^2-bundle and $\mathbb{Z}^2$-bundle of P with respect to the natural action of $SL_2(\mathbb{Z})$ on T^2 and on $\mathbb{Z}^2$ are denoted by $\pi_T : T^2_P \to B$ and $\pi_{\mathbb{Z}} : \mathbb{Z}^2_P \to B$, respectively. Consider a four-dimensional manifold X, surjective maps $\nu : T^2_P \to X$ and $\mu : X \to B$ such that the following diagram is commutative

$$\begin{array}{ccc}
T^2_P & \overset{\nu}{\twoheadrightarrow} & X \\
{\scriptstyle \pi_T}\searrow & \circlearrowright & \swarrow{\scriptstyle \mu} \\
& B &
\end{array}\ .$$

Definition 3.1. The above tuple $\{X, \nu, \mu\}$ is called a four-dimensional *twisted toric manifold associated with the principal* $SL_2(\mathbb{Z})$*-bundle* $\pi_P : P \to B$ (or if there are no confusions, we call simply X a twisted toric manifold), if for arbitrary $b \in B$, there exist

(i) a coordinate neighborhood (U, φ^B) of $b \in B$, i.e. U is an open neighborhood of b in B and φ^B is a diffeomorphism from U to $\mathbb{R}^2_{\geq 0} \cap D^2_\epsilon(\xi_0)$ which sends b to ξ_0, where $D^2_\epsilon(\xi_0) = \{\xi \in \mathbb{R}^2 \colon \|\xi - \xi_0\| < \epsilon\}$
(ii) a local trivialization $\varphi^P : \pi_P^{-1}(U) \cong U \times SL_2(\mathbb{Z})$ of P (then φ^P induces the local trivializations $\varphi^T : \pi_T^{-1}(U) \cong U \times T^2$ and $\varphi^{\mathbb{Z}} : \pi_{\mathbb{Z}}^{-1}(U) \cong U \times \mathbb{Z}^2$ of T^n_P and $\mathbb{Z}^n_P$, respectively)
(iii) a diffeomorphism $\varphi^X : \mu^{-1}(U) \cong \mu_{\mathbb{C}^2}^{-1}(D^2_\epsilon(\xi_0))$

such that the following diagram commutes



where $\mu_{\mathbb{C}^2}$ is the moment map of T^2-action on $\mathbb{C}^2$ in Example 2.2 and $\nu_{\mathbb{C}^2}$ is the map which is defined by

$$\nu_{\mathbb{C}^2}(\xi, \theta) = (\sqrt{\xi_i} e^{2\pi\sqrt{-1}\theta_i}). \tag{3.1}$$

The tuple $(U, \varphi^P, \varphi^X, \varphi^B)$ is called a *locally toric chart*.

Remark 3.2. We do not assume the existence of the symplectic structure on X whose restriction to $\mu^{-1}(U)$ is equal to $(\varphi^X)^*\omega_{\mathbb{C}^2}$. This condition may be too strong. See [12].

Example 3.3 (Torus bundle)**.** Let $\pi_P : P \to B$ be a principal $SL_2(\mathbb{Z})$-bundle on a closed surface B. Then the associated T^2-bundle $\pi_T : T_P^2 \to B$ itself is an example of a twisted toric manifold associated with $\pi_P : P \to B$. In particular, the four-dimensional torus T^4 is a twisted toric manifold, which is a T^2-bundle on T^2.

Example 3.4 (Symplectic toric manifold)**.** A four-dimensional symplectic toric manifold X with Delzant polytope Δ has a twisted toric structure associated with the trivial $SL_2(\mathbb{Z})$-bundle on Δ.

Example 3.5. $S^1 \times S^3$ has a twisted toric structure associated with the trivial $SL_2(\mathbb{Z})$-bundle on a two-dimensional unit disc. For more details, see [12].

Example 3.6. The four-dimensional sphere S^4 has a twisted toric structure associated with the trivial $SL_2(\mathbb{Z})$-bundle on a two-dimensional disc with two corner points (the shape of leaf). For more details, see [12].

We also have another interesting examples associated with non trivial $SL_2(\mathbb{Z})$-bundles on a surface with corners, see [12] for more details.

4. Classification

In this section, we shall prove the classification theorem for twisted toric manifolds. Let $\mathcal{S}^{(0)}B$ be the set of corner points, $\mathcal{S}^{(1)}B = \partial B \backslash \mathcal{S}^{(0)}B$, and $\mathcal{S}^{(2)}B = B \backslash \partial B$. $\{\mathcal{S}^{(k)}B\}$ defines the natural stratification of B. Let $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(1)}B$ be a rank one sub-lattice bundle of the restriction $\pi_{\mathbb{Z}} : \mathbb{Z}_P^2\big|_{\mathcal{S}^{(1)}B} \to \mathcal{S}^{(1)}B$ of the lattice bundle $\pi_{\mathbb{Z}} : \mathbb{Z}_P^2 \to B$.

