ADIABATIC LIMITS, THETA FUNCTIONS, AND GEOMETRIC QUANTIZATION

TAKAHIKO YOSHIDA

Abstract. Let $\pi\colon (M,\omega)\to B$ be a (non-singular) Lagrangian torus fibration on a compact, complete base B with prequantum line bundle $(L,\nabla^L)\to(M,\omega)$. For a positive integer N and a compatible almost complex structure J on (M,ω) invariant along the fiber of π, let D be the associated Spin^c Dirac operator with coefficients in $L^{\otimes N}$. Then, we give an orthogonal family $\{\widetilde{\vartheta}_b\}_{b\in B_{BS}}$ of sections of $L^{\otimes N}$ indexed by the Bohr-Sommerfeld points B_{BS}, and show that each $\widetilde{\vartheta}_b$ converges to a delta-function section supported on the corresponding Bohr-Sommerfeld fiber $\pi^{-1}(b)$ and the L^2-norm of $D\widetilde{\vartheta}_b$ converges to 0 by the adiabatic(-type) limit. Moreover, if J is integrable, we also give an orthogonal basis $\{\vartheta_b\}_{b\in B_{BS}}$ of the space of holomorphic sections of $L^{\otimes N}$ indexed by B_{BS}, and show that each ϑ_b converges to a delta-function section supported on the corresponding Bohr-Sommerfeld fiber $\pi^{-1}(b)$ by the adiabatic(-type) limit. We also explain the relation of ϑ_b with Jacobi's theta functions when (M,ω) is T^{2n}.

Contents

1. Motivation and Main Theorems

The purpose of this paper is to show the Spin^c quantization converges to the real quantization by the adiabatic(-type) limit for a Lagrangian torus fibration on a compact complete base and a compatible almost complex structures on its total space which is invariant along the fiber. In this paper, a Lagrangian torus fibration is assumed to be non-singular unless otherwise stated. First let us explain the motivation which comes from geometric quantization. For geometric quantization, see [17, 22, 32, 38]. In physics, quantization is the procedure for building quantum mechanics starting from classical mechanics. In the mathematical context, it is often thought of as a representation of the Poisson algebra consisting of certain functions on a symplectic manifold to some Hilbert space, so called the quantum Hilbert space, and the geometric quantization gives us the method to construct a quantum Hilbert space and a

2010 *Mathematics Subject Classification.* Primary 53D50; Secondary 58H15, 58J05.
Key words and phrases. adiabatic limit, Theta function, Lagrangian fibration, geometric quantization.

representation from a given symplectic manifold (M,ω) and a prequantum line bundle $(L,\nabla^L)\to(M,\omega)$ in the geometric way. In the theory of geometric quantization by Kostant and Souriau [25, 34, 33], we need an additional structure which is called a polarization to obtain the quantum Hilbert space. By definition, a polarization is an integrable Lagrangian distribution P of the complexified tangent bundle $TM\otimes\mathbb{C}$ of (M,ω). For a polarization P, the quantum Hilbert space is naively given as the closure of the space of smooth, square-integrable sections of (L,∇^L) which are covariant constant along P.

A common example is the Kähler polarization. When (M,ω) is Kähler and (L,∇^L) is a holomorphic line bundle with canonical connection, we can take $T^{0,1}M$ as a polarization, and the obtained quantum Hilbert space is nothing but the space of holomorphic sections $H^0(M,\mathcal{O}_L)$. This polarization is called the Kähler polarization and the quantization procedure is called the Kähler quantization. Note that when M is compact and the Kodaira vanishing theorem holds, its dimension is equal to the index of the Dolbeault operator with coefficients in L.

Another example is a real polarization. Suppose (M,ω) admits a structure of a Lagrangian torus fibration $\pi\colon(M,\omega)\to B$. For each point $b\in B$ of the base manifold B, the restriction $\left(L,\nabla^L\right)|_{\pi^{-1}(b)}$ of (L,∇^L) to the fiber $\pi^{-1}(b)$ is a flat line bundle. Let $H^0\left(\pi^{-1}(b);\left(L,\nabla^L\right)|_{\pi^{-1}(b)}\right)$ be the space of covariant constant sections of $\left(L,\nabla^L\right)|_{\pi^{-1}(b)}$. Then, an element $b\in B$ is said to be Bohr-Sommerfeld if $H^0\left(\pi^{-1}(b);\left(L,\nabla^L\right)|_{\pi^{-1}(b)}\right)\neq\{0\}$. It is well-known that Bohr-Sommerfeld points appear discretely. In this case, we can take $T_\pi M\otimes\mathbb{C}$, the complexified tangent bundle along fibers of π as a polarization, and if M is compact, the quantum Hilbert space is defined by $\oplus_{b\in B_{BS}}H^0\left(\pi^{-1}(b);\left(L,\nabla^L\right)|_{\pi^{-1}(b)}\right)$, where the sum is taken over all Bohr-Sommerfeld points. See [32] for more details. In this paper, we call this quantization the real quantization.

When a Lagrangian torus fibration $\pi\colon(M,\omega)\to B$ with closed total space M and a prequantum line bundle $(L,\nabla^L)\to(M,\omega)$ are given, it is natural to ask whether the quantum Hilbert space obtained by the Kähler quantization is isomorphic to the one obtained by the real quantization. A completely integrable system can be thought of as a Lagrangian fibration with singular fibers. In the case of the the moment map of a projective toric variety, Danilov has shown in [9] that $H^0(M,\mathcal{O}_L)$ has the irreducible decomposition $H^0(M,\mathcal{O}_L)=\oplus_{m\in\Delta\cap\mathfrak{t}^*_{\mathbb{Z}}}\mathbb{C}_m$ as a compact torus representation, where Δ is the moment polytope, $\mathfrak{t}^*_{\mathbb{Z}}$ is the weight lattice, and $\mathbb{C}_m$ is the irreducible representation of the torus with weight m. Since $\Delta\cap\mathfrak{t}^*_{\mathbb{Z}}$ is identified with the set of Bohr-Sommerfeld points, this implies the dimensions of the quantum Hilbert spaces obtained by the above quantizations agree. A similar equality of the dimensions has been also shown for the Gelfand-Cetlin system on the flag variety [16], the Goldman system on the moduli space of flat $SU(2)$ connections on a surface [20], and the Kapovich-Millson system on the polygon space. [21].

Moreover, in the case of smooth projective toric varieties, not only the numerical equality for the dimensions, but also a geometric correspondence between the Kähler and the real quantizations has been shown concretely by Baier-Florentino-Mourão-Nunes in [3]. Namely, they have given a one-parameter family of complex structures $\{J_t\}_{t\in[0,\infty)}$ and a basis $\{s_t^m\}_{m\in\Delta\cap\mathfrak{t}^*_{\mathbb{Z}}}$ of the space of holomorphic sections associated with the complex structure J_t for each t such that each section s_t^m converges to the delta function section supported on the corresponding Bohr-Sommerfeld fiber as t goes to ∞. The similar result has been also obtained for flag manifolds in [19] and smooth irreducible complex algebraic varieties by [18]. But in [19] and [18] the convergence has been shown only for the non-singular Bohr-Sommerfeld fibers whereas in [3] it has been shown for all Bohr-Sommerfeld fibers.

The Kähler quantization can be generalized to a non-integrable compatible almost complex structure on a closed (M,ω). When a compatible almost complex structure J on (M,ω) is given, we can consider the associated Spin^c Dirac operator D acting on $\Gamma\left(\wedge^\bullet T^*M^{0,1}\otimes L\right)$. It is well-known that D is a formally self-adjoint, first order, elliptic differential operator of degree-one, and if J is integrable, D agrees with the Dolbeault operator up to constant. If J is not integrable, $T^{0,1}M$ is no more polarization. But, even in this case, since D is Fredholm, we can still take the element of the K-theory of a point

$$\ker\left(D|_{\wedge^{even}T^*M^{0,1}\otimes L}\right)-\ker\left(D|_{\wedge^{odd}T^*M^{0,1}\otimes L}\right)\in K(pt) \tag{1.1}$$

as a (virtual) quantum Hilbert space. Its virtual dimension is equal to the index of D. We call this quantization the Spin^c quantization. It has been shown in [1, 13, 26] that the above equality between dimensions of two quantum Hilbert spaces still holds by replacing the Kähler quantization with the Spin^c quantization by using the index theory.

In this paper, we generalize the approach taken in [3] for the Kähler quantization to the Spin^c quantization of Lagrangian torus fibrations. Let $\pi\colon (M,\omega)\to B$ be a Lagrangian torus fibration on a compact, complete base B with prequantum line bundle $(L,\nabla^L)\to(M,\omega)$. For a positive integer N and a compatible almost complex structure J of (M,ω) invariant along the fiber of π (in the sense of Lemma 3.6), let D be the associated Spin^c Dirac operator with coefficients in $L^{\otimes N}$. Then, the main result is as follows. This theorem is a combination of Theorem 5.2 and Theorem 5.3.

Theorem 1.1. *For the above data, we give one-parameter families of*

- $\{J^t\}_{t>0}$ *compatible almost complex structures of* (M,ω) *with* $J^1=J$
- $\{\widetilde{\vartheta}_b^t\}_{b\in B_{BS}}$ *sets of sections on* $L^{\otimes N}$ *indexed by the Bohr-Sommerfeld points* B_{BS}

such that

1. *any pair in* $\{\widetilde{\vartheta}_b^t\}_{b\in B_{BS}}$ *are orthogonal to each other,*
2. *each* $\widetilde{\vartheta}_b^t$ *converges as a delta-function section supported on* $\pi^{-1}(b)$ *as* $t\to\infty$ *in the following sense, for any section* s *of* $L^{\otimes N}$,
$$\lim_{t\to\infty}\int_M\left\langle s,\frac{\widetilde{\vartheta}_b^t}{\|\widetilde{\vartheta}_b^t\|_{L^1}}\right\rangle_{L^{\otimes N}}(-1)^{\frac{n(n-1)}{2}}\frac{\omega^n}{n!}=\int_{\pi^{-1}(b)}\langle s,\delta_b\rangle_{L^{\otimes N}}\,|dy|,$$
where $\langle\,,\,\rangle_{L^{\otimes N}}$ *is the Hermitian metric of* $L^{\otimes N}$, δ_b *is the covariant constant section of* $L^{\otimes N}|_{\pi^{-1}(b)}$ *defined by* (4.9), *and* $|dy|$ *is the natural one-density on* $\pi^{-1}(b)$,
3. $\lim_{t\to\infty}\|D^t\widetilde{\vartheta}_b^t\|_{L^2}=0$.

By the Spin^c Dirac vanishing theorem due to Borthwick-Uribe [7], $\ker\left(D|_{\wedge^{odd}T^*M^{0,1}\otimes L^{\otimes N}}\right)$ vanishes for a sufficiently large N. So, (3) implies the the complex vector space spanned by $\{\widetilde{\vartheta}_b^t\}_{b\in B_{BS}}$ approximates the quantum Hilbert space of the Spin^c quantization for a sufficiently large N.

If J is integrable, we also give the following refinement of Theorem 1.1, which is immediately obtained by putting Corollary 4.3 and Theorem 4.12 together.

Theorem 1.2. *Under the above setting, assume* J *is integrable. Then, with a technical assumption, we give one-parameter families of*

- $\{J^t\}_{t>0}$ *compatible complex structures of* (M,ω) *with* $J^1=J$
- $\{\vartheta_b^t\}_{b\in B_{BS}}$ *orthogonal bases of holomorphic sections of* $L^{\otimes N}\to(M,N\omega,J^t)$ *indexed by* B_{BS}

such that each ϑ_b^t *converges as a delta-function section supported on* $\pi^{-1}(b)$ *as* $t\to\infty$ *in the following sense, for any section* s *of* $L^{\otimes N}$,
$$\lim_{t\to\infty}\int_M\left\langle s,\frac{\vartheta_b^t}{\|\vartheta_b^t\|_{L^1}}\right\rangle_{L^{\otimes N}}(-1)^{\frac{n(n-1)}{2}}\frac{\omega^n}{n!}=\int_{\pi^{-1}(b)}\langle s,\delta_b\rangle_{L^{\otimes N}}\,|dy|.$$

One of examples of the total space of a Lagrangian torus fibration with complete base is an abelian variety. In this case, we show that each ϑ_b coincides with Jacobi's theta functions up to function on the base space (Theorem 4.7). For the theta functions, see [28, 29].

We should remark there are several works which deal with theta functions from the viewpoint of geometric quantization of Lagrangian fibrations, for example, [29], [4], [30, 31]. In [7], Borthwick-Uribe have introduced another approach to generalize the Kähler quantization to non-integrable almost complex structures by using the metric Laplacian of the connection on the prequantum line bundle instead of Spin^c Dirac operator. Their approach is called the almost Kähler quantization. In the almost Kähler quantization of the Kodaira-Thurston manifold, Kirwin-Uribe and Egorov have constructed an analog of the theta function as an element of the quantum Hilbert space [23], [12]. In [11], Egorov has also shown the similar result for Lagrangian T^2-fibrations on T^2 with zero Euler class.

The idea used in this paper is quite simple. One of two key facts is Corollary 2.25 which claims that any Lagrangian torus fibration $\pi\colon(M,\omega)\to B$ with complete base B and a prequantum line bundle $\left(L,\nabla^L\right)\to(M,\omega)$ can be obtained as the quotient of a $\pi_1(B)$-action on the standard Lagrangian fibration $\left(\widetilde{M},\omega_0\right):=(\mathbb{R}^n\times T^n,\sum_{i=1}^n dx_i\wedge dy_i)\to\mathbb{R}^n$ with the standard prequantum line bundle. In particular, any compatible almost complex structure on (M,ω) is induced from a $\pi_1(B)$-equivariant one on $\left(\widetilde{M},\omega_0\right)$, and the set of compatible almost complex structures on $\left(\widetilde{M}, \omega_0\right)$ corresponds one-to-one to the set of smooth maps from $\widetilde{M}$ to the Siegel upper half space. We show that there exists a $\pi_1(B)$-invariant compatible almost complex structure J whose corresponding map is invariant along the fiber (Lemma 3.6). For the Spin^c Dirac operator D associated with such an almost complex structure J, we consider the problem on the existence of non-trivial degree-zero harmonic spinors, i.e., sections of $\widetilde{L}^{\otimes N}$ contained in the kernel of D. By taking the Fourier series expansion of a section s of $\widetilde{L}^{\otimes N}$ with respect to the fiber coordinates, the equation $Ds = 0$ can be reduced to a system of partial differential equations on $\mathbb{R}^n$. The other key fact is Proposition 3.12 in which we give a necessary and sufficient condition in order that the system of partial differential equations has non-trivial solutions and show that it is equal to the integrability condition for J, i.e., $\left(\widetilde{M}, \omega_0, J\right)$ is Kähler. Moreover, in this case, we give a family of $\pi_1(B)$-equivariant solutions of $Ds = 0$ indexed by the Bohr-Sommerfeld points, each of which is expressed by the formal Fourier series. If they converge absolutely and uniformly, this gives a linear basis of the space of holomorphic sections of $\left(L, \nabla^L\right)^{\otimes N} \to (M, N\omega, J) \to B$. We also give a sufficient condition for their convergence. Even if J is not integrable, by considering an approximation of D, we can obtain an orthogonal family of sections of $L^{\otimes N}$ indexed by the Bohr-Sommerfeld points B_{BS}. The limit used in this paper is slightly different from the adiabatic limit in Riemannian geometry. When a fiber bundle $\pi \colon M \to B$ and a Riemannian metric g on M are given, we can consider the decomposition $(TM, g) = (V, g_V) \oplus (H, g_H)$, where V is the tangent bundle along the fiber with fiber metric $g_V := g|_V$ and H is the orthogonal complement of V with respect to g with fiber metric $g_H := g|_H$. For each $t > 0$, define the Riemannian metric g^t to be $g^t := g_V \oplus t g_H$. Then, in Riemannian geometry, the adiabatic limit is the procedure for taking the limit of geometric objects depending on g^t as $t \to \infty$. But, since such a deformation of Riemannian metrics does not fit into our symplectic context, we modify the deformation. Namely, in this paper, we use a one-parameter family $\{J^t\}_{t>0}$ of compatible almost complex structures on (M, ω) such that the corresponding one-parameter family of Riemannian metrics is $\{g^t = \frac{1}{t} g_V \oplus t g_H\}_{t>0}$, and investigate the behavior of $\widetilde{\vartheta}_b^t$ (resp. ϑ_b^t) when t goes to ∞.

The paper is organized as follows. In section 2, we first briefly review some well-known facts about integral affine geometry and Lagrangian fibrations. Then, by using these, we prove Corollary 2.25. In Section 3, we discuss the $\pi_1(B)$-equivariant Spin^c quantization of $(\mathbb{R}^n \times T^n, \sum_{i=1}^n dx_i \wedge dy_i) \to \mathbb{R}^n$ with the standard prequantum line bundle and prove Proposition 3.12. In Section 4, we prove Theorem 1.2 step by step, and explain the relation between ϑ_b^t and Jacobi's classical theta function. Finally, in Section 5 we prove Theorem 1.1.

Acknowledgment The most part of this work had been done while the author stayed in McMaster university. The author would like to thank the department of Mathematics and Statistics, McMaster university and especially Megumi Harada for their hospitality. This work is supported by Grant-in-Aid for Scientific Research (C) 15K04857.

1.1. **Notations.** For $x = {}^t(x_1, \dots, x_n)$ and $y = {}^t(y_1, \dots, y_n) \in \mathbb{R}^n$, let us denote the standard inner product $\sum_{i=1}^n x_i y_i$ by $x \cdot y$. ∂_{x_i} denotes $\dfrac{\partial}{\partial x_i}$. In this paper, all manifolds and maps are supposed to be smooth.

2. Developing Lagrangian fibrations

2.1. **Integral affine structures.** Let B be a manifold.

Definition 2.1. An **integral affine atlas** of B is an atlas $\{(U_\alpha, \phi_\alpha)\}$ of B on each of whose non-empty overlap $U_{\alpha\beta}$, the transition function $\phi_\alpha \circ \phi_\beta^{-1}$ is an integral affine transformation, namely, $\phi_\alpha \circ \phi_\beta^{-1}$ is of the form $\phi_\alpha \circ \phi_\beta^{-1}(x) = A_{\alpha\beta} x + c_{\alpha\beta}$ for some locally constant maps $A_{\alpha\beta} \colon U_{\alpha\beta} \to \mathrm{GL}_n(\mathbb{Z})$ and $c_{\alpha\beta} \colon U_{\alpha\beta} \to \mathbb{R}^n$. Two integral affine atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(U'_\beta, \phi'_\beta)\}$ of B are said to be **equivalent** if on each non-empty overlap $U_\alpha \cap U'_\beta$, the transition function $\phi_\alpha \circ (\phi'_\beta)^{-1}$ is an integral affine transformation. An **integral affine structure** on B is an equivalence class of integral affine atlases of B. A manifold equipped with integral affine structure is called an **integral affine manifold**.

Example 2.2. An n-dimensional Euclidean space $\mathbb{R}^n$ is equipped with a natural integral affine structure.

Let us give examples of integral affine manifolds obtained from integral affine actions on $\mathbb{R}^n$.

Example 2.3. (1) Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be a linear basis of $\mathbb{R}^n$ and $C = (v_1 \ \cdots \ v_n) \in \mathrm{GL}_n(\mathbb{R})$ the matrix whose ith column vector is v_i for $i = 1, \ldots, n$. $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ by

$$\rho_\gamma(x) := x + C\gamma$$

for $\gamma \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$. Since the action preserves the natural integral affine structure on $\mathbb{R}^n$, the quotient space, which is topologically T^n, is equipped with an integral affine structure.
(2) Let $\lambda \in \mathbb{N}$ be a positive integer and $a, b \in \mathbb{R}_{>0}$ positive real numbers. Define $\mathbb{Z}^2$-action on $\mathbb{R}^2$ as follows. First, for the standard basis e_1, e_2 of $\mathbb{Z}^2$, let us define the integral affine transform ρ_{e_1}, ρ_{e_2} by

$$\rho_{e_1}(x) := x + \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad \rho_{e_2}(x) := \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ b \end{pmatrix}$$

for $x \in \mathbb{R}^2$. Since ρ_{e_1} and ρ_{e_2} are commutative, they form the $\mathbb{Z}^2$-action on $\mathbb{R}^2$ by

$$\rho_\gamma(x) := \rho_{e_1}^{\gamma_1} \circ \rho_{e_2}^{\gamma_2}(x)$$

for each $\gamma = {}^t(\gamma_1, \gamma_2) \in \mathbb{Z}^2$. In the same manner as in (1), the quotient space is equipped with an integral affine structure. It is shown in [27, Theorem A] that the quotient space is topologically T^2, but the induced integral affine structure is not isomorphic to that obtained in (1) for $n = 2$ and there are only these two integral affine structures on T^2 up to isomorphism.

Example 2.4. For $\gamma = {}^t(\gamma_1, \gamma_2, \gamma_3), \gamma' = {}^t(\gamma_1', \gamma_2', \gamma_3') \in \mathbb{Z}^3$ define the product $\gamma \circ \gamma' \in \mathbb{Z}^3$ by

$$\gamma \circ \gamma' := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}^{\gamma_1} \gamma' + \gamma.$$

Then, $\mathbb{Z}^3$ with product $\circ$ is a non abelian group $\left(\mathbb{Z}^3, \circ\right)$. $\left(\mathbb{Z}^3, \circ\right)$ acts on $\mathbb{R}^3$ by

$$\rho_\gamma(x) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}^{\gamma_1} x + \gamma.$$

Then, the quotient space $\mathbb{R}^3 / \left(\mathbb{Z}^3, \circ\right)$ is equipped with the integral affine structure induced from that of $\mathbb{R}^3$.

Example 2.5. Let $n \geq 2$. For $\gamma = {}^t(\gamma_1, \ldots, \gamma_n), \gamma' = {}^t(\gamma_1', \ldots, \gamma_n') \in \mathbb{Z}^n$ define the product $\gamma \circ \gamma' \in \mathbb{Z}^n$ by

$$\gamma \circ \gamma' := \begin{pmatrix} 1 & & & \\ & (-1)^{\gamma_1} & & \\ & & \ddots & \\ & & & (-1)^{\gamma_{n-1}} \end{pmatrix} \gamma' + \gamma.$$

Then, $\mathbb{Z}^n$ with product $\circ$ is a non abelian group $(\mathbb{Z}^n, \circ)$. $(\mathbb{Z}^n, \circ)$ acts on $\mathbb{R}^n$ by

$$\rho_\gamma(x) := \begin{pmatrix} 1 & & & \\ & (-1)^{\gamma_1} & & \\ & & \ddots & \\ & & & (-1)^{\gamma_{n-1}} \end{pmatrix} x + \gamma.$$

Then, the quotient space $\mathbb{R}^n / (\mathbb{Z}^n, \circ)$ is equipped with the integral affine structure induced from that of $\mathbb{R}^n$. For $n = 2$, the quotient space is topologically a Klein bottle.

Example 2.6. Let $n \geq 2$ and $\lambda_1, \ldots \lambda_{n-1} \in \mathbb{Z}$. For $\gamma = {}^t(\gamma_1, \ldots, \gamma_n), \gamma' = {}^t(\gamma_1', \ldots, \gamma_n') \in \mathbb{Z}^n$ define the product $\gamma \circ \gamma' \in \mathbb{Z}^n$ by

$$\gamma \circ \gamma' := \begin{pmatrix} 1 & \lambda_1 & & & \\ & 1 & \lambda_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda_{n-1} \\ & & & & 1 \end{pmatrix}^{\gamma_n} \gamma' + \gamma.$$

$\mathbb{Z}^n$ with product $\circ$ is a group $(\mathbb{Z}^n, \circ)$, which is non abelian for $n \geq 3$. $(\mathbb{Z}^n, \circ)$ acts on $\mathbb{R}^n$ by

$$\rho_\gamma(x) := \begin{pmatrix} 1 & \lambda_1 & & & \\ & 1 & \lambda_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda_{n-1} \\ & & & & 1 \end{pmatrix}^{\gamma_n} x + \gamma.$$

Then, the quotient space $\mathbb{R}^n / (\mathbb{Z}^n, \circ)$ is equipped with the integral affine structure induced from that of $\mathbb{R}^n$. In the case where $n = 2$ and $\lambda_1 > 0$, it coincides with the one given in Example 2.3 (2) with $a = b = 1$.

Example 2.7. Let $\mathbb{Z}/4\mathbb{Z} \cong \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$ act on $(\mathbb{R}^2)^n \smallsetminus \{0\}$ naturally. Then, the quotient space is a non-compact manifold and equipped with the integral affine structure induced from that of $(\mathbb{R}^2)^n \smallsetminus \{0\}$.

As we can guess from above examples, every integral affine manifold is obtained from a group action. Let B be an n-dimensional connected integral affine manifold, $p\colon \widetilde{B} \to B$ the universal covering of B. It is clear that $\widetilde{B}$ is also equipped with the integral affine structure so that p is an integral affine map. We set $\Gamma := \pi_1(B)$. Γ acts on $\widetilde{B}$ from the right as a deck transformation. For each $\gamma \in \Gamma$ we denote by σ_γ the inverse of the deck transformation corresponding to γ. Then, $\sigma\colon \gamma \mapsto \sigma_\gamma$ defines a left action $\sigma \in \mathrm{Hom}\left(\Gamma, \mathrm{Aut}(\widetilde{B})\right)$.

We assume that all the actions considered in this paper are left actions unless otherwise stated.

The following proposition is well known in affine geometry.

Proposition 2.8. *There exists an integral affine immersion* $\mathrm{dev}\colon \widetilde{B} \to \mathbb{R}^n$ *and a homomorphism* $\rho\colon \Gamma \to GL_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ *such that the image of* dev *is an open set of* $\mathbb{R}^n$ *and* dev *is equivariant with respect to* σ *and* ρ. *Such an integral affine immersion is unique up to the composition of an integral affine transformation on* $\mathbb{R}^n$.

See [15, p.641] for a proof. We will prove a version of this proposition (Proposition 2.22) when B is equipped with a Lagrangian fibration on it in Section 2.

Proposition 2.9. *Let* B, $p\colon \widetilde{B} \to B$, $\mathrm{dev}\colon \widetilde{B} \to \mathbb{R}^n$, *and* $\rho\colon \Gamma \to GL_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ *be as in Proposition 2.8. Suppose that* B *is compact and the* Γ*-action* ρ *on* $\mathbb{R}^n$ *is properly discontinuous. Then,* dev *is surjective.*

Proof. We denote the image of dev by O. By proposition 2.8, O is an open set in $\mathbb{R}^n$. So, it is sufficient to show that O is closed in $\mathbb{R}^n$. Since the Γ-action ρ on $\mathbb{R}^n$ is properly discontinuous, the quotient space $\mathbb{R}^n/\Gamma$ becomes a Hausdorff space and the natural projection $q\colon \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ is continuous. O is preserved by the Γ-action ρ on $\mathbb{R}^n$ since dev is Γ-equivariant. Then, dev induces a continuous surjective map $\overline{\mathrm{dev}}\colon B = \widetilde{B}/\Gamma \to O/\Gamma$. Since B is compact, O/Γ is a compact subset in the Hausdorff space $\mathbb{R}^n/\Gamma$. In particular, it is also closed. Hence, $O = q^{-1}(O/\Gamma)$ is also closed in $\mathbb{R}^n$. □

Corollary 2.10. *Let* B, $p\colon \widetilde{B} \to B$, $\mathrm{dev}\colon \widetilde{B} \to \mathbb{R}^n$, *and* $\rho\colon \Gamma \to GL_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ *be as in Proposition 2.8 and assume that* B *compact. If the image of* ρ *lies in* $(\mathrm{GL}_n(\mathbb{Z}) \cap O(n)) \ltimes \mathbb{R}^n$ *and the subgroup* $\rho(\Gamma)$ *of* $(\mathrm{GL}_n(\mathbb{Z}) \cap O(n)) \ltimes \mathbb{R}^n$ *is discrete, then,* dev *is surjective.*

Proof. It follows from [37, Theorem 3.1.3]. □

Definition 2.11. The integral affine immersion dev is called a **developing map**. B is said to be **complete** if dev is bijective. B is said to be **incomplete** if B is not complete.

Example 2.12. All the above examples are complete other than Example 2.7 for $n \geq 2$.

Example 2.13. Let B be an n-dimensional compact integral affine manifold B with integral affine atlas $\{(U_\alpha, \phi_\alpha)\}$ as in Definition 2.1. If on each non-empty overlap $U_{\alpha\beta}$, the Jacobi matrix of the coordinate changing map $D\left(\phi_\alpha \circ \phi_\beta^{-1}\right)$ lies in $GL_n(\mathbb{Z}) \cap O(n)$, then, B has a flat Riemannian metric. Hence, by Bieberbach's theorem [5, 6], B is finitely covered by T^n. In particular, B is complete. For flat Riemannian manifolds, see [37, Chapter 3].

2.2. **Lagrangian fibrations.** In this section let us recall Lagrangian fibrations and explain how integral affine structures are associated with Lagrangian fibrations. After that let us recall their classification by Duistermaat. For more details, see [10, 39].

Let (M, ω) be a symplectic manifold.

Definition 2.14. A map $\pi\colon (M, \omega) \to B$ from (M, ω) to a manifold B is called a *Lagrangian fibration* if π is a fiber bundle whose fiber is a Lagrangian submanifold of (M, ω).

Example 2.15. Let $T^n = (\mathbb{R}/\mathbb{Z})^n$ be an n-dimensional torus. $\mathbb{R}^n \times T^n$ admits a standard symplectic structure $\omega_0 = \sum_i dx_i \wedge dy_i$, where $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ are the coordinates of $\mathbb{R}^n$, T^n, respectively. Then, the projection $\pi_0\colon (\mathbb{R}^n \times T^n, \omega_0) \to \mathbb{R}^n$ to $\mathbb{R}^n$ is a Lagrangian fibration.

The following theorem shows that Example 2.15 is a local model of Lagrangian fibration.

Theorem 2.16 (Arnold-Liouville's theorem [2])**.** *Let $\pi\colon (M, \omega) \to B$ be a Lagrangian fibration with compact, path-connected fibers. Then, for each $b \in B$, there exists a chart (U, ϕ) containing b and a symplectomorphism $\varphi\colon (\pi^{-1}(U),\, \omega|_{\pi^{-1}(U)}) \to (\phi(U) \times T^n, \omega_0)$ such that the following diagram commutes*

$$\begin{array}{ccc} \big(\pi^{-1}(U), \omega|_{\pi^{-1}(U)}\big) & \xrightarrow{\ \varphi\ } & (\phi(U) \times T^n, \omega_0) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_0} \\ U & \xrightarrow{\quad \phi \quad} & \phi(U). \end{array}$$

In the rest of this paper we assume that every Lagrangian fibration has compact, path-connected fibers.

Now we investigate automorphisms of the local model. By the direct computation shows the following lemma. See also [35, Lemma 2.5].

Lemma 2.17. *Let $\varphi\colon (\mathbb{R}^n \times T^n, \omega_0) \to (\mathbb{R}^n \times T^n, \omega_0)$ be a fiber-preserving symplectomorphism of $\pi_0\colon (\mathbb{R}^n \times T^n, \omega_0) \to \mathbb{R}^n$ which covers a map $\phi\colon R^n \to \mathbb{R}^n$. Then, there exists a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$, a constant $c \in \mathbb{R}^n$, and a map $u\colon \mathbb{R}^n \to T^n$ with ${}^tADu$ symmetric such that φ is written as*

$$\varphi(x, y) = \big(Ax + c, {}^tA^{-1}y + u(x)\big)$$

for any $(x, y) \in \mathbb{R}^n \times T^n$, where Du is the Jacobi matrix of u.

By Theorem 2.16 and Lemma 2.17 we can obtain the following proposition.

Proposition 2.18. *Let $\pi\colon (M, \omega) \to B$ be a Lagrangian fibration. Then, there exists an atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ of B and for each $\alpha \in A$ there exists a symplectomorphism $\varphi_\alpha\colon \big(\pi^{-1}(U_\alpha), \omega|_{\pi^{-1}(U_\alpha)}\big) \to (\phi_\alpha(U_\alpha) \times T^n, \omega_0)$ such that the following diagram commutes*

$$\begin{array}{ccc} (\pi^{-1}(U_\alpha), \omega|_{\pi^{-1}(U_\alpha)}) & \xrightarrow{\ \varphi_\alpha\ } & (\phi_\alpha(U_\alpha) \times T^n, \omega_0) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_0} \\ U_\alpha & \xrightarrow{\quad \phi_\alpha \quad} & \phi_\alpha(U_\alpha). \end{array}$$

Moreover, on each non-empty overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$ there exist locally constant maps $A_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_n(\mathbb{Z})$, $c_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{R}^n$, and a map $u_{\alpha\beta}\colon U_{\alpha\beta} \to T^n$ with ${}^tA_{\alpha\beta}D\left(u_{\alpha\beta} \circ \phi_\beta^{-1}\right)$ symmetric such that the overlap map is written as

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, y) = \left(A_{\alpha\beta}x + c_{\alpha\beta}, {}^tA_{\alpha\beta}^{-1}y + u_{\alpha\beta} \circ \phi_\beta^{-1}(x)\right) \tag{2.1}$$

for any $(x, y) \in \phi_\beta(U_{\alpha\beta}) \times T^n$.

Proposition 2.18 implies that the base manifold of a Lagrangian fibration has an integral affine structure. Conversely, suppose that a manifold B admits an integral affine structure and let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ be an integral affine atlas of B. Then, we can construct a Lagrangian fibration on B in the following way. For each $\alpha \in A$ let $\overline{\phi}_\alpha \colon T^*B|_{U_\alpha} \to \phi_\alpha(U_\alpha) \times \mathbb{R}^n$ be the local trivialization of the cotangent bundle T^*B induced from (U_α, ϕ_α). On each nonempty overlap $U_{\alpha\beta}$, suppose that $\phi_\alpha \circ \phi_\beta^{-1}$ is written by $\phi_\alpha \circ \phi_\beta^{-1}(x) = A_{\alpha\beta}x + c_{\alpha\beta}$ as in Definition 2.1. Then, the overlap map is written as

$$\overline{\phi}_\alpha \circ (\overline{\phi}_\beta)^{-1}(x, y) = (A_{\alpha\beta}x + c_{\alpha\beta}, {}^tA_{\alpha\beta}^{-1}y). \tag{2.2}$$

Since $A_{\alpha\beta}$ lies in $GL_n(\mathbb{Z})$, (2.2) preserves the integer lattice $\mathbb{Z}^n$ of the fiber $\mathbb{R}^n$, hence, induces the fiber-preserving symplectomorphism from $\pi_0 \colon (\phi_\beta(U_{\alpha\beta}) \times T^n, \omega_0) \to \phi_\beta(U_{\alpha\beta})$ to $\pi_0 \colon (\phi_\alpha(U_{\alpha\beta}) \times T^n, \omega_0) \to \phi_\alpha(U_{\alpha\beta})$. Then, the Lagrangian fibrations $\{\pi_0 \colon (\phi_\alpha(U_\alpha) \times T^n, \omega_0) \to \phi_\alpha(U_\alpha)\}_{\alpha\in A}$ are patched together by the symplectomorphisms to form a Lagrangian fibration $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$, namely

$$(T^*B_T, \omega_{T^*B_T}) := \coprod_{\alpha \in A} (\phi_\alpha(U_\alpha) \times T^n, \omega_0) \,/ \sim$$

and

$$\pi_T([x_\alpha, y_\alpha]) := \phi_\alpha^{-1}(x_\alpha)$$

for $(x_\alpha, y_\alpha) \in \phi_\alpha(U_\alpha) \times T^n$. This construction does not depend on the choice of equivalent integral affine structures and depends only on the integral affine structure on B. We call $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ the **canonical model**.

We summarize the above argument to the following proposition.

Proposition 2.19. *Let B be a manifold. B is a base space of a Lagrangian fibration if and only if B admits an integral affine structure.*

Let us give a classification theorem of Lagrangian fibrations in the required form in this paper. Let $\pi \colon (M, \omega) \to B$ be a Lagrangian fibration. Then, B has an integral affine structure by Proposition 2.19. We take and fix an integral affine atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ on B and let $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ be the canonical model associated with the integral affine structure on B. On each U_α, let $\varphi_\alpha \colon \left(\pi^{-1}(U_\alpha), \omega|_{\pi^{-1}(U_\alpha)}\right) \to (\phi_\alpha(U_\alpha) \times T^n, \omega_0)$ be a local trivialization of $\pi \colon (M, \omega) \to B$ as in Proposition 2.18, and $\overline{\phi}_\alpha \colon \left(\pi_T^{-1}(U_\alpha), \omega_{T^*B_T}\right) \to (\phi_\alpha(U_\alpha) \times T^n, \omega_0)$ be the local trivialization of $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ naturally induced from (U_α, ϕ_α) as explained above.[1] Then their composition

$$h_\alpha := \overline{\phi}_\alpha^{\,-1} \circ \varphi_\alpha \colon \left(\pi^{-1}(U_\alpha), \omega|_{\pi^{-1}(U_\alpha)}\right) \to \left(\pi_T^{-1}(U_\alpha), \omega_{T^*B_T}\right)$$

gives a local identification between them.

On each $U_\alpha \cap U_\beta$, suppose that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is written as in (2.1). Then, $h_\alpha \circ h_\beta^{-1}$ can be written as

$$h_\alpha \circ h_\beta^{-1}(p) = \overline{\phi}_\alpha^{\,-1}\left(A_{\alpha\beta}x + c_{\alpha\beta}, {}^tA_{\alpha\beta}^{-1}y + u_{\alpha\beta}(\pi(p))\right),$$

where $\overline{\phi}_\beta(p) = (x, y)$. $u_{\alpha\beta}$ induces the local section $\widetilde{u}_{\alpha\beta}$ of $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ on $U_{\alpha\beta}$ by

$$\widetilde{u}_{\alpha\beta}(b) := [\phi_\alpha(b), u_{\alpha\beta}(b)]$$

for $b \in U_{\alpha\beta}$. It is easy to see that $\widetilde{u}_{\alpha\beta}$ satisfies $\widetilde{u}_{\alpha\beta}^*\omega_{T^*B_T} = 0$. A section with this condition is said to be **Lagrangian**.

Let $\mathscr{S}$ be the sheaf of germs of Lagrangian section of $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$. $\mathscr{S}$ is the sheaf of Abelian groups since the fiber of $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ has the structure of an Abelian group by construction. By definition $\{\widetilde{u}_{\alpha\beta}\}$ forms a Čech one-cocycle on B with coefficients in $\mathscr{S}$. The cohomology class determined by $\{\widetilde{u}_{\alpha\beta}\}$ does not depend on the choice of a specific integral affine structure and depends only on $\pi \colon (M, \omega) \to B$. We denote the cohomology class by $u \in H^1(B; \mathscr{S})$. u is called the **Chern class** of $\pi \colon (M, \omega) \to B$ in [10].

Lagrangian fibrations on the same integral affine manifold are classified with the Chern classes.

Theorem 2.20 ([10])**.** *For two Lagrangian fibrations $\pi_1 \colon (M_1, \omega_1) \to B$ and $\pi_2 \colon (M_2, \omega_2) \to B$ on the same integral affine manifold B, there exists a fiber-preserving symplectomorphism between them which covers the identity if and only if their Chern classes u_1 and u_2 agree with each other. Moreover, if an integral affine manifold B and the cohomology class $u \in H^1(B; \mathscr{S})$ are given, then, there exists a Lagrangian fibration $\pi \colon (M, \omega) \to B$ that realizes them.*

[1]Here we use the same notation as the local trivialization of T^*B because we have no confusion.

Remark 2.21. By the construction of u, there exists a fiber-preserving symplectomorphism between $\pi\colon (M,\omega) \to B$ and $\pi_T\colon (T^*B_T, \omega_{T^*B_T}) \to B$ that covers the identity of B if and only if u vanishes. Since $\pi_T\colon (T^*B_T, \omega_{T^*B_T}) \to B$ has the zero section which is Lagrangian, u is the obstruction class in order that $\pi\colon (M,\omega) \to B$ posses a Lagrangian section. In particular, any Lagrangian fibration with Lagrangian section is identified with the canonical model.

2.3. **Lagrangian fibrations with complete bases.** Let $\pi\colon (M,\omega) \to B$ be a Lagrangian fibration with n-dimensional connected base manifold B, $p\colon \widetilde{B} \to B$ the universal covering of B. We denote by $\widetilde{\pi}\colon (\widetilde{M}, \widetilde{\omega}) \to \widetilde{B}$ the pullback of $\pi\colon (M,\omega) \to B$ to $\widetilde{B}$. Let Γ be the fundamental group of B and $\sigma \in \operatorname{Hom}\left(\Gamma, \operatorname{Aut}(\widetilde{B})\right)$ the action of Γ defined as the inverse of the deck transformation as in Proposition 2.8. By definition, $\widetilde{M}$ admits a natural lift of σ which preserves $\widetilde{\omega}$. The Γ-action on $(\widetilde{M}, \widetilde{\omega})$ is denoted by $\widetilde{\sigma}$. By Proposition 2.8 we have a developing map $\operatorname{dev}\colon \widetilde{B} \to \mathbb{R}^n$ and the homomorphism $\rho\colon \Gamma \to \operatorname{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$. We denote the image of dev by O. Note that the Γ-action ρ on $\mathbb{R}^n$ preserves O since dev is Γ-equivariant.

Proposition 2.22. *There exists a Lagrangian fibration $\pi'\colon (M',\omega') \to O$, a fiber-preserving symplectic immersion $\widetilde{\operatorname{dev}}\colon (\widetilde{M}, \widetilde{\omega}) \to (M',\omega')$ which covers* dev*, and a lift $\widetilde{\rho}$ of the Γ-action ρ on O to (M',ω') such that $\widetilde{\operatorname{dev}}$ is Γ-equivariant with respect to $\widetilde{\sigma}$ and $\widetilde{\rho}$.*

Proof. By Proposition 2.19 B admits an integral affine structure determined by π, and it also induces the integral affine structure on $\widetilde{B}$. Let $\{(U_\alpha, \phi''_\alpha)\}$ be the integral affine atlas of $\widetilde{B}$ and $\{(\widetilde{\pi}^{-1}(U_\alpha), \omega|_{\widetilde{\pi}^{-1}(U_\alpha)}, \varphi''_\alpha)\}$ the local trivializations of $\widetilde{\pi}\colon (\widetilde{M}, \widetilde{\omega}) \to \widetilde{B}$ as in Proposition 2.18 so that on each non-empty overlap $U_{\alpha\beta}$, there exist locally constant maps $A_{\alpha\beta}\colon U_{\alpha\beta} \to \operatorname{GL}_n(\mathbb{Z})$ and $c'_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{R}^n$, and a map $u'_{\alpha\beta}\colon U_{\alpha\beta} \to T^n$ with ${}^tA_{\alpha\beta} D\left(u'_{\alpha\beta} \circ (\phi''_\beta)^{-1}\right)$ symmetric such that $\varphi''_\alpha \circ (\varphi''_\beta)^{-1}$ is written as in (2.1). Then, $A_{\alpha\beta}$'s form a Čech one-cocycle $\{A_{\alpha\beta}\} \in C^1(\{U_\alpha\}; \operatorname{GL}_n(\mathbb{Z}))$ and defines a cohomology class $[\{A_{\alpha\beta}\}] \in H^1(\widetilde{B}; GL_n(\mathbb{Z}))$. It is well known that $H^1(\widetilde{B}; GL_n(\mathbb{Z}))$ is identified with the moduli space of homomorphisms from $\pi_1(\widetilde{B})$ to $GL_n(\mathbb{Z})$. Since $\pi_1(\widetilde{B})$ is trivial, there exists a Čech zero-cocycle $\{A_\alpha\} \in C^0(\{U_\alpha\}; \operatorname{GL}_n(\mathbb{Z}))$ such that $A_{\alpha\beta} = A_\alpha A_\beta^{-1}$ on each $U_{\alpha\beta}$. By using the cocycle we modify the local trivializations $\{(\widetilde{\pi}^{-1}(U_\alpha), \omega|_{\widetilde{\pi}^{-1}(U_\alpha)}, \varphi''_\alpha)\}$ and the integral affine atlas $\{(U_\alpha, \phi''_\alpha)\}$ by replacing φ''_α, ϕ''_α by

$$\varphi'_\alpha(\widetilde{p}) := (A_\alpha^{-1} \times {}^tA_\alpha) \circ \phi''_\alpha(\widetilde{p}), \quad \phi'_\alpha := A_\alpha^{-1}\phi''_\alpha$$

for each $\alpha \in A$, respectively. Then, on each $U_{\alpha\beta}$, $\varphi'_\alpha \circ (\varphi'_\beta)^{-1}$ is written as

$$\varphi'_\alpha \circ (\varphi'_\beta)^{-1}(\widetilde{x}, y) = \left(\widetilde{x} + c_{\alpha\beta}, y + u_{\alpha\beta} \circ (\phi'_\beta)^{-1}(\widetilde{x})\right),$$

where we set $c_{\alpha\beta} := A_\alpha^{-1} c'_{\alpha\beta}$ and $u_{\alpha\beta} := {}^tA_\alpha u'_{\alpha\beta}$. Then, $c_{\alpha\beta}$'s form a Čech one-cocycle $\{c_{\alpha\beta}\} \in C^1(\{U_\alpha\}; \mathbb{R}^n)$ and defines a cohomology class $[\{c_{\alpha\beta}\}] \in H^1(\widetilde{B}; \mathbb{R}^n)$. By the universal coefficients theorem, $H^1(\widetilde{B}; \mathbb{R}^n)$ is identified with $\operatorname{Hom}(H_1(\widetilde{B}; \mathbb{Z}), \mathbb{R}^n)$, which is trivial. So there exists a Čech zero-cocycle $\{c_\alpha\} \in C^0(\{U_\alpha\}; \mathbb{R}^n)$ such that $c_{\alpha\beta} = c_\alpha - c_\beta$ on each $U_{\alpha\beta}$. By using the cocycle, we again modify $\{(\widetilde{\pi}^{-1}(U_\alpha), \omega|_{\widetilde{\pi}^{-1}(U_\alpha)}, \varphi'_\alpha)\}$ and $\{(U_\alpha, \phi'_\alpha)\}$ by replacing φ'_α, ϕ'_α by

$$\varphi_\alpha(\widetilde{p}) := \varphi'_\alpha(\widetilde{p}) - (c_\alpha, 0), \quad \phi_\alpha(\widetilde{b}) := \phi'_\alpha(\widetilde{b}) - c_\alpha,$$

respectively for each $\alpha \in A$. Then, on each $U_{\alpha\beta}$, ϕ_α coincides with ϕ_β and $\varphi_\alpha \circ \varphi_\beta^{-1}$ is written as

$$\varphi_\alpha \circ \varphi_\beta^{-1}(\widetilde{x}, y) = (\widetilde{x}, y + u_{\alpha\beta} \circ \phi_\beta^{-1}(\widetilde{x})).$$

Now we define the map $\operatorname{dev}\colon \widetilde{B} \to \mathbb{R}^n$ by

$$\operatorname{dev}(\widetilde{b}) := \phi_\alpha(\widetilde{b})$$

if $\widetilde{b}$ lies in U_α. It is well defined, and by construction, it is an integral affine immersion whose image is $\cup_{\alpha \in A}\phi_\alpha(U_\alpha)$. (M',ω') is defined by

$$(M',\omega') := \coprod_{\alpha \in A} (\phi_\alpha(U_\alpha) \times T^n, \omega_0) / \sim,$$

where $(x_\alpha, y_\alpha) \in \phi_\alpha(U_\alpha) \times T^n$ and $(x_\beta, y_\beta) \in \phi_\beta(U_\beta) \times T^n$ are in the relation $(x_\alpha, y_\alpha) \sim (x_\beta, y_\beta)$ if they satisfy $(x_\alpha, y_\alpha) = \varphi_\alpha \circ \varphi_\beta^{-1}(x_\beta, y_\beta)$, and $\pi' \colon (M', \omega') \to O$ is defined to be the first projection. $\widetilde{\mathrm{dev}} \colon (\widetilde{M}, \widetilde{\omega}) \to (M', \omega')$ is defined by

$$\widetilde{\mathrm{dev}}(\widetilde{p}) := [\varphi_\alpha(\widetilde{p})]$$

if $\widetilde{p}$ lies in $\widetilde{\pi}^{-1}(U_\alpha)$.

Without loss of generality, we can assume that each U_α is connected,and for each $\gamma \in \Gamma$ and $\alpha \in A$ there uniquely exists $\alpha' \in A$ such that the deck transformation σ_γ maps U_α onto $U_{\alpha'}$. Then, its lift $\widetilde{\sigma}_\gamma$ to $(\widetilde{M}, \widetilde{\omega})$ maps $\widetilde{\pi}^{-1}(U_\alpha)$ to $\widetilde{\pi}^{-1}(U_{\alpha'})$. By Lemma 2.17, $\phi_{\alpha'} \circ \sigma_\gamma \circ \phi_\alpha^{-1}$ can be written as

$$\phi_{\alpha'} \circ \sigma_\gamma \circ \phi_\alpha^{-1}(\widetilde{x}) = A_\gamma^{\alpha'\alpha} \widetilde{x} + c_\gamma^{\alpha'\alpha}$$

for some $A_\gamma^{\alpha'\alpha} \in \mathrm{GL}_n(\mathbb{Z})$, $c_\gamma^{\alpha'\alpha} \in \mathbb{R}^n$. Since ϕ_α coincides with ϕ_β on each overlap $U_{\alpha\beta}$, $\phi_{\alpha'} \circ \phi_\gamma \circ \phi_\alpha(\widetilde{x}) = A_\gamma^{\alpha'\alpha}\widetilde{x} + c_\gamma^{\alpha'\alpha}$ also agrees with $\phi_{\beta'} \circ \phi_\gamma \circ \phi_\beta(\widetilde{x}) = A_\gamma^{\beta'\beta}\widetilde{x} + c_\gamma^{\beta'\beta}$ on the overlap $\phi_\alpha(U_{\alpha\beta}) = \phi_\beta(U_{\alpha\beta})$. This implies $A_\gamma^{\alpha'\alpha}$'s and $c_\gamma^{\alpha'\alpha}$'s do not depend on α and depends only on γ. In fact, for each $\gamma \in \Gamma$ and $\alpha_0 \in A$, we set

$$A_0 := \{\alpha \in A \mid A_\gamma^{\alpha_0'\alpha_0} = A_\gamma^{\alpha'\alpha} \text{ and } c_\gamma^{\alpha_0'\alpha_0} = c_\gamma^{\alpha'\alpha}\}.$$

A_0 contains all $\beta \in A$ with $U_{\alpha_0\beta} \neq \emptyset$. In particular, A_0 is not empty since $\alpha_0 \in A_0$. Then, we have

$$(\cup_{\alpha\in A_0} U_\alpha) \cup (\cup_{\alpha \in A\smallsetminus A_0} U_\alpha) = \widetilde{B}, \quad (\cup_{\alpha\in A_0} U_\alpha) \cap (\cup_{\alpha \in A\smallsetminus A_0} U_\alpha) = \emptyset.$$

If the compliment $A \smallsetminus A_0$ is not empty, this contradicts to the connectedness of $\widetilde{B}$. So we denote them by A_γ and c_γ, respectively. Thus, we define the homomorphism $\rho : \Gamma \to \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ by

$$\rho_\gamma := (A_\gamma, c_\gamma).$$

Γ acts on $\mathbb{R}^n$ by $\rho_\gamma(x) = A_\gamma x + c_\gamma$ for $\gamma \in \Gamma$ and $x \in \mathbb{R}^n$. The lift $\widetilde{\rho}_\gamma$ of ρ_γ to (M', ω') is defined by

$$\widetilde{\rho}_\gamma([x_\alpha, y_\alpha]) := [\varphi_{\alpha'} \circ \widetilde{\sigma}_\gamma \circ \varphi_\alpha^{-1}(x_\alpha, y_\alpha)]$$

if (x_α, y_α) lies in $\phi_\alpha(U_\alpha) \times T^n$. By construction, $\widetilde{\rho}$ is a lift of ρ, and $\widetilde{\rho}$ and ρ satisfy $\widetilde{\mathrm{dev}}(\widetilde{\sigma}_\gamma(\widetilde{p})) = \widetilde{\rho}_\gamma(\widetilde{\mathrm{dev}}(\widetilde{p}))$ and $\mathrm{dev}(\sigma_\gamma(\widetilde{b})) = \rho_\gamma(\mathrm{dev}(\widetilde{b}))$, respectively. ☐

Remark 2.23. (1) By construction, the n-dimensional torus T^n acts freely on M' preserving ω' from the right so that $\pi' \colon M' \to O$ is a principal T^n-bundle.
(2) When $\pi \colon (M, \omega) \to B$ admits a Lagrangian section, the restriction of $\pi_0 \colon (\mathbb{R}^n \times T^n, \omega_0) \to \mathbb{R}^n$ to O can be taken as $\pi' \colon (M', \omega') \to O$. In fact, in this case, since $\pi \colon (M, \omega) \to B$ is identified with the canonical model, we can take a system of local trivializations $\{(\pi^{-1}(U_\alpha), \varphi_\alpha)\}$ with $u_{\alpha\beta} = 0$ on each overlaps $U_{\alpha\beta}$. By applying the construction of $\pi' \colon (M', \omega') \to O$ given in the proof of Proposition 2.22 to such a $\{(\pi^{-1}(U_\alpha), \varphi_\alpha)\}$ we can show the claim.

Suppose that (M, ω) is prequantizable and let $(L, \nabla^L) \to (M, \omega)$ be a prequantum line bundle. We denote by $(\widetilde{L}, \nabla^{\widetilde{L}}) \to (\widetilde{M}, \widetilde{\omega})$ the pullback of $(L, \nabla^L) \to (M, \omega)$ to $(\widetilde{M}, \widetilde{\omega})$. By definition, $\widetilde{L}$ admits a natural lift of the Γ-action $\widetilde{\sigma}$ on $(\widetilde{M}, \widetilde{\omega})$ which preserves $\nabla^{\widetilde{L}}$. The Γ-action on $(\widetilde{L}, \nabla^{\widetilde{L}})$ is denoted by $\widetilde{\widetilde{\sigma}}$. Then, we have the following prequantum version of Proposition 2.22.

Proposition 2.24. *There exists a prequantum line bundle $(L', \nabla^{L'}) \to (M', \omega')$, a bundle immersion $\widetilde{\widetilde{\mathrm{dev}}} \colon (\widetilde{L}, \nabla^{\widetilde{L}}) \to (L', \nabla^{L'})$ which covers $\widetilde{\mathrm{dev}}$, and a lift $\widetilde{\widetilde{\rho}}$ of the Γ-action $\widetilde{\rho}$ on (M', ω') to $(L', \nabla^{L'})$ such that $\widetilde{\widetilde{\mathrm{dev}}}$ is equivariant with respect to $\widetilde{\widetilde{\sigma}}$ and $\widetilde{\widetilde{\rho}}$.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ and $\{(\widetilde{\pi}^{-1}(U_\alpha), \omega|_{\widetilde{\pi}^{-1}(U_\alpha)}, \varphi_\alpha)\}$ be the integral affine atlas of $\widetilde{B}$ and the local trivializations of $\widetilde{\pi} \colon (\widetilde{M}, \widetilde{\omega}) \to \widetilde{B}$ obtained in the proof of Proposition 2.22, respectively. Then, for each $\alpha \in A$ there exists a prequantum line bundle $(\phi_\alpha(U_\alpha) \times T^n \times \mathbb{C}, \nabla^{\widetilde{L}_\alpha}) \to (\phi_\alpha(U_\alpha) \times T^n, \omega_0)$ and a bundle isomorphism $\psi_\alpha \colon (\widetilde{L}, \nabla^{\widetilde{L}})|_{\widetilde{\pi}^{-1}(U_\alpha)} \to (\phi_\alpha(U_\alpha) \times T^n \times \mathbb{C}, \nabla^{\widetilde{L}_\alpha})$ which covers φ. Now we define $(L', \nabla^{L'})$ by

$$(L', \nabla^{L'}) := \coprod_{\alpha \in A} \left(\phi_\alpha(U_\alpha) \times T^n \times \mathbb{C}, \nabla^{\widetilde{L}_\alpha}\right) / \sim,$$

where $(x_\alpha, y_\alpha, z_\alpha) \in \phi_\alpha(U_\alpha) \times T^n \times \mathbb{C}$ and $(x_\beta, y_\beta, z_\beta) \in \phi_\beta(U_\beta) \times T^n \times \mathbb{C}$ are in the relation $(x_\alpha, y_\alpha, z_\alpha) \sim (x_\beta, y_\beta, z_\beta)$ if they satisfy $(x_\beta, y_\beta, z_\beta) = \psi_\alpha \circ \psi_\beta^{-1}(x_\beta, y_\beta, z_\beta)$. $\widetilde{\widetilde{\widetilde{\mathrm{dev}}}}\colon (\widetilde{L}, \nabla^{\widetilde{L}}) \to (L', \nabla^{L'})$ is defined by

$$\widetilde{\widetilde{\widetilde{\mathrm{dev}}}}(\widetilde{v}) := [\psi_\alpha(\widetilde{v})]$$

if $\widetilde{v}$ lies in $(\widetilde{L}, \nabla^{\widetilde{L}})|_{\widetilde{\pi}^{-1}(U_\alpha)}$.

Suppose that for each $\gamma \in \Gamma$ the deck transformation σ_γ maps each U_α to some $U_{\alpha'}$ as before. Then, $\widetilde{\widetilde{\sigma}}_\gamma$ maps $\widetilde{L}_{\widetilde{\pi}^{-1}(U_\alpha)}$ to $\widetilde{L}_{\widetilde{\pi}^{-1}(U_{\alpha'})}$. Then, the Γ-action $\widetilde{\widetilde{\rho}}$ is defined by

$$\widetilde{\widetilde{\rho}}_\gamma(x_\alpha, y_\alpha, z_\alpha) := [\psi_{\alpha'} \circ \widetilde{\widetilde{\sigma}}_\gamma \circ \psi_\alpha^{-1}(x_\alpha, y_\alpha, z_\alpha)]$$

if $(x_\alpha, y_\alpha, z_\alpha)$ lies in $\phi_\alpha(U_\alpha) \times T^n \times \mathbb{C}$. □

In the case where B is complete, we obtain the following corollary.

Corollary 2.25. *Let $\pi\colon (M, \omega) \to B$ be a Lagrangian fibration with connected n-dimensional base B and $(L, \nabla^L) \to (M, \omega)$ a prequantum line bundle on (M, ω). Let $p\colon \widetilde{B} \to B$ be the universal covering of B. Let us denote by $(\widetilde{M}, \widetilde{\omega})$ the pullback of (M, ω) to $\widetilde{B}$ and denote by $(\widetilde{L}, \nabla^{\widetilde{L}})$ the pullback of (L, ∇^L) to $(\widetilde{M}, \widetilde{\omega})$. If B is complete, there exist an integral affine isomorphism* dev: $\widetilde{B} \to \mathbb{R}^n$*, a fiber-preserving symplectomorphism* $\widetilde{\mathrm{dev}}\colon (\widetilde{M}, \widetilde{\omega}) \to (\mathbb{R}^n \times T^n, \omega_0)$*, and a bundle isomorphism* $\widetilde{\widetilde{\widetilde{\mathrm{dev}}}}\colon (\widetilde{L}, \nabla^{\widetilde{L}}) \to \left(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}x \cdot dy\right)$ *such that* $\widetilde{\mathrm{dev}}$ *covers* dev *and* $\widetilde{\widetilde{\widetilde{\mathrm{dev}}}}$ *covers* $\widetilde{\mathrm{dev}}$*, respectively. Here $x \cdot dy$ denotes $\sum_{i=1}^n x_i dy_i$. Moreover, let σ be the Γ-action on $\widetilde{B}$ defined as the inverse of deck transformations, $\widetilde{\sigma}$ the natural lift of σ to $(\widetilde{M}, \widetilde{\omega})$, and $\widetilde{\widetilde{\sigma}}$ the natural lift of $\widetilde{\sigma}$ to $(\widetilde{L}, \nabla^{\widetilde{L}})$, respectively. Then, there exist an integral affine Γ-action $\rho\colon \Gamma \to \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ on $\mathbb{R}^n$, its lifts $\widetilde{\rho}$ and $\widetilde{\widetilde{\rho}}$ to $(\mathbb{R}^n \times T^n, \omega_0)$ and $\left(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}x \cdot dy\right)$, respectively such that* dev, $\widetilde{\mathrm{dev}}$*, and* $\widetilde{\widetilde{\widetilde{\mathrm{dev}}}}$ *are Γ-equivariant.*

Proof. By construction of $\widetilde{\mathrm{dev}}$ given in the proof of Proposition 2.22, if dev is bijective, so is $\widetilde{\mathrm{dev}}$. The argument in [10, p.696] and Theorem 2.20 also show that $\pi_0\colon (\mathbb{R}^n \times T^n, \omega_0) \to \mathbb{R}^n$ is the unique Lagrangian fibration on $\mathbb{R}^n$ up to fiber-preserving symplectomorphism covering the identity. In particular, $\pi'\colon (M', \omega') \to \mathbb{R}^n$ is identified with $\pi_0\colon (\mathbb{R}^n \times T^n, \omega_0) \to \mathbb{R}^n$.

Concerning the prequantum line bundle, it is sufficient to show that $(\mathbb{R}^n \times T^n, \omega_0)$ has a unique prequantum line bundle $\left(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}x \cdot dy\right)$ up to bundle isomorphism. Since ω_0 is exact, any prequantum line bundle on $(\mathbb{R}^n \times T^n, \omega_0)$ is trivial as a complex line bundle. Let $\left(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}\alpha\right)$ be a prequantum line bundle on $(\mathbb{R}^n \times T^n, \omega_0)$ with connection $d - 2\pi\sqrt{-1}\alpha$. Then, $\alpha - x \cdot dy$ defines a de Rham cohomology class in $H^1(\mathbb{R}^n \times T^n; \mathbb{R})$. Since $H^1(\mathbb{R}^n \times T^n; \mathbb{R})$ is isomorphic to $H^1(T^n; \mathbb{R})$, in terms of the generators dy_i's of $H^1(T^n; \mathbb{R})$, $\alpha - x \cdot dy$ can be described as

$$\alpha - x \cdot dy = \sum_{i=1}^n \tau_i dy_i + df$$

for some $\tau_1, \ldots, \tau_n \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R}^n \times T^n)$. Now we define the bundle isomorphism $\psi\colon \mathbb{R}^n \times T^n \times \mathbb{C} \to \mathbb{R}^n \times T^n \times \mathbb{C}$ by

$$\psi(x, y, z) := \left(x + (\tau_i), y, e^{-2\pi\sqrt{-1}f(x,y)}z\right).$$

Then, ψ satisfies $\psi^*\left(d - 2\pi\sqrt{-1}x \cdot dy\right) = d - 2\pi\sqrt{-1}\alpha$. □

Remark 2.26. By Corollary 2.25, any Lagrangian fibration $\pi\colon (M, \omega) \to B$ on a connected, complete B with prequantum line bundle $(L, \nabla^L) \to (M, \omega)$ is obtained as the quotient space of the Γ-action on $\pi_0\colon (\mathbb{R}^n \times T^n, \omega_0) \to \mathbb{R}^n$ with prequantum line bundle $(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}x \cdot dy) \to (\mathbb{R}^n \times T^n, \omega_0)$. By definition, the prequantum line bundle $(L, \nabla^L) \to (M, \omega)$ is equipped with a Hermitian metric $\langle \cdot, \cdot \rangle_L$ compatible with ∇^L.[2] The pull-back of $\langle \cdot, \cdot \rangle_L$ to $(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}x \cdot dy) \to (\mathbb{R}^n \times T^n, \omega_0)$ coincides with the one induced from the standard Hermitian inner product on $\mathbb{C}$ up to constant. In fact, it is easy to see that, up to constant, it is the unique Hermitian metric on $(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}x \cdot dy) \to (\mathbb{R}^n \times T^n, \omega_0)$ compatible with $d - 2\pi\sqrt{-1}x \cdot dy$. In the rest of this paper, we assume

[2] A Hermitian metric $\langle \cdot, \cdot \rangle_L$ on L is compatible with ∇^L if it satisfies $d\left(\langle s_1, s_2 \rangle_L\right) = \left\langle \nabla^L s_1, s_2 \right\rangle_L + \left\langle s_1, \nabla^L s_2 \right\rangle_L$ for all $s_1, s_2 \in \Gamma(L)$.

that $(\mathbb{R}^n\times T^n\times\mathbb{C}, d-2\pi\sqrt{-1}x\cdot dy)\to(\mathbb{R}^n\times T^n,\omega_0)$ is always equipped with the Hermitian metric though we do not specify it.

2.4. **The lifting problem of the Γ-action to the prequantum line bundle.** In the rest of this section, we investigate the condition for the Γ-action on $\pi_0\colon(\mathbb{R}^n\times T^n,\omega_0)\to\mathbb{R}^n$ to have a lift to $(\mathbb{R}^n\times T^n\times\mathbb{C},\, d-2\pi\sqrt{-1}x\cdot dy)\to(\mathbb{R}^n\times T^n,\omega_0)$ in detail. Let $\rho\colon\Gamma\to\mathrm{GL}_n(\mathbb{Z})\ltimes\mathbb{R}^n$ be a Γ-action on $\mathbb{R}^n$ and $\widetilde{\rho}$ its lift to $(\mathbb{R}^n\times T^n,\omega_0)$. By Lemma 2.17, for each $\gamma\in\Gamma$, there exist $A_\gamma\in\mathrm{GL}_n(\mathbb{Z})$, $c_\gamma\in\mathbb{R}^n$, and a map $u_\gamma\colon\mathbb{R}^n\to T^n$ with ${}^tA_\gamma Du_\gamma$ symmetric such that ρ_γ and $\widetilde{\rho}_\gamma$ can be described as follows

$$\rho_\gamma(x)=A_\gamma x+c_\gamma,\quad \widetilde{\rho}_\gamma(x,y)=\left(A_\gamma x+c_\gamma,{}^tA_\gamma^{-1}y+u_\gamma(x)\right). \tag{2.3}$$

Note that since (2.3) is a Γ-action, A_γ, c_γ, and u_γ satisfy the following conditions

$$\begin{cases} A_{\gamma_1\gamma_2} &= A_{\gamma_1}A_{\gamma_2}\\ c_{\gamma_1\gamma_2} &= A_{\gamma_1}c_{\gamma_2}+c_{\gamma_1}\\ u_{\gamma_1\gamma_2}(x) &= {}^tA_{\gamma_1}^{-1}u_{\gamma_2}(x)+u_{\gamma_1}\left(\rho_{\gamma_2}(x)\right)\end{cases} \tag{2.4}$$

for γ_1, $\gamma_2\in\Gamma$, and $x\in\mathbb{R}^n$. Let $\widetilde{u}_\gamma={}^t(\widetilde{u}_\gamma^1,\dots,\widetilde{u}_\gamma^n)\colon\mathbb{R}^n\to\mathbb{R}^n$ be a lift of u_γ. For $\widetilde{u}_\gamma$ and $i=1,\dots,n$, we put

$$\int_0^{x_i}\widetilde{u}_\gamma(x)dx_i:={}^t\left(\int_0^{x_i}\widetilde{u}_\gamma^1(x)dx_i,\dots,\int_0^{x_i}\widetilde{u}_\gamma^n(x)dx_i\right)$$

and

$$F_\gamma^i(x):=\left({}^tA_\gamma\int_0^{x_i}\widetilde{u}_\gamma(x)dx_i\right)_i=\sum_{j=1}^n\left({}^tA_\gamma\right)_{ij}\int_0^{x_i}\widetilde{u}_\gamma^j(x)dx_i.$$

Let $N\in\mathbb{N}$ be a positive integer. The Γ-action $\widetilde{\rho}$ also preserves $N\omega_0$. Then, we can show the following lemma.

Lemma 2.27. (1) *For each $\gamma\in\Gamma$, there exists a bundle automorphism $\widetilde{\widetilde{\rho}}_\gamma$ of $\left(\mathbb{R}^n\times T^n\times\mathbb{C},d-2\pi\sqrt{-1}Nx\cdot dy\right)$ which covers $\widetilde{\rho}_\gamma$ and preserves the connection if and only if c_γ lies in $\frac{1}{N}\mathbb{Z}^n$. Moreover, in this case, $\widetilde{\widetilde{\rho}}_\gamma$ can be described as follows*

$$\widetilde{\widetilde{\rho}}_\gamma(x,y,z)=\left(\widetilde{\rho}_\gamma(x,y),\ g_\gamma e^{2\pi\sqrt{-1}N\left\{\widetilde{g}_\gamma(x)+c_\gamma\cdot({}^tA_\gamma^{-1}y)\right\}}z\right), \tag{2.5}$$

where g_γ is an arbitrary element in $U(1)$ and

$$\widetilde{g}_\gamma(x):=\rho_\gamma(x)\cdot\widetilde{u}_\gamma(x)-c_\gamma\cdot\widetilde{u}_\gamma(0)-\sum_{i=1}^nF_\gamma^i(0,\dots,0,x_i,\dots,x_n). \tag{2.6}$$

The formula (2.5) does not depend on the choice of $\widetilde{u}_\gamma$.[3]
(2) *Under the condition given in (1), the map $\widetilde{\widetilde{\rho}}\colon\Gamma\to\mathrm{Aut}\left(\left(\mathbb{R}^n\times T^n\times\mathbb{C},d-2\pi\sqrt{-1}Nx\cdot dy\right)\right)$ defined by (2.5) is a homomorphism if and only if the map $g\colon\Gamma\ni\gamma\mapsto g_\gamma\in U(1)$ is a homomorphism and for all $\gamma_1,\gamma_2\in\Gamma$ and $x\in\mathbb{R}^n$, the following condition holds*

$$\begin{aligned}&\left\{-c_{\gamma_1}\cdot u_{\gamma_1}(0)+c_{\gamma_1}\cdot{}^tA_{\gamma_1}^{-1}u_{\gamma_2}(0)+\rho_{\gamma_1}(c_{\gamma_2})\cdot u_{\gamma_1}\left(\rho_{\gamma_2}(0)\right)\right\}\\ &\qquad-\sum_{i=1}^n\left({}^tA_{\gamma_1}\int_0^{(\rho_{\gamma_2}(x))_i}u_{\gamma_1}\left(0,\dots,0,\tau_i,(\rho_{\gamma_2}(x))_{i+1},\dots,(\rho_{\gamma_2}(x))_n\right)d\tau_i\right)_i\\ &\qquad\qquad+\sum_{i=1}^n\left({}^tA_{\gamma_2}{}^tA_{\gamma_1}\int_0^{x_i}u_{\gamma_1}\left(\rho_{\gamma_2}(0,\dots0,\tau_i,x_{i+1},\dots,x_n)\right)d\tau_i\right)_i\in\frac{1}{N}\mathbb{Z}.\end{aligned}$$

Proof. For each $\gamma\in\Gamma$ we put

$$\widetilde{\widetilde{\rho}}_\gamma(x,y,z)=\left(\widetilde{\rho}_\gamma(x,y),e^{2\pi\{\widetilde{g}_\gamma^R(x,y)+\sqrt{-1}\widetilde{g}_\gamma^I(x,y)\}}z\right),$$

[3]In the rest of this paper, we often use the notation u_γ instead of $\widetilde{u}_\gamma$.

where $\widetilde{g}_\gamma^R$ and $\widetilde{g}_\gamma^I$ are real valued functions on $\mathbb{R}^n \times T^n$. By the direct computation, it is easy to see that $\widetilde{\widetilde{\rho}}_\gamma$ preserves $d - 2\pi\sqrt{-1}Nx \cdot dy$ if and only if $\widetilde{g}_\gamma^R$ is constant and $\widetilde{g}_\gamma^I$ satisfy the following conditions

$$\partial_{x_i}\widetilde{g}_\gamma^I = N(A_\gamma x + c_\gamma) \cdot \partial_{x_i}\widetilde{u}_\gamma \tag{2.7}$$

$$\partial_{y_i}\widetilde{g}_\gamma^I = N(A_\gamma^{-1}c_\gamma)_i \tag{2.8}$$

for $i = 1, \dots, n$. From (2.7) we obtain

$$\widetilde{g}_\gamma^I(x, y) = \widetilde{g}_\gamma^I(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n, y) + N\left\{[\rho_\gamma(x) \cdot \widetilde{u}_\gamma(x)]_{x_i=0}^{x_i=x_i} - F_\gamma^i(x)\right\}. \tag{2.9}$$

Using (2.9) recursively, we obtain

$$\widetilde{g}_\gamma^I(x, y) = \widetilde{g}_\gamma^I(0, y) + N\left\{\rho_\gamma(x) \cdot \widetilde{u}_\gamma(x) - c_\gamma \cdot \widetilde{u}_\gamma(0) - \sum_{i=1}^n F_\gamma^i(0, \dots, 0, x_i, \dots, x_n)\right\}. \tag{2.10}$$

(2.10) does not depend on the order of applying (2.9) to x_i's. In fact, by applying (2.9) first to x_i, then next to x_j, we obtain

$$\begin{aligned}
\frac{1}{N}\widetilde{g}_\gamma^I(x, y) =& \frac{1}{N}\widetilde{g}_\gamma^I(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n, y) + \int_0^{x_i} (A_\gamma x + c_\gamma) \cdot \partial_{x_i} u_\gamma(x) dx_i \\
=& \frac{1}{N}\widetilde{g}_\gamma^I(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n, y) \\
&+ \int_0^{x_j} (A_\gamma x + c_\gamma) \cdot \partial_{x_j} u_\gamma(x) dx_j\Big|_{x_i=0} + \int_0^{x_i} (A_\gamma x + c_\gamma) \cdot \partial_{x_i} u_\gamma(x) dx_i.
\end{aligned}$$

So, in order to see this, it is sufficient to show

$$\begin{aligned}
&\int_0^{x_j} (A_\gamma x + c_\gamma) \cdot \partial_{x_j} u_\gamma(x) dx_j\Big|_{x_i=0} + \int_0^{x_i} (A_\gamma x + c_\gamma) \cdot \partial_{x_i} u_\gamma(x) dx_i \\
&- \int_0^{x_i} (A_\gamma x + c_\gamma) \cdot \partial_{x_i} u_\gamma(x) dx_i\Big|_{x_j=0} - \int_0^{x_j} (A_\gamma x + c_\gamma) \cdot \partial_{x_j} u_\gamma(x) dx_j
\end{aligned} \tag{2.11}$$

vanishes. Since ${}^tA_\gamma Du_\gamma$ is symmetric, we have $\left({}^tA_\gamma \partial_{x_i} u_\gamma(x)\right)_j = ({}^tA_\gamma Du_\gamma(x))_{ji} = ({}^tA_\gamma Du_\gamma(x))_{ij} = \left({}^tA_\gamma \partial_{x_j} u_\gamma(x)\right)_i$ for all $i, j = 1, \dots, n$. By using this, we can show

$$\begin{aligned}
(2.11) =& \int_0^{x_j} \partial_{x_j}\left(\int_0^{x_i} (A_\gamma x + c_\gamma) \cdot \partial_{x_i} u_\gamma(x) dx_i\right) dx_j - \int_0^{x_i} \partial_{x_i}\left(\int_0^{x_j} (A_\gamma x + c_\gamma) \cdot \partial_{x_j} u_\gamma(x) dx_j\right) dx_i \\
=& \int_0^{x_j}\int_0^{x_i} \left(\partial_{x_j}(A_\gamma x + c_\gamma)\right) \cdot \partial_{x_i} u_\gamma(x) dx_i dx_j + \int_0^{x_j}\int_0^{x_i} (A_\gamma x + c_\gamma) \cdot \partial_{x_j}\partial_{x_i} u_\gamma(x) dx_i dx_j \\
&- \int_0^{x_i}\int_0^{x_j} \left(\partial_{x_i}(A_\gamma x + c_\gamma)\right) \cdot \partial_{x_j} u_\gamma(x) dx_j dx_i - \int_0^{x_i}\int_0^{x_j} (A_\gamma x + c_\gamma) \cdot \partial_{x_i}\partial_{x_j} u_\gamma(x) dx_j dx_i \\
=& \int_0^{x_j}\int_0^{x_i} \left({}^tA_\gamma \partial_{x_i} u_\gamma(x)\right)_j dx_i dx_j - \int_0^{x_i}\int_0^{x_j} \left({}^tA_\gamma \partial_{x_j} u_\gamma(x)\right)_i dx_j dx_i \\
=& 0.
\end{aligned}$$

By the same way, from (2.8) we obtain

$$\widetilde{g}_\gamma^I(x, y) = \widetilde{g}_\gamma^I(x, 0) + Nc_\gamma \cdot {}^tA_\gamma^{-1} y. \tag{2.12}$$

Thus, from (2.10) and (2.12) we have

$$\widetilde{g}_\gamma^I(x, y) = \widetilde{g}_\gamma^I(0, 0) + N\left\{\rho_\gamma(x) \cdot \widetilde{u}_\gamma(x) - c_\gamma \cdot \widetilde{u}_\gamma(0) - \sum_{i=1}^n F_\gamma^i(0, \dots, 0, x_i, \dots, x_n) + c_\gamma \cdot {}^tA_\gamma^{-1} y\right\}. \tag{2.13}$$

Since $y \in T^n$, $\widetilde{g}_\gamma^I$ should satisfies $e^{2\pi\sqrt{-1}\widetilde{g}_\gamma^I(0, e_i)} = e^{2\pi\sqrt{-1}\widetilde{g}_\gamma^I(0,0)}$ for all $i = 1, \dots, n$ and $\gamma \in \Gamma$. This holds if and only if $A_\gamma^{-1} Nc_\gamma \cdot e_i \in \mathbb{Z}$ for all $i = 1, \dots, n$ and $\gamma \in \Gamma$. Since $A_\gamma \in \mathrm{GL}_n(\mathbb{Z})$ this is equivalent to the condition $Nc_\gamma \in \mathbb{Z}^n$. In this case, we put $g_\gamma := e^{2\pi(\widetilde{g}_\gamma^R(0,0) + \sqrt{-1}\widetilde{g}_\gamma^I(0,0))}$. Since $\widetilde{\widetilde{\rho}}_\gamma$ preserves the Hermitian metric on $(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}x \cdot dy) \to (\mathbb{R}^n \times T^n, \omega_0)$, g_γ lies in $U(1)$. The formula (2.5) does not depend on the choice of $\widetilde{u}_\gamma$ since the difference of two lifts of u_γ lies in $\mathbb{Z}^n$. This proves (1).

The map $\widetilde{\widetilde{\rho}}$ defined in (2) is a homomorphism if and only if $\widetilde{g}^I_\gamma(x,y)-\widetilde{g}^I_\gamma(0,0)$ defined by (2.13) satisfies the cocycle condition. By a direct computation using (2.4), it is equivalent to the ones given in (2). $\square$

Example 2.28. Let B be the n-dimensional integral affine torus given in Example 2.3 (1) for a linear basis $v_1,\dots,v_n\in\mathbb{R}^n$. The product $B\times T^n$ admits a symplectic structure ω so that the trivial torus bundle $\pi\colon(B\times T^n,\omega)\to B$ becomes a Lagrangian fibration. This is obtained as the quotient space of the action of $\Gamma:=\mathbb{Z}^n$ on $\pi_0\colon(\mathbb{R}^n\times T^n,\omega_0)\to\mathbb{R}^n$ which is defined by

$$\widetilde{\rho}_\gamma(x,y)=(x+C\gamma,\ y)$$

for $\gamma\in\Gamma$ and $(x,y)\in\mathbb{R}^n\times T^n$, where $C=(v_1\ \cdots\ v_n)\in GL_n(\mathbb{R})$. Let $N\in\mathbb{N}$ be a positive number. The Γ-action $\widetilde{\rho}$ on $(\mathbb{R}^n\times T^n,N\omega_0)$ has a lift to the prequantum line bundle $(\mathbb{R}^n\times T^n\times\mathbb{C},d-2\pi\sqrt{-1}Nx\cdot dy)\to(\mathbb{R}^n\times T^n,N\omega_0)$ if and only if all v_i's lie in $\frac{1}{N}\mathbb{Z}^n$, and in this case $\widetilde{\widetilde{\rho}}$ is given by

$$\widetilde{\widetilde{\rho}}_\gamma(x,y,z)=\left(\widetilde{\rho}_\gamma(x,y),g_\gamma e^{2\pi\sqrt{-1}NC\gamma\cdot y}z\right)$$

for $\gamma\in\Gamma$ and $(x,y,z)\in\mathbb{R}^n\times T^n\times\mathbb{C}$, where $g\colon\Gamma\ni\gamma\mapsto g_\gamma\in U(1)$ is an arbitrary homomorphism.

Example 2.29 (The Kodaira-Thurston manifold)**.** Let Γ be $\mathbb{Z}^2$. Consider the Γ-action on $\pi_0\colon\left(\mathbb{R}^2\times T^2,\omega_0\right)\to\mathbb{R}^2$ which is defined by

$$\rho_\gamma(x):=x+\gamma,\quad \widetilde{\rho}_\gamma(x,y):=(\rho_\gamma(x),y+u_\gamma(x))$$

for $\gamma\in\Gamma$ and $(x,y)\in\mathbb{R}^2\times T^2$, where $u_\gamma(x)={}^t(0,\gamma_1x_2)$. The Lagrangian fibration given by the quotient of this action is denoted by $\pi\colon(M,\omega)\to B$. M was first observed by Kodaira in [24] and Thurston [36] pointed out in [36] that (M,ω) does not admits any Kähler structure. M is nowadays called the Kodaira-Thurston manifold. Let $N\in\mathbb{N}$ be a positive number. The Γ-action $\widetilde{\rho}$ on $\left(\mathbb{R}^2\times T^2,N\omega_0\right)$ has a lift to the prequantum line bundle $(\mathbb{R}^2\times T^2\times\mathbb{C},d-2\pi\sqrt{-1}Nx\cdot dy)\to\left(\mathbb{R}^2\times T^2,N\omega_0\right)$ if and only if N is even, and in this case the lift $\widetilde{\widetilde{\rho}}$ is given by

$$\widetilde{\widetilde{\rho}}_\gamma(x,y,z)=\left(\widetilde{\rho}_\gamma(x,y),g_\gamma e^{2\pi\sqrt{-1}N\{\frac{1}{2}\gamma_1x_2^2+\gamma_1\gamma_2x_2+\gamma\cdot y\}}z\right)$$

for $\gamma\in\Gamma$ and $(x,y,z)\in\mathbb{R}^n\times T^n\times\mathbb{C}$, where $g\colon\Gamma\ni\gamma\mapsto g_\gamma\in U(1)$ is an arbitrary homomorphism.

Example 2.30. Let B be the n-dimensional integral affine torus given in Example 2.3 (2) for a linear basis $v_1,\dots,v_n\in\mathbb{R}^n$. When all v_i's are integer vectors, i.e., $v_1,\dots,v_n\in\mathbb{Z}^n$, we can generalize Example 2.28 and Example 2.29 in the following way. Namely, for $i,j=1,\dots,n$, let u_{ij} be an integer vector with $u_{ij}=u_{ji}$. For each $\gamma\in\Gamma:=\mathbb{Z}^n$, define the map $u_\gamma\colon\mathbb{R}^n\to T^n$ by

$$u_\gamma(x):=\begin{pmatrix}u_{11}\cdot\gamma & \cdots & u_{1n}\cdot\gamma\\ \vdots & & \vdots\\ u_{n1}\cdot\gamma & \cdots & u_{nn}\cdot\gamma\end{pmatrix}x,$$

and define the action of Γ on $\pi_0\colon(\mathbb{R}^n\times T^n,\omega_0)\to\mathbb{R}^n$ by

$$\widetilde{\rho}_\gamma(x,y)=(x+C\gamma,\ y+u_\gamma(x)) \tag{2.14}$$

for $\gamma\in\Gamma$ and $(x,y)\in\mathbb{R}^n\times T^n$, where $C=(v_1\ \cdots\ v_n)$. Then, the quotient $\pi\colon(M,\omega)\to B$ obtained as the Γ-action (2.14) is a Lagrangian fibration on B. Let $N\in\mathbb{N}$ be a positive number. The Γ-action $\widetilde{\rho}$ on $(\mathbb{R}^n\times T^n,N\omega_0)$ has a lift to the prequantum line bundle $(\mathbb{R}^n\times T^n\times\mathbb{C},d-2\pi\sqrt{-1}Nx\cdot dy)\to(\mathbb{R}^n\times T^n,N\omega_0)$ if and only if $\frac{N}{2}v_i\cdot U_jv_i\in\mathbb{Z}$ for all $i,j=1,\dots,n$, where

$$U_j:=\begin{pmatrix}(u_{11})_j & \cdots & (u_{1n})_j\\ \vdots & & \vdots\\ (u_{n1})_j & \cdots & (u_{nn})_j\end{pmatrix}.$$

And in this case the lift $\widetilde{\widetilde{\rho}}$ is given by

$$\widetilde{\widetilde{\rho}}_\gamma(x,y,z)=\left(\widetilde{\rho}_\gamma(x,y),\ g_\gamma e^{2\pi\sqrt{-1}N\left[\frac{1}{2}\{\rho_\gamma(x)\cdot u_\gamma(\rho_\gamma(x))-\rho_\gamma(0)\cdot u_\gamma(\rho_\gamma(0))\}+\rho_\gamma(0)\cdot y\right]}z\right)$$

for $\gamma\in\Gamma$ and $(x,y,z)\in\mathbb{R}^n\times T^n\times\mathbb{C}$, where $g\colon\Gamma\ni\gamma\mapsto g_\gamma\in U(1)$ is an arbitrary homomorphism.

Example 2.31. Let $n \geq 2$ and $\lambda_1, \dots \lambda_{n-1} \in \mathbb{Z}$. Let Γ be the group $(\mathbb{Z}^n, \circ)$ given in Example 2.6. For each $\gamma \in \Gamma$, let A_γ be the matrix

$$A_\gamma := \begin{pmatrix} 1 & \lambda_1 & & & \\ & 1 & \lambda_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda_{n-1} \\ & & & & 1 \end{pmatrix}^{\gamma_n}$$

and $u_\gamma \colon \mathbb{R}^n \to T^n$ the map defined by

$$u_\gamma(x) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma_n x_n \end{pmatrix}.$$

Consider the Γ-action $\widetilde{\rho}$ on $\pi_0 \colon (\mathbb{R}^n \times T^n, \omega) \to \mathbb{R}^n$ which is defined by

$$\widetilde{\rho}_\gamma(x, y) := \left(A_\gamma x + \gamma,\ {}^t A_\gamma^{-1} y + u_\gamma(x)\right) \tag{2.15}$$

for $\gamma \in \Gamma$ and $(x, y) \in \mathbb{R}^n \times T^n$, where $C = (v_1 \ \cdots \ v_n)$. Then, the quotient $\pi \colon (M, \omega) \to B$ obtained as the Γ-action (2.15) is a Lagrangian fibration on the integral affine manifold B obtained in Example 2.6. Let $N \in \mathbb{N}$ be a positive number. The Γ-action $\widetilde{\rho}$ on $(\mathbb{R}^n \times T^n, N\omega_0)$ has a lift to the prequantum line bundle $(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}Nx \cdot dy) \to (\mathbb{R}^n \times T^n, N\omega_0)$ if and only if N is even, and in this case the lift $\widetilde{\widetilde{\rho}}$ is given by

$$\widetilde{\widetilde{\rho}}_\gamma(x, y, z) = \left(\widetilde{\rho}_\gamma(x, y),\ g_\gamma e^{2\pi\sqrt{-1}N\left\{\gamma_n x_n\left(\frac{1}{2}x_n + \gamma_n\right) + \gamma \cdot \left({}^t A_\gamma^{-1} y\right)\right\}} z\right)$$

for $\gamma \in \Gamma$ and $(x, y, z) \in \mathbb{R}^n \times T^n \times \mathbb{C}$, where $g \colon \Gamma \ni \gamma \mapsto g_\gamma \in U(1)$ is an arbitrary homomorphism.

3. Degree-zero harmonic spinors and integrability of almost complex structures

Let $N \in \mathbb{N}$ be a positive integer. For a compatible almost complex structure J on the total space of the Lagrangian fibration $\pi_0 \colon (\mathbb{R}^n \times T^n, N\omega_0) \to \mathbb{R}^n$, let D be the associated Spin^c Dirac operator with coefficients in the prequantum line bundle $\left(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}Nx \cdot dy\right) \to (\mathbb{R}^n \times T^n, N\omega_0)$. An element in the kernel $\ker D$ of D is called a harmonic spinor. In this section, for J which is invariant along the fiber, we investigate the existence condition of non-trivial degree-zero harmonic spinors, i.e., non-trivial sections which lie in $\ker D$. In the rest of this paper, we put $\widetilde{M} := \mathbb{R}^n \times T^n$ and $\left(\widetilde{L}, \nabla^{\widetilde{L}}\right) := (\mathbb{R}^n \times T^n \times \mathbb{C},\ d - 2\pi\sqrt{-1}x \cdot dy)$ for simplicity.

3.1. Bohr-Sommerfeld points.

Let $\pi \colon (M, \omega) \to B$ be a Lagrangian fibration with prequantum line bundle $(L, \nabla^L) \to (M, \omega)$. We recall the definition of Bohr-Sommerfeld points.

Definition 3.1. A point $b \in B$ is said to be **Bohr-Sommerfeld** if $\left(L, \nabla^L\right)|_{\pi^{-1}(b)}$ admits a non-trivial covariant constant section. We denote the set of Bohr-Sommerfeld points by B_{BS}.

Let us detect Bohr-Sommerfeld points for $\pi_0 \colon (\widetilde{M}, N\omega_0) \to \mathbb{R}^n$ with prequantum line bundle $\left(\widetilde{L}, \nabla^{\widetilde{L}}\right)^{\otimes N} \to (\widetilde{M}, N\omega_0)$.

Lemma 3.2. *$x \in \mathbb{R}^n$ is a Bohr-Sommerfeld if and only if x lies in $\frac{1}{N}\mathbb{Z}^n$, i.e., $\mathbb{R}^n_{BS} = \frac{1}{N}\mathbb{Z}^n$. Moreover, for a Bohr-Sommerfeld point $x \in \frac{1}{N}\mathbb{Z}^n$, a covariant constant section s of $\left(\widetilde{L}, \nabla^{\widetilde{L}}\right)^{\otimes N}\Big|_{\pi_0^{-1}(x)}$ is of the form $s(y) = s(0)e^{2\pi\sqrt{-1}Nx \cdot y}$.*

Proof. For a fixed $x \in \mathbb{R}^n$, $\left(\widetilde{L}, \nabla^{\widetilde{L}}\right)^{\otimes N}\Big|_{\pi_0^{-1}(x)} \to \pi_0^{-1}(x)$ admits a non-trivial covariant constant section s if and only if s satisfies

$$0 = \nabla^{\widetilde{L}^{\otimes N}}_{\partial_{y_i}} s = \partial_{y_i} s - 2\pi\sqrt{-1}Nx_i s$$

for $i = 1, \dots, n$. Hence, s should be of the form $s(y) = s(0)e^{2\pi\sqrt{-1}Nx\cdot y}$. Since s is global, $s(0) = s(e_i) = s(0)e^{2\pi\sqrt{-1}Nx_i}$. This implies $Nx_i \in \mathbb{Z}$ for $i = 1, \dots, n$. □

Remark 3.3. Suppose that $\pi_0\colon (\widetilde{M}, N\omega_0) \to \mathbb{R}^n$ is equipped with an action of a group Γ which preserves all the data, and its lift $\widetilde{\widetilde{\rho}}$ to $\left(\widetilde{L}, \nabla^{\widetilde{L}}\right)^{\otimes N}$ is given by (2.5). Then, by Lemma 2.27 (1), the Γ-action ρ on $\mathbb{R}^n$ preserves $\mathbb{R}^n_{BS}$. When the Γ-action ρ on $\mathbb{R}^n$ is properly discontinuous and free, let $F \subset \mathbb{R}^n$ be a fundamental domain of the Γ-action ρ on $\mathbb{R}^n$. Then, the map

$$\Gamma \times \left(F \cap \frac{1}{N}\mathbb{Z}^n\right) \ni \left(\gamma, \frac{m}{N}\right) \mapsto N\rho_\gamma\left(\frac{m}{N}\right) \in \mathbb{Z}^n \tag{3.1}$$

can be defined and is bijective. In particular, let $\pi\colon (M, N\omega) \to B$ be a Lagrangian fibration with prequantum line bundle $(L, \nabla^L)^{\otimes N} \to (M, N\omega)$ obtained as the quotient space of the Γ-action. Then, $F \cap \frac{1}{N}\mathbb{Z}^n$ is identified with B_{BS}.

3.2. **Almost complex structures.** Let $\mathcal{S}_n$ be the Siegel upper half space, namely, the space of $n \times n$ symmetric complex matrices whose imaginary parts are positive definite

$$\mathcal{S}_n := \{Z = X + \sqrt{-1}Y \in M_n(\mathbb{C}) \mid X, Y \in M_n(\mathbb{R}), {}^tZ = Z, \text{and } Y \text{ is positive definite}\}.$$

It is well known that $\mathcal{S}_n$ is identified with the space of compatible complex structures on the $2n$-dimensional standard symplectic vector space.

For a tangent vector $u = \sum_{i=1}^n \{(u_x)_i\partial_{x_i} + (u_y)_i\partial_{y_i}\} \in T_{(x,y)}\widetilde{M}$ at a point $(x, y) \in \widetilde{M}$ we use the following notation

$$u = (\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n})\begin{pmatrix}(u_x)_1\\ \vdots\\ (u_x)_n\\ (u_y)_1\\ \vdots\\ (u_y)_n\end{pmatrix} = (\partial_x, \partial_y)\begin{pmatrix}u_x\\ u_y\end{pmatrix},$$

where

$$\partial_x = (\partial_{x_1}, \dots, \partial_{x_n}), \quad \partial_y = (\partial_{y_1}, \dots, \partial_{y_n}), \quad u_x = \begin{pmatrix}(u_x)_1\\ \vdots\\ (u_x)_n\end{pmatrix}, \quad u_y = \begin{pmatrix}(u_y)_1\\ \vdots\\ (u_y)_n\end{pmatrix}.$$

In terms of the notations of tangent vectors $u = (\partial_x, \partial_y)\begin{pmatrix}u_x\\ u_y\end{pmatrix}$ and $v = (\partial_x, \partial_y)\begin{pmatrix}v_x\\ v_y\end{pmatrix} \in T_{(x,y)}\widetilde{M}$, ω_0 can be described by

$$\omega_0(u, v) = \left({}^tu_x, {}^tu_y\right)\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}v_x\\ v_y\end{pmatrix}.$$

Since the tangent bundle $T\widetilde{M}$ is trivial, the space of C^∞ maps from $\widetilde{M}$ to $\mathcal{S}_n$ is identified with the space of compatible almost complex structures on $(\widetilde{M}, \omega_0)$. For $Z = X + \sqrt{-1}Y \in C^\infty\left(\widetilde{M}, \mathcal{S}_n\right)$, the corresponding almost complex structure J_Z is given as follows

$$J_Zu := (\partial_x, \partial_y)\begin{pmatrix}XY^{-1} & -Y - XY^{-1}X\\ Y^{-1} & -Y^{-1}X\end{pmatrix}_{(x,y)}\begin{pmatrix}u_x\\ u_y\end{pmatrix} \tag{3.2}$$

for $u = (\partial_x, \partial_y)\begin{pmatrix} u_x \\ u_y \end{pmatrix} \in T_{(x,y)}\widetilde{M}$.[4] Then, the Riemannian metric g determined by ω_0 and J_Z can be described by

$$\begin{aligned} g(u,v) &:= \omega_0(u, Jv) \\ &= ({}^t u_x, {}^t u_y)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} XY^{-1} & -Y - XY^{-1}X \\ Y^{-1} & -Y^{-1}X \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ &= ({}^t u_x, {}^t u_y)\begin{pmatrix} Y^{-1} & -Y^{-1}X \\ -XY^{-1} & Y + XY^{-1}X \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix}. \end{aligned} \tag{3.3}$$

Let $J = J_Z$ be the almost complex structure on $\left(\widetilde{M}, \omega_0\right)$ corresponding to a given $Z = X + \sqrt{-1}Y \in C^\infty\left(\widetilde{M}, \mathcal{S}_n\right)$. Then, $(-J\partial_y,\, \partial_y) = (-J\partial_{y_1}, \dots, -J\partial_{y_n}, \partial_{y_1}, \dots, \partial_{y_n})$ is also a basis of the tangent space of $\left(\widetilde{M}, \omega_0\right)$. With this basis, each tangent vector $u \in T_{(x,y)}\widetilde{M}$ can be written as

$$u = \sum_i \{(u_H)_i(-J\partial_{y_i}) + (u_V)_i\partial_{y_i}\} = (-J\partial_y,\, \partial_y)\begin{pmatrix} u_H \\ u_V \end{pmatrix}.$$

Then, we have the following transition formula between $(\partial_x,\, \partial_y)$ and $(-J\partial_y,\, \partial_y)$

$$u = (-J\partial_y,\, \partial_y)\begin{pmatrix} u_H \\ u_V \end{pmatrix} = (\partial_x, \partial_y)\left(\begin{pmatrix} -XY^{-1} & Y + XY^{-1}X \\ -Y^{-1} & Y^{-1}X \end{pmatrix}\begin{pmatrix} 0 \\ u_H \end{pmatrix} + \begin{pmatrix} 0 \\ u_V \end{pmatrix}\right).$$

By this formula, we obtain the following lemma.

Lemma 3.4. *In terms of this notation, the Riemannian metric g defined by* (3.3) *can be described by*

$$g(u,v) = (0, {}^t u_H)\begin{pmatrix} Y^{-1} & -Y^{-1}X \\ -XY^{-1} & Y + XY^{-1}X \end{pmatrix}\begin{pmatrix} 0 \\ v_H \end{pmatrix} + (0, {}^t u_V)\begin{pmatrix} Y^{-1} & -Y^{-1}X \\ -XY^{-1} & Y + XY^{-1}X \end{pmatrix}\begin{pmatrix} 0 \\ v_V \end{pmatrix}.$$

Suppose that a group Γ acts on $\pi_0 \colon (\widetilde{M}, \omega_0) \to \mathbb{R}^n$ and the Γ-actions ρ on $\mathbb{R}^n$ and $\widetilde{\rho}$ on $(\widetilde{M}, \omega_0)$ are written as in (2.3). Then, it is easy to see the following lemma.

Lemma 3.5. *The Γ-action $\widetilde{\rho}$ on $(\widetilde{M}, \omega_0)$ preserves the almost complex structure $J = J_Z$ on $(\widetilde{M}, \omega_0)$ corresponding to $Z = X + \sqrt{-1}Y \in C^\infty\left(\widetilde{M}, \mathcal{S}_n\right)$ if and only if the following conditions hold*

$$A_\gamma(XY^{-1})_{(x,y)} = (XY^{-1})_{\widetilde{\rho}_\gamma(x,y)}A_\gamma - \left(Y + XY^{-1}X\right)_{\widetilde{\rho}_\gamma(x,y)}(Du_\gamma)_x \tag{3.4}$$

$$A_\gamma(Y + XY^{-1}X)_{(x,y)} = \left(Y + XY^{-1}X\right)_{\widetilde{\rho}_\gamma(x,y)}{}^t A_\gamma^{-1} \tag{3.5}$$

$$(Du_\gamma)_x(XY^{-1})_{(x,y)} + {}^t A_\gamma^{-1}Y^{-1}_{(x,y)} = Y^{-1}_{\widetilde{\rho}_\gamma(x,y)}A_\gamma - (Y^{-1}X)_{\widetilde{\rho}_\gamma(x,y)}(Du_\gamma)_x. \tag{3.6}$$

Proof. For all $\gamma \in \Gamma$ and $(x,y) \in (\widetilde{M}, \omega_0)$, the condition $(d\widetilde{\rho}_\gamma)_{(x,y)} \circ J_{(x,y)} = J_{\widetilde{\rho}_\gamma(x,y)} \circ (d\widetilde{\rho}_\gamma)_{(x,y)}$ implies above three equalities together with the following equality

$$(Du_\gamma)_x(Y + XY^{-1}X)_{(x,y)} + {}^t A_\gamma^{-1}(Y^{-1}X)_{(x,y)} = (Y^{-1}X)_{\widetilde{\rho}_\gamma(x,y)}{}^t A_\gamma^{-1}.$$

But, this can be obtained from (3.4), (3.5), and ${}^t\left({}^t A_\gamma(Du_\gamma)_x\right) = {}^t A_\gamma(Du_\gamma)_x$. □

Let $\pi \colon (M, \omega) \to B$ be a Lagrangian fibration with connected n-dimensional complete base B and $p \colon \widetilde{B} \to B$ the universal covering of B. By Corollary 2.25, the pullback of $\pi \colon (M, \omega) \to B$ to $\widetilde{B}$ is identified with $\pi_0 \colon (\widetilde{M}, \omega_0) \to \mathbb{R}^n$ and $\pi \colon (M, \omega) \to B$ can be obtained as the quotient of the $\Gamma = \pi_1(B)$-action on $\pi_0 \colon (\widetilde{M}, \omega_0) \to \mathbb{R}^n$. In particular, for each compatible almost complex structure J on (M, ω), there exists a map $Z_J = X + \sqrt{-1}Y \in C^\infty\left(\widetilde{M}, \mathcal{S}_n\right)$ such that the pullback p^*J of J to $p^*(M, \omega)$ coincides with J_{Z_J}. Then, we have the following lemma.

[4] $\begin{pmatrix} XY^{-1} & -Y - XY^{-1}X \\ Y^{-1} & -Y^{-1}X \end{pmatrix}_{(x,y)}$, $(XY^{-1})_{(x,y)}$ etc. are the values of the maps $\begin{pmatrix} XY^{-1} & -Y - XY^{-1}X \\ Y^{-1} & -Y^{-1}X \end{pmatrix}$, XY^{-1} etc. at (x, y). We will often omit the subscript "$_{(x,y)}$" for simplicity unless it causes confusion.

Lemma 3.6 ([14, Corollary 9.15])**.** *For any Lagrangian fibration $\pi\colon (M,\omega)\to B$, there exists a compatible almost complex structure J such that the corresponding map Z_J does not depend on $y_1,\dots,y_n$. We say such J to be invariant along the fiber.*

Proof. Take a Riemannian metric g' on (M,ω). Then, the pullback p^*g' is $\pi_1(B)$-invariant. Moreover, $p^*(M,\omega)$ admits a free T^n-action, and this T^n-action together with the $\pi_1(B)$-action forms an action of the semi-direct product $\pi_1(B)\ltimes T^n$ of T^n and $\pi_1(B)$. By averaging p^*g' over T^n, we obtain a Riemannian metric on p^*M invariant under the $\pi_1(B)\ltimes T^n$-action. It is easy to see that $p^*\omega$ is also $\pi_1(B)\ltimes T^n$-invariant, so by the standard method using the $\pi_1(B)\ltimes T^n$-invariant Riemannian metric and $p^*\omega$, we can obtain a $\pi_1(B)\ltimes T^n$-invariant compatible almost complex structure on $p^*(M,\omega)$. In particular, since the almost complex structure is still invariant under $\pi_1(B)$-action, it descends to (M,ω). This is the required almost complex structure. □

3.3. **The existence condition of non-trivial harmonic spinors of degree-zero.** For a map $Z = X+\sqrt{-1}Y \in C^\infty\left(\widetilde{M},\mathcal{S}_n\right)$, we set

$$\Omega := \left(Y+XY^{-1}X\right)^{-1} ZY^{-1}. \tag{3.7}$$

Ω has the following properties.

Lemma 3.7. (1) $\Omega=\overline{Z}^{-1}$*, where* $\overline{Z}=X-\sqrt{-1}Y$*.*
(2) Ω *is symmetric, i.e.,* ${}^t\Omega=\Omega$*.*

Proof. A direct computation shows that $\Omega\overline{Z}=I$. This proves (1). (2) follows from (1) since Z is symmetric. □

Let $N\in\mathbb{N}$ be a positive integer. Let $J=J_Z$ be the compatible almost complex structure on $\left(\widetilde{M},N\omega_0\right)$ corresponding to a given $Z=X+\sqrt{-1}Y\in C^\infty\left(\widetilde{M},\mathcal{S}_n\right)$. Then, the Riemannian metric $Ng:=N\omega_0(\cdot,\ J\cdot)$ defines an isomorphism $f\colon T^*\widetilde{M}\cong T\widetilde{M}$ by $\tau = Ng\left(f(\tau),\ \cdot\right)$ for $\tau\in T^*\widetilde{M}$. For $i=1,\dots,n$, let Ω_i denote the ith column vector of Ω, and $\operatorname{Re}\Omega_i$ and $\operatorname{Im}\Omega_i$ be the real and imaginary parts of Ω_i, respectively. Then, we can show the following lemma.

Lemma 3.8. *For* $i=1,\dots,n$,

$$f(dx_i) = -\frac{1}{N}J\partial_{y_i},\quad f(dy_i) = (-J\partial_y,\partial_y)\begin{pmatrix}\frac{1}{N}\operatorname{Re}\Omega_i\\ \frac{1}{N}\operatorname{Im}\Omega_i\end{pmatrix}.$$

Proof. We prove the latter. The former can be proved by the same way. Put $f(dy_i)=(-J\partial_y,\partial_y)\begin{pmatrix}Y_H^i\\ Y_V^i\end{pmatrix}$. By definition, for each $i,j=1,\dots,n$, we have

$$dy_i(-J\partial_{y_j}) = Ng\left((-J\partial_y,\partial_y)\begin{pmatrix}Y_H^i\\ Y_V^i\end{pmatrix},(-J\partial_y,\partial_y)\begin{pmatrix}e_j\\ 0\end{pmatrix}\right) \tag{3.8}$$

$$dy_i(\partial_{y_j}) = Ng\left((-J\partial_y,\partial_y)\begin{pmatrix}Y_H^i\\ Y_V^i\end{pmatrix},(-J\partial_y,\partial_y)\begin{pmatrix}0\\ e_j\end{pmatrix}\right). \tag{3.9}$$

Since $-J\partial_{y_j}$ is written as

$$-J\partial_{y_j} = (\partial_x,\partial_y)\begin{pmatrix}-XY^{-1} & Y+XY^{-1}X\\ -Y^{-1} & Y^{-1}X\end{pmatrix}\begin{pmatrix}0\\ e_j\end{pmatrix}$$

by (3.2), the left hand side of (3.8) is $\left(Y^{-1}X\right)_{ij}$. On the other hand, by Lemma 3.4, the right hand side of (3.8) can be described as $NY_H^i\cdot(Y+XY^{-1}X)e_j$. This implies $Y^{-1}X=N^t(Y_H^1\cdots Y_H^n)(Y+XY^{-1}X)$. Since Y is positive definite, so is $Y+XY^{-1}X$. In particular, $N(Y+XY^{-1}X)$ is invertible. By using ${}^tX=X$, ${}^tY=Y$ together with this fact, we can obtain $(Y_H^1\cdots Y_H^n)=\frac{1}{N}(Y+XY^{-1}X)^{-1}XY^{-1}$. By the same way, from (3.9), we obtain $I=N^t(Y_V^1\cdots Y_V^n)(Y+XY^{-1}X)$, i.e., $(Y_V^1\cdots Y_V^n)=\frac{1}{N}(Y+XY^{-1}X)^{-1}$. Hence, $\frac{1}{N}\Omega=(Y_H^1\cdots Y_H^n)+\sqrt{-1}(Y_V^1\cdots Y_V^n)$. □

Define the Hermitian metric on $(\widetilde{M}, N\omega_0, Ng, J)$ by

$$h(u,v) := Ng(u,v) + \sqrt{-1}Ng(u, Jv) \tag{3.10}$$

for $u, v \in T_{(x,y)}\widetilde{M}$. Let (W, c) be the Clifford module bundle associated with (Ng, J), i.e., as a complex vector bundle, W is defined by

$$W := \wedge^{\bullet}\left(T\widetilde{M}, J\right) \otimes_{\mathbb{C}} \left(\widetilde{L}^{\otimes N}\right).$$

W is equipped with the Hermitian metric induced from h and that on $\widetilde{L}$, and also equipped with the Hermitian connection, which is denoted by ∇^W, induced from the Levi-Civita connection ∇^{LC} of $\left(\widetilde{M}, g\right)$ and $\nabla^{\widetilde{L}}$. c is the Clifford multiplication $c\colon T\widetilde{M} \to \mathrm{End}_{\mathbb{C}}(W)$ defined by

$$c(u)(\tau) := u \wedge \tau - u \llcorner_h \tau$$

for $u \in T\widetilde{M}$ and $\tau \in W$, where $\llcorner_h$ is the contraction with respect to the Hermitian metric h on $(\widetilde{M}, N\omega_0, Ng, J)$. It is well known that W is identified with $\wedge^{\bullet}(T^*\widetilde{M})^{0,1} \otimes_{\mathbb{C}} \left(\widetilde{L}^{\otimes N}\right)$ as a Clifford module bundle.

Now let us define the Spin^c Dirac operator $D\colon \Gamma(W) \to \Gamma(W)$ by the composition of the following maps

$$D\colon \Gamma(W) \xrightarrow{\nabla^W} \Gamma(T^*\widetilde{M} \otimes W) \xrightarrow{f \otimes \mathrm{id}_W} \Gamma(T\widetilde{M} \otimes W) \xrightarrow{c} \Gamma(W).$$

We compute the action of D on a degree zero element in $\Gamma(W)$. We identify a section of $\widetilde{L}$ with a complex valued function on $\widetilde{M}$. By using Lemma 3.8, for a section s of $\widetilde{L}^{\otimes N}$, Ds can be computed as

$$\begin{aligned}
Ds &= c \circ (f \otimes \mathrm{id}_W) \circ \nabla^W s \\
&= c \circ (f \otimes \mathrm{id}_W)(ds - 2\pi\sqrt{-1}Nx \cdot dys) \\
&= \sum_{i=1}^{n} \left\{ c\left(f(dx_i)\right)\left(\partial_{x_i} s\right) + c\left(f(dy_i)\right)\left(\partial_{y_i} s - 2\pi\sqrt{-1}Nx_i s\right)\right\} \\
&= -\frac{\sqrt{-1}}{N} \sum_{i=1}^{n} \partial_{y_i} \otimes_{\mathbb{C}} \left\{ \partial_{x_i} s + \sum_{j=1}^{n} \Omega_{ij}\left(\partial_{y_j} s - 2\pi\sqrt{-1}Nx_j s\right)\right\}.
\end{aligned}$$

In particular, the equality $Ds = 0$ is equivalent to

$$0 = \begin{pmatrix} \partial_{x_1} s \\ \vdots \\ \partial_{x_n} s \end{pmatrix} + \Omega \begin{pmatrix} \partial_{y_1} s - 2\pi\sqrt{-1}Nx_1 s \\ \vdots \\ \partial_{y_n} s - 2\pi\sqrt{-1}Nx_n s \end{pmatrix}. \tag{3.11}$$

Suppose that Z does not depend on $y_1, \dots, y_n$ as in Lemma 3.6. Then, by substituting a Fourier expansion $s = \sum_{m \in \mathbb{Z}^n} a_m(x) e^{2\pi\sqrt{-1}m\cdot y}$ of s with respect to y_i's into (3.11), (3.11) can be reduced to the following system of differential equations for a_m's with variables $x_1, \dots, x_n$

$$0 = \begin{pmatrix} \partial_{x_1} a_m \\ \vdots \\ \partial_{x_n} a_m \end{pmatrix} + 2\pi\sqrt{-1}a_m \Omega(m - Nx) \tag{3.12}$$

for all $m \in \mathbb{Z}^n$.

Lemma 3.9. *Let a_m be a solution of* (3.12) *for some $m \in \mathbb{Z}^n$. If there exists $p \in \mathbb{R}^n$ such that $a_m(p) = 0$. Then, $a_m(x) = 0$ for all $x \in \mathbb{R}^n$.*

Proof. First, fix variables $x_2, \dots, x_n$ with $p_2, \dots, p_n$. Then, the first entry of (3.12), i.e., $0 = \partial_{x_1} a_m + 2\pi\sqrt{-1}a_m\left(\Omega(m - Nx)\right)_1$ can be thought of as an ordinary differential equation on x_1, and $a_m(x_1, p_2, \dots, p_n)$ is its solution with initial condition $a_m(p) = 0$. On the other hand, the trivial solution also has the same initial condition. By the uniqueness of the solution of the ordinary differential equation, $a_m(x_1, p_2, \dots, p_n) = 0$ for any x_1. Next, by fixing variables $x_3, \dots, x_n$ with $p_3, \dots, p_n$ and fixing x_1 with arbitrary value, $a_m(x_1, x_2, p_3, \dots, p_n)$ is a solution of $0 = \partial_{x_2} a_m + 2\pi\sqrt{-1}a_m\left(\Omega(m - Nx)\right)_2$ with initial condition $a_m(x_1, p_2, \dots, p_n) = 0$. Then, $a_m(x_1, x_2, p_3, \dots, p_n) = 0$ for any x_1, x_2. By repeating the process for $x_3, \dots, x_n$, we can show that $a_m(x) = 0$. $\square$

Lemma 3.10. *If a_m is a non trivial smooth solution of* (3.12) *for some $m \in \mathbb{Z}^n$, then, the condition*

$$((\partial_{x_i}\Omega)_x (m - Nx))_j = \left(\left(\partial_{x_j}\Omega\right)_x (m - Nx)\right)_i \text{ for all } i, j = 1, \dots, n, \text{ and all } x \in \mathbb{R}^n \tag{3.13}$$

holds.Conversely, if there exists $m \in \mathbb{Z}^n$ such that (3.13) *holds, then,* (3.12) *has a unique non trivial solution up to constant. Moreover, in this case, each solution a_m of* (3.12) *has the following form*

$$a_m(x) = a_m\left(\frac{m}{N}\right) e^{-2\pi\sqrt{-1}\sum_{i=1}^n G_m^i\left(\frac{m_1}{N}, \dots, \frac{m_{i-1}}{N}, x_i, \dots, x_n\right)}, \tag{3.14}$$

where $a_m\left(\frac{m}{N}\right)$ can be taken as an arbitrary constant in $\mathbb{C}$ and

$$G_m^i(x) := \left(\int_{\frac{m_i}{N}}^{x_i} \Omega(m - Nx) dx_i\right)_i.$$

Proof. Since a_m is smooth, a_m satisfies $\partial_{x_i}\partial_{x_j}a_m = \partial_{x_j}\partial_{x_i}a_m$ for all $i, j = 1, \dots, n$. By differentiating (3.12), we have

$$\partial_{x_i}\partial_{x_j}a_m = -2\pi\sqrt{-1}a_m\left\{-2\pi\sqrt{-1}\sum_{k=1}^n \Omega_{ik}(m_k - Nx_k)\sum_{l=1}^n \Omega_{jl}(m_l - Nx_l) + \sum_{l=1}^n (\partial_{x_i}\Omega_{jl})(m_l - Nx_l) - N\Omega_{ji}\right\}$$

for $i, j = 1, \dots, n$ and $x \in \mathbb{R}^n$. The condition (3.13) is obtained from this equation.

Conversely, suppose there exists $m \in \mathbb{Z}^n$ such that (3.13) holds. By solving the differential equation appeared as the ith component of (3.12) for $i = 1, \dots, n$, we have

$$a_m(x) = a_m\left(x_1, \dots, x_{i-1}, \frac{m_i}{N}, x_{i+1}, \dots, x_n\right) e^{-2\pi\sqrt{-1}G_m^i(x)}. \tag{3.15}$$

Using (3.15) recursively, we obtain the formula (3.14). By using (3.13), we can show that (3.14) does not depend on the order of applying (3.15) to x_i's as in the proof of Lemma 2.27. Hence, (3.14) is well-defined. $\square$

For each $m \in \mathbb{Z}^n$ for which the condition (3.13) holds, define the section $s_m \in \Gamma\left(\widetilde{L}^{\otimes N}\right)$ by

$$s_m(x, y) := e^{2\pi\sqrt{-1}\left\{-\sum_{i=1}^n G_m^i\left(\frac{m_1}{N}, \dots, \frac{m_{i-1}}{N}, x_i, \dots, x_n\right) + m \cdot y\right\}}. \tag{3.16}$$

By the elliptic regularity of D and Lemma 3.10, we can obtain the following.

Proposition 3.11. *If $s = \sum_{m \in \mathbb{Z}^n} a_m(x) e^{2\pi\sqrt{-1}m \cdot y} \in \Gamma\left(\widetilde{L}^{\otimes N}\right)$ is a non trivial solution of $0 = Ds$, then, the condition* (3.13) *holds for all $m \in \mathbb{Z}^n$ with $a_m \neq 0$. Conversely, suppose that there exists $m \in \mathbb{Z}^n$ such that* (3.13) *holds. Then, the section s_m defined by* (3.16) *satisfies $0 = Ds_m$. In particular, if* (3.13) *holds for all $m \in \mathbb{Z}^n$, then, $\{s_m\}_{m \in \mathbb{Z}^n}$ is a linear basis of $\Gamma\left(\widetilde{L}^{\otimes N}\right) \cap \ker D$.*

The following proposition gives a geometric interpretation of the condition (3.13).

Proposition 3.12. *The following conditions are equivalent:*
(1) *The condition* (3.13) *holds for all $m \in \mathbb{Z}^n$.*
(2) $\partial_{x_i}\Omega_{jk} = \partial_{x_j}\Omega_{ik}$ *for all $i, j, k = 1, \dots, n$.*
(3) $\nabla^{LC}J = 0$*, where ∇^{LC} is the Levi-Civita connection with respect to g.*

Proof. If (3.13) holds for all $m \in \mathbb{Z}^n$, then, by putting $m = 0$, we have $((\partial_{x_i}\Omega)_x x)_j = \left(\left(\partial_{x_j}\Omega\right)_x x\right)_i$. By substituting this to (3.13), we can see the condition $((\partial_{x_i}\Omega)_x m)_j = \left(\left(\partial_{x_j}\Omega\right)_x m\right)_i$ holds for all $m \in \mathbb{Z}^n$. In particular, by substituting $m = e_k$ to this condition for each $k = 1, \dots, n$, we can obtain (2). (2) $\Rightarrow$ (1) is trivial.

We show (2) $\Leftrightarrow$ (3). (2) is equivalent to the following two conditions

$$\left(\left(Y + XY^{-1}X\right)^{-1}\partial_{x_i}\left(XY^{-1}\right)\right)_{jk} = \left(\left(Y + XY^{-1}X\right)^{-1}\partial_{x_j}\left(XY^{-1}\right)\right)_{ik} \tag{3.17}$$

$$\partial_{x_i}\left(Y + XY^{-1}X\right)^{-1}_{jk} = \partial_{x_j}\left(Y + XY^{-1}X\right)^{-1}_{ik} \tag{3.18}$$

for $i, j, k = 1, \dots, n$. For $i = 1, \dots, 2n$, we set

$$\Gamma_i := \begin{pmatrix} \Gamma^1_{i\,1} & \cdots & \Gamma^1_{i\,2n} \\ \vdots & & \vdots \\ \Gamma^{2n}_{i\,1} & \cdots & \Gamma^{2n}_{i\,2n} \end{pmatrix},$$

where Γ^k_{ij} is the Christoffel symbol. Then, (3) is equivalent to

$$0 = \partial_i J + \Gamma_i J - J\Gamma_i \quad (i = 1, \dots, 2n),$$

where

$$\partial_i = \begin{cases} \partial_{x_i} & (i = 1, \dots, n) \\ \partial_{y_{i-n}} & (i = n+1, \dots, 2n). \end{cases}$$

It is also equivalent to the following conditions

$$\begin{aligned} & XY^{-1} \begin{pmatrix} \partial_{x_1}(XY^{-1})_{1i} & \cdots & \partial_{x_n}(XY^{-1})_{1i} \\ \vdots & & \vdots \\ \partial_{x_1}(XY^{-1})_{ni} & \cdots & \partial_{x_n}(XY^{-1})_{ni} \end{pmatrix} \\ & - (Y + XY^{-1}X) \begin{pmatrix} \partial_{x_1}(Y^{-1})_{1i} - \partial_{x_1}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{1i} - \partial_{x_1}(Y^{-1})_{ni} \\ \vdots & & \vdots \\ \partial_{x_1}(Y^{-1})_{ni} - \partial_{x_n}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{ni} - \partial_{x_n}(Y^{-1})_{ni} \end{pmatrix} \\ & = \begin{pmatrix} \partial_{x_1}(XY^{-1})_{1i} & \cdots & \partial_{x_n}(XY^{-1})_{1i} \\ \vdots & & \vdots \\ \partial_{x_1}(XY^{-1})_{ni} & \cdots & \partial_{x_n}(XY^{-1})_{ni} \end{pmatrix} XY^{-1}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} & Y^{-1} \begin{pmatrix} \partial_{x_1}(XY^{-1})_{1i} & \cdots & \partial_{x_n}(XY^{-1})_{1i} \\ \vdots & & \vdots \\ \partial_{x_1}(XY^{-1})_{ni} & \cdots & \partial_{x_n}(XY^{-1})_{ni} \end{pmatrix} \\ & - Y^{-1}X \begin{pmatrix} \partial_{x_1}(Y^{-1})_{1i} - \partial_{x_1}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{1i} - \partial_{x_1}(Y^{-1})_{ni} \\ \vdots & & \vdots \\ \partial_{x_1}(Y^{-1})_{ni} - \partial_{x_n}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{ni} - \partial_{x_n}(Y^{-1})_{ni} \end{pmatrix} \\ & = \begin{pmatrix} \partial_{x_1}(Y^{-1})_{1i} - \partial_{x_1}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{1i} - \partial_{x_1}(Y^{-1})_{ni} \\ \vdots & & \vdots \\ \partial_{x_1}(Y^{-1})_{ni} - \partial_{x_n}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{ni} - \partial_{x_n}(Y^{-1})_{ni} \end{pmatrix} XY^{-1} \\ & + \begin{pmatrix} \partial_{x_1}(Y^{-1}X)_{i1} & \cdots & \partial_{x_1}(Y^{-1}X)_{in} \\ \vdots & & \vdots \\ \partial_{x_n}(Y^{-1}X)_{i1} & \cdots & \partial_{x_n}(Y^{-1}X)_{in} \end{pmatrix} Y^{-1}, \end{aligned} \tag{3.20}$$

$$\begin{aligned} & (Y + XY^{-1}X) \begin{pmatrix} \partial_{x_1}(Y^{-1}X)_{i1} & \cdots & \partial_{x_1}(Y^{-1}X)_{in} \\ \vdots & & \vdots \\ \partial_{x_n}(Y^{-1}X)_{i1} & \cdots & \partial_{x_n}(Y^{-1}X)_{in} \end{pmatrix} \\ & = \begin{pmatrix} \partial_{x_1}(XY^{-1})_{1i} & \cdots & \partial_{x_n}(XY^{-1})_{1i} \\ \vdots & & \vdots \\ \partial_{x_1}(XY^{-1})_{ni} & \cdots & \partial_{x_n}(XY^{-1})_{ni} \end{pmatrix} (Y + XY^{-1}X), \end{aligned} \tag{3.21}$$

$$
\begin{aligned}
(3.22)\qquad & Y^{-1}X\begin{pmatrix}\partial_{x_1}(Y^{-1}X)_{i1} & \cdots & \partial_{x_1}(Y^{-1}X)_{in}\\ \vdots & & \vdots\\ \partial_{x_n}(Y^{-1}X)_{i1} & \cdots & \partial_{x_n}(Y^{-1}X)_{in}\end{pmatrix}\\
&=\begin{pmatrix}\partial_{x_1}(Y^{-1})_{1i}-\partial_{x_1}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{1i}-\partial_{x_1}(Y^{-1})_{ni}\\ \vdots & & \vdots\\ \partial_{x_1}(Y^{-1})_{ni}-\partial_{x_n}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{ni}-\partial_{x_n}(Y^{-1})_{ni}\end{pmatrix}(Y+XY^{-1})\\
&+\begin{pmatrix}\partial_{x_1}(Y^{-1}X)_{i1} & \cdots & \partial_{x_1}(Y^{-1}X)_{in}\\ \vdots & & \vdots\\ \partial_{x_n}(Y^{-1}X)_{i1} & \cdots & \partial_{x_n}(Y^{-1}X)_{in}\end{pmatrix}Y^{-1}X,
\end{aligned}
$$

$$
\begin{aligned}
(3.23)\qquad & XY^{-1}\begin{pmatrix}\partial_{x_1}(Y+XY^{-1}X)_{1i} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{1i}\\ \vdots & & \vdots\\ \partial_{x_1}(Y+XY^{-1}X)_{ni} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{ni}\end{pmatrix}\\
&+(Y+XY^{-1}X)\begin{pmatrix}-\partial_{x_1}(Y^{-1}X)_{1i}+\partial_{x_1}(XY^{-1})_{i1} & \cdots & -\partial_{x_n}(Y^{-1}X)_{1i}+\partial_{x_1}(XY^{-1})_{in}\\ \vdots & & \vdots\\ -\partial_{x_1}(Y^{-1}X)_{ni}+\partial_{x_n}(XY^{-1})_{i1} & \cdots & -\partial_{x_n}(Y^{-1}X)_{ni}+\partial_{x_n}(XY^{-1})_{in}\end{pmatrix}\\
&=\begin{pmatrix}\partial_{x_1}(Y+XY^{-1}X)_{1i} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{1i}\\ \vdots & & \vdots\\ \partial_{x_1}(Y+XY^{-1}X)_{ni} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{ni}\end{pmatrix}XY^{-1},
\end{aligned}
$$

$$
\begin{aligned}
(3.24)\qquad & Y^{-1}\begin{pmatrix}\partial_{x_1}(Y+XY^{-1}X)_{1i} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{1i}\\ \vdots & & \vdots\\ \partial_{x_1}(Y+XY^{-1}X)_{ni} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{ni}\end{pmatrix}\\
&+Y^{-1}X\begin{pmatrix}-\partial_{x_1}(Y^{-1}X)_{1i}+\partial_{x_1}(XY^{-1})_{i1} & \cdots & -\partial_{x_n}(Y^{-1}X)_{1i}+\partial_{x_1}(XY^{-1})_{in}\\ \vdots & & \vdots\\ -\partial_{x_1}(Y^{-1}X)_{ni}+\partial_{x_n}(XY^{-1})_{i1} & \cdots & -\partial_{x_n}(Y^{-1}X)_{ni}+\partial_{x_n}(XY^{-1})_{in}\end{pmatrix}\\
&=-\begin{pmatrix}-\partial_{x_1}(Y^{-1}X)_{1i}+\partial_{x_1}(XY^{-1})_{i1} & \cdots & -\partial_{x_n}(Y^{-1}X)_{1i}+\partial_{x_1}(XY^{-1})_{in}\\ \vdots & & \vdots\\ -\partial_{x_1}(Y^{-1}X)_{ni}+\partial_{x_n}(XY^{-1})_{i1} & \cdots & -\partial_{x_n}(Y^{-1}X)_{ni}+\partial_{x_n}(XY^{-1})_{in}\end{pmatrix}XY^{-1}\\
&+\begin{pmatrix}\partial_{x_1}(Y+XY^{-1}X)_{i1} & \cdots & \partial_{x_1}(Y+XY^{-1}X)_{in}\\ \vdots & & \vdots\\ \partial_{x_n}(Y+XY^{-1}X)_{i1} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{in}\end{pmatrix}Y^{-1},
\end{aligned}
$$

$$
\begin{aligned}
(3.25)\qquad & (Y+XY^{-1}X)\begin{pmatrix}\partial_{x_1}(Y+XY^{-1}X)_{i1} & \cdots & \partial_{x_1}(Y+XY^{-1}X)_{in}\\ \vdots & & \vdots\\ \partial_{x_n}(Y+XY^{-1}X)_{i1} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{in}\end{pmatrix}\\
&=\begin{pmatrix}\partial_{x_1}(Y+XY^{-1}X)_{1i} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{1i}\\ \vdots & & \vdots\\ \partial_{x_1}(Y+XY^{-1}X)_{ni} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{ni}\end{pmatrix}(Y+XY^{-1}X),
\end{aligned}
$$

$$
\begin{aligned}
(3.26)\quad & Y^{-1}X \begin{pmatrix} \partial_{x_1}(Y+XY^{-1}X)_{i1} & \cdots & \partial_{x_1}(Y+XY^{-1}X)_{in} \\ \vdots & & \vdots \\ \partial_{x_n}(Y+XY^{-1}X)_{i1} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{in} \end{pmatrix} \\
&= - \begin{pmatrix} -\partial_{x_1}(Y^{-1}X)_{1i} + \partial_{x_1}(XY^{-1})_{i1} & \cdots & -\partial_{x_n}(Y^{-1}X)_{1i} + \partial_{x_1}(XY^{-1})_{in} \\ \vdots & & \vdots \\ -\partial_{x_1}(Y^{-1}X)_{ni} + \partial_{x_n}(XY^{-1})_{i1} & \cdots & -\partial_{x_n}(Y^{-1}X)_{ni} + \partial_{x_n}(XY^{-1})_{in} \end{pmatrix} (Y+XY^{-1}X) \\
&+ \begin{pmatrix} \partial_{x_1}(Y+XY^{-1}X)_{i1} & \cdots & \partial_{x_1}(Y+XY^{-1}X)_{in} \\ \vdots & & \vdots \\ \partial_{x_n}(Y+XY^{-1}X)_{i1} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{in} \end{pmatrix} Y^{-1}X.
\end{aligned}
$$

for $i = 1, \ldots, n$. It is easy to see that (3.22) and (3.26) are obtained by transposing (3.19) and (3.23), respectively. First, we show that (3.17) is equivalent to (3.21). In fact, (3.17) implies

$$
\begin{pmatrix} \partial_{x_1}(XY^{-1})_{1k} & \cdots & \partial_{x_1}(XY^{-1})_{nk} \\ \vdots & & \vdots \\ \partial_{x_n}(XY^{-1})_{1k} & \cdots & \partial_{x_n}(XY^{-1})_{nk} \end{pmatrix} (Y+XY^{-1}X)^{-1}
$$

is symmetric for $k = 1, \ldots, n$. Since X, Y is symmetric, this implies (3.21). Next, we show (3.25) is equivalent to (3.18). (3.25) is equivalent to

$$
\begin{aligned}
(3.27)\quad & \begin{pmatrix} \partial_{x_1}(Y+XY^{-1}X)_{i1} & \cdots & \partial_{x_1}(Y+XY^{-1}X)_{in} \\ \vdots & & \vdots \\ \partial_{x_n}(Y+XY^{-1}X)_{i1} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{in} \end{pmatrix} (Y+XY^{-1}X)^{-1} \\
&= (Y+XY^{-1}X)^{-1} \begin{pmatrix} \partial_{x_1}(Y+XY^{-1}X)_{1i} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{1i} \\ \vdots & & \vdots \\ \partial_{x_1}(Y+XY^{-1}X)_{ni} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{ni} \end{pmatrix}.
\end{aligned}
$$

By computing the (j,k)-components of the both sides of (3.27), we obtain

$$
\sum_{l=1}^{n} \left(\partial_{x_j}(Y+XY^{-1}X)^{-1}_{kl}\right)(Y+XY^{-1}X)_{li} = \sum_{l=1}^{n} \left(\partial_{x_k}(Y+XY^{-1}X)^{-1}_{jl}\right)(Y+XY^{-1}X)_{li}
$$

for $i, j, k = 1, \ldots, n$. Here, we used

$$
\begin{aligned}
0 =& \partial_{x_j}\left(\left(Y+XY^{-1}X\right)\left(Y+XY^{-1}X\right)^{-1}\right) \\
=& \left(\partial_{x_j}\left(Y+XY^{-1}X\right)\right)\left(Y+XY^{-1}X\right)^{-1} + \left(Y+XY^{-1}X\right)\partial_{x_j}\left(Y+XY^{-1}X\right)^{-1}
\end{aligned}
$$

and so on. Thus,

$$
\begin{aligned}
\partial_{x_j}(Y+XY^{-1}X)^{-1}_{km} &= \sum_{i=1}^{n}\sum_{l=1}^{n} \partial_{x_j}\left((Y+XY^{-1}X)^{-1}_{kl}\right)(Y+XY^{-1}X)_{li}(Y+XY^{-1}X)^{-1}_{im} \\
&= \sum_{i=1}^{n}\sum_{l=1}^{n} \left(\partial_{x_k}(Y+XY^{-1}X)^{-1}_{jl}\right)(Y+XY^{-1}X)_{li}(Y+XY^{-1}X)^{-1}_{im} \\
&= \partial_{x_k}(Y+XY^{-1}X)^{-1}_{jm}.
\end{aligned}
$$

This implies (3.18). In particular, this means (3) $\Rightarrow$ (2).

We show (3.19), (3.20), (3.23), and (3.24) are obtained from (2). To show (3.23), it is sufficient to show

$$
\begin{aligned}
0 =&(Y+XY^{-1}X)^{-1}XY^{-1}\begin{pmatrix}\partial_{x_1}(Y+XY^{-1}X)_{1i} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{1i}\\ \vdots & & \vdots \\ \partial_{x_1}(Y+XY^{-1}X)_{ni} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{ni}\end{pmatrix}\\
&-\begin{pmatrix}\partial_{x_1}(Y^{-1}X)_{1i} & \cdots & \partial_{x_n}(Y^{-1}X)_{1i}\\ \vdots & & \vdots \\ \partial_{x_1}(Y^{-1}X)_{ni} & \cdots & \partial_{x_n}(Y^{-1}X)_{ni}\end{pmatrix}\\
&-(Y+XY^{-1}X)^{-1}\begin{pmatrix}\partial_{x_1}(Y+XY^{-1}X)_{1i} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{1i}\\ \vdots & & \vdots \\ \partial_{x_1}(Y+XY^{-1}X)_{ni} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{ni}\end{pmatrix}XY^{-1}\\
&+\begin{pmatrix}\partial_{x_1}(XY^{-1})_{i1} & \cdots & \partial_{x_1}(XY^{-1})_{in}\\ \vdots & & \vdots \\ \partial_{x_n}(XY^{-1})_{i1} & \cdots & \partial_{x_n}(XY^{-1})_{in}\end{pmatrix}.
\end{aligned}
\tag{3.28}
$$

Since Ω is symmetric, so is its real part $\operatorname{Re}\Omega = (Y+XY^{-1}X)^{-1}XY^{-1}$. By taking the real part of (2) we also have

$$\partial_{x_i}\left((Y+XY^{-1}X)^{-1}XY^{-1}\right)_{jk} = \partial_{x_j}\left((Y+XY^{-1}X)^{-1}XY^{-1}\right)_{ik}.$$

By using these as well as (3.17) and (3.18), the (j,k)-component of the first two terms of the right hand side of (3.28) can be computed as

$$
\begin{aligned}
&\sum_l\left((Y+XY^{-1}X)^{-1}XY^{-1}\right)_{jl}\partial_{x_k}(Y+XY^{-1}X)_{li}-\partial_{x_k}(Y^{-1}X)_{ji}\\
=&\sum_l\left(Y^{-1}X(Y+XY^{-1}X)^{-1}\right)_{jl}\partial_{x_k}(Y+XY^{-1}X)_{li}-\partial_{x_k}(Y^{-1}X)_{ji}\\
=&\partial_{x_k}\left(\sum_l\left(Y^{-1}X(Y+XY^{-1}X)^{-1}\right)_{jl}(Y+XY^{-1}X)_{li}\right)\\
&-\sum_l\left(\partial_{x_k}\left(Y^{-1}X(Y+XY^{-1}X)^{-1}\right)_{jl}\right)(Y+XY^{-1}X)_{li}-\partial_{x_k}(Y^{-1}X)_{ji}\\
=&-\sum_l\left(\partial_{x_k}\left(Y^{-1}X(Y+XY^{-1}X)^{-1}\right)_{jl}\right)(Y+XY^{-1}X)_{li}\\
=&-\sum_l\left(\partial_{x_j}\left((Y+XY^{-1}X)^{-1}XY^{-1}\right)_{kl}\right)(Y+XY^{-1}X)_{li}.
\end{aligned}
$$

On the other hand, the (j,k)-component of the last two terms of the right hand side of (3.28) can be computed as

$$
\begin{aligned}
&-\sum_{m,l}(Y+XY^{-1}X)^{-1}_{jl}\left(\partial_{x_m}(Y+XY^{-1}X)_{li}\right)(XY^{-1})_{mk}+\partial_{x_j}(XY^{-1})_{ik}\\
=&\sum_{m,l}\left(\partial_{x_m}(Y+XY^{-1}X)^{-1}_{jl}\right)(Y+XY^{-1}X)_{li}(XY^{-1})_{mk}+\partial_{x_j}(XY^{-1})_{ik}\\
=&\sum_{m,l}(Y+XY^{-1}X)_{li}\left(\partial_{x_j}(Y+XY^{-1}X)^{-1}_{ml}\right)(XY^{-1})_{mk}+\sum_{m,l}(Y+XY^{-1}X)_{li}(Y+XY^{-1}X)^{-1}_{ml}\partial_{x_j}(XY^{-1})_{mk}\\
=&\sum_l\left(\partial_{x_j}\left((Y+XY^{-1}X)^{-1}XY^{-1}\right)_{kl}\right)(Y+XY^{-1}X)_{li}.
\end{aligned}
$$

This proves (3.28). We show (3.24). We put

$$W := \begin{pmatrix} \partial_{x_1}(Y+XY^{-1}X)_{1i} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{1i} \\ \vdots & & \vdots \\ \partial_{x_1}(Y+XY^{-1}X)_{ni} & \cdots & \partial_{x_n}(Y+XY^{-1}X)_{ni} \end{pmatrix}.$$

By (3.23) and (3.25), we obtain

$$\begin{aligned} &\begin{pmatrix} -\partial_{x_1}(Y^{-1}X)_{1i} + \partial_{x_1}(XY^{-1})_{i1} & \cdots & -\partial_{x_n}(Y^{-1}X)_{1i} + \partial_{x_1}(XY^{-1})_{in} \\ \vdots & & \vdots \\ -\partial_{x_1}(Y^{-1}X)_{ni} + \partial_{x_n}(XY^{-1})_{i1} & \cdots & -\partial_{x_n}(Y^{-1}X)_{ni} + \partial_{x_n}(XY^{-1})_{in} \end{pmatrix} \\ &= (Y+XY^{-1}X)^{-1}WXY^{-1} - (Y+XY^{-1}X)^{-1}XY^{-1}W \end{aligned}$$

and

$$(Y+XY^{-1}X)^t W = W(Y+XY^{-1}X).$$

In order to show (3.24) it is sufficient to check

$$\begin{aligned} 0 =& Y^{-1}W + Y^{-1}X(Y+XY^{-1}X)^{-1}WXY^{-1} - Y^{-1}X(Y+XY^{-1}X)^{-1}XY^{-1}W \\ &+ (Y+XY^{-1}X)^{-1}WXY^{-1}XY^{-1} - (Y+XY^{-1}X)^{-1}XY^{-1}WXY^{-1} - {}^tWY^{-1}. \end{aligned} \tag{3.29}$$

By using above equalities, the right hand side of (3.29) can be computed as

$$\begin{aligned} &Y^{-1}W - Y^{-1}X(Y+XY^{-1}X)^{-1}XY^{-1}W + (Y+XY^{-1}X)^{-1}WXY^{-1}XY^{-1} - {}^tWY^{-1} \\ =& Y^{-1}W - (Y+XY^{-1}X)^{-1}XY^{-1}XY^{-1}W + (Y+XY^{-1}X)^{-1}WXY^{-1}XY^{-1} \\ &- (Y+XY^{-1}X)^{-1}W(Y+XY^{-1}X)Y^{-1} \\ =& Y^{-1}W - (Y+XY^{-1}X)^{-1}XY^{-1}XY^{-1}W + (Y+XY^{-1}X)^{-1}WXY^{-1}XY^{-1} \\ &- (Y+XY^{-1}X)^{-1}W - (Y+XY^{-1}X)^{-1}WXY^{-1}XY^{-1} \\ =& Y^{-1}W - \{(Y+XY^{-1}X)^{-1}XY^{-1}X + (Y+XY^{-1}X)^{-1}Y\}Y^{-1}W \\ =& 0. \end{aligned}$$

This proves (3.24).

We show (3.19). To see this, we show

$$\begin{aligned} 0 =& (Y+XY^{-1}X)^{-1}XY^{-1} \begin{pmatrix} \partial_{x_1}(XY^{-1})_{1i} & \cdots & \partial_{x_n}(XY^{-1})_{1i} \\ \vdots & & \vdots \\ \partial_{x_1}(XY^{-1})_{ni} & \cdots & \partial_{x_n}(XY^{-1})_{ni} \end{pmatrix} \\ &- \begin{pmatrix} \partial_{x_1}(Y^{-1})_{1i} - \partial_{x_1}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{1i} - \partial_{x_1}(Y^{-1})_{ni} \\ \vdots & & \vdots \\ \partial_{x_1}(Y^{-1})_{ni} - \partial_{x_n}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{ni} - \partial_{x_n}(Y^{-1})_{ni} \end{pmatrix} \\ &- (Y+XY^{-1}X)^{-1} \begin{pmatrix} \partial_{x_1}(XY^{-1})_{1i} & \cdots & \partial_{x_n}(XY^{-1})_{1i} \\ \vdots & & \vdots \\ \partial_{x_1}(XY^{-1})_{ni} & \cdots & \partial_{x_n}(XY^{-1})_{ni} \end{pmatrix} XY^{-1}. \end{aligned} \tag{3.30}$$

The (j,k)-component of the right hand side of (3.30) is

$$\sum_l \left((Y+XY^{-1}X)^{-1}XY^{-1}\right)_{jl}\partial_{x_k}(XY^{-1})_{li} - \partial_{x_k}Y^{-1}_{ji} + \partial_{x_j}Y^{-1}_{ki} - \sum_{l,m}(Y+XY^{-1}X)^{-1}_{jm}\partial_{x_l}(XY^{-1})_{mi}(XY^{-1})_{lk}$$
$$= \left((Y+XY^{-1}X)^{-1}XY^{-1}\partial_{x_k}(XY^{-1})\right)_{ji} - \partial_{x_k}Y^{-1}_{ji} + \partial_{x_j}Y^{-1}_{ki} - \sum_{l,m}(Y+XY^{-1}X)^{-1}_{lm}\partial_{x_j}(XY^{-1})_{mi}(XY^{-1})_{lk}$$
$$= \left((Y+XY^{-1}X)^{-1}\left\{\partial_{x_k}\left((Y+XY^{-1}X)Y^{-1}\right) - \partial_{x_k}(XY^{-1})XY^{-1}\right\}\right)_{ji} - \partial_{x_k}Y^{-1}_{ji} + \partial_{x_j}Y^{-1}_{ki}$$
$$- \sum_{l,m}(Y+XY^{-1}X)^{-1}_{ml}(XY^{-1})_{lk}\partial_{x_j}(XY^{-1})_{mi}$$
$$= \left((Y+XY^{-1}X)^{-1}\left(\partial_{x_k}(Y+XY^{-1}X)\right)Y^{-1} + \partial_{x_k}Y^{-1} - (Y+XY^{-1}X)^{-1}\partial_{x_k}(XY^{-1})XY^{-1}\right)_{ji}$$
$$- \partial_{x_k}Y^{-1}_{ji} + \partial_{x_j}Y^{-1}_{ki} - \sum_m \left((Y+XY^{-1}X)^{-1}XY^{-1}\right)_{mk}\partial_{x_j}(XY^{-1})_{mi}$$
$$= \left((Y+XY^{-1}X)^{-1}\left(\partial_{x_k}(Y+XY^{-1}X)\right)Y^{-1}\right)_{ji} - \left((Y+XY^{-1}X)^{-1}\partial_{x_k}(XY^{-1})XY^{-1}\right)_{ji} + \partial_{x_j}Y^{-1}_{ki}$$
$$- \sum_m \left((Y+XY^{-1}X)^{-1}XY^{-1}\right)_{km}\partial_{x_j}(XY^{-1})_{mi}$$
$$= \left((Y+XY^{-1}X)^{-1}\left(\partial_{x_k}(Y+XY^{-1}X)\right)Y^{-1}\right)_{ji} - \left((Y+XY^{-1}X)^{-1}\partial_{x_k}(XY^{-1})XY^{-1}\right)_{ji}$$
$$+ \left(\partial_{x_j}Y^{-1} - (Y+XY^{-1}X)^{-1}XY^{-1}\partial_{x_j}(XY^{-1})\right)_{ki}$$
$$= \left(-\left(\partial_{x_k}(Y+XY^{-1}X)^{-1}\right)(Y+XY^{-1}X)Y^{-1}\right)_{ji} - \sum_l \left((Y+XY^{-1}X)^{-1}\partial_{x_k}(XY^{-1})\right)_{jl}XY^{-1}_{li}$$
$$+ \left(\partial_{x_j}Y^{-1} - (Y+XY^{-1}X)^{-1}XY^{-1}\partial_{x_j}(XY^{-1})\right)_{ki}$$
$$= -\sum_l \partial_{x_k}(Y+XY^{-1}X)^{-1}_{jl}\left((Y+XY^{-1}X)Y^{-1}\right)_{li} - \sum_l \left((Y+XY^{-1}X)^{-1}\partial_{x_j}(XY^{-1})\right)_{kl}XY^{-1}_{li}$$
$$+ \left(\partial_{x_j}Y^{-1} - (Y+XY^{-1}X)^{-1}XY^{-1}\partial_{x_j}(XY^{-1})\right)_{ki}$$
$$= -\sum_l \partial_{x_j}(Y+XY^{-1}X)^{-1}_{kl}\left((Y+XY^{-1}X)Y^{-1}\right)_{li} - \left((Y+XY^{-1}X)^{-1}\partial_{x_j}(XY^{-1})XY^{-1}\right)_{ki}$$
$$+ \left(\partial_{x_j}Y^{-1} - (Y+XY^{-1}X)^{-1}XY^{-1}\partial_{x_j}(XY^{-1})\right)_{ki}$$
$$= \left(-\left(\partial_{x_j}(Y+XY^{-1}X)^{-1}\right)(Y+XY^{-1}X)Y^{-1}\right)_{ki} - \left((Y+XY^{-1}X)^{-1}\partial_{x_j}(XY^{-1})XY^{-1}\right)_{ki}$$
$$+ \left(\partial_{x_j}Y^{-1} - (Y+XY^{-1}X)^{-1}XY^{-1}\partial_{x_j}(XY^{-1})\right)_{ki}$$
$$= \left((Y+XY^{-1}X)^{-1}\left(\partial_{x_j}(Y+XY^{-1}X)\right)Y^{-1}\right)_{ki} - \left((Y+XY^{-1}X)^{-1}\partial_{x_j}(XY^{-1})XY^{-1}\right)_{ki}$$
$$+ \left(\partial_{x_j}Y^{-1} - (Y+XY^{-1}X)^{-1}XY^{-1}\partial_{x_j}(XY^{-1})\right)_{ki}$$
$$= \left((Y+XY^{-1}X)^{-1}\left\{\left(\partial_{x_j}(Y+XY^{-1}X)\right)Y^{-1} + (Y+XY^{-1}X)\partial_{x_j}Y^{-1} - \partial_{x_j}(XY^{-1})XY^{-1} - XY^{-1}\partial_{x_j}(XY^{-1})\right\}\right)_{ki}$$
$$= \left((Y+XY^{-1}X)^{-1}\left\{\partial_{x_j}((Y+XY^{-1}X)Y^{-1}) - \partial_{x_j}(XY^{-1}XY^{-1})\right\}\right)_{ki}$$
$$= 0.$$

This proves (3.19).

Finally, we show (3.20). We put

$$V := \begin{pmatrix} \partial_{x_1}(XY^{-1})_{1i} & \cdots & \partial_{x_n}(XY^{-1})_{1i} \\ \vdots & & \vdots \\ \partial_{x_1}(XY^{-1})_{ni} & \cdots & \partial_{x_n}(XY^{-1})_{ni} \end{pmatrix}.$$

By (3.19) and (3.21), we obtain

$$\begin{pmatrix} \partial_{x_1}(Y^{-1})_{1i} - \partial_{x_1}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{1i} - \partial_{x_1}(Y^{-1})_{ni} \\ \vdots & & \vdots \\ \partial_{x_1}(Y^{-1})_{ni} - \partial_{x_n}(Y^{-1})_{1i} & \cdots & \partial_{x_n}(Y^{-1})_{ni} - \partial_{x_n}(Y^{-1})_{ni} \end{pmatrix}$$
$$= (