Definition 4.1. $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(1)}B$ is **primitive**, if for arbitrary $b \in B$ included in $\mathcal{S}^{(k)}B$, there exist

(i) $U(\subset B)$: a locally toric chart whose intersection $U \cap \mathcal{S}^{(1)}B$ with $\mathcal{S}^{(1)}B$ has exactly $2-k$ connected components $(U \cap \mathcal{S}^{(1)}B)_1, \cdots, (U \cap \mathcal{S}^{(1)}B)_{2-k}$

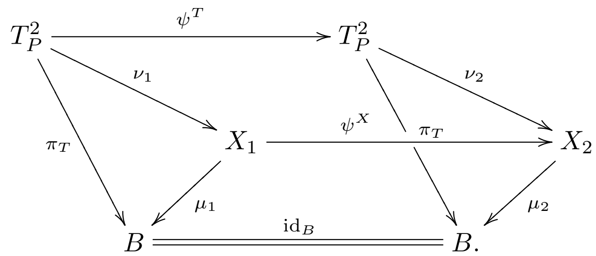
(ii) $\{u_1, \cdots, u_{2-k}\} \subset \mathbb{Z}^2$: a *primitive* tuple of vectors, i.e. they generate over $\mathbb{Z}$ a rank $(2-k)$ direct summand of $\mathbb{Z}^2$

such that for $j = 1, \cdots, 2-k$, the following diagram is commutative

$$\begin{array}{ccc}
\mathbb{Z}^2_P\big|_U & \overset{\varphi^{\mathbb{Z}}}{\cong} & U \times \mathbb{Z}^2 \\
\cup & \circlearrowleft & \cup \\
\mathbb{Z}^2_P\big|_{\left(U\cap\mathcal{S}^{(1)}B\right)_j} & \cong & \left(U \cap \mathcal{S}^{(1)}B\right)_j \times \mathbb{Z}^2 \\
\cup & \circlearrowleft & \cup \\
\mathcal{L}\big|_{\left(U\cap\mathcal{S}^{(1)}B\right)_j} & \cong & \left(U \cap \mathcal{S}^{(1)}B\right)_j \times \mathbb{Z}u_j.
\end{array}$$

Remark 4.2. (1) Definition 4.1 does not depend on the choice of a locally toric chart U, since the notion of primitivity is invariant under the action of $SL_2(\mathbb{Z})$.
(2) The automorphism group $\mathrm{Aut}(P)$ of P acts on the set of primitive rank one sub-lattice bundles of $\pi_{\mathbb{Z}} : \mathbb{Z}^2_P\big|_{\mathcal{S}^{(1)}B} \to \mathcal{S}^{(1)}B$ as the automorphisms of the restriction of the associated lattice bundle $\pi_{\mathbb{Z}} : \mathbb{Z}^2_P \to B$ to $\mathcal{S}^{(1)}B$.

Definition 4.3. Two twisted toric manifolds $\{X_1, \nu_1, \mu_1\}$ and $\{X_2, \nu_2, \mu_2\}$ associated with $\pi_P : P \to B$ are *isomorphic*, if there exist an automorphism ψ^P of $\pi_P : P \to B$ which covers identity map of B (then ψ^P induces the automorphism ψ^T of $\pi_T : T^2_P \to B$), and a diffeomorphism ψ^X from X_1 to X_2 such that the following diagram commutes



Fix a principal $SL_2(\mathbb{Z})$-bundle on a surface B with corners.

Theorem 4.4 ([12])**.** *(1) For any twisted toric manifold $\{X, \nu, \mu\}$ associated with $\pi_P : P \to B$, there is the primitive rank one sub-lattice bundle $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(1)}B$ of $\pi_{\mathbb{Z}} : \mathbb{Z}^2_P\big|_{\mathcal{S}^{(1)}B} \to \mathcal{S}^{(1)}B$ which is determined uniquely by $\{X, \nu, \mu\}$. $\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{S}^{(1)}B$ is called a characteristic bundle of $\{X, \nu, \mu\}$.*
(2) By associating the characteristic bundle to a twisted toric manifold, the set of isomorphism classes of twisted toric manifolds associated with $\pi_P : P \to B$ corresponds one-to-one to the set of equivalent classes of primitive rank one sub-lattice bundles on $\mathcal{S}^{(1)}B$ of $\mathbb{Z}^2_P\big|_{\mathcal{S}^{(1)}B}$ by the action of the automorphism group of P.

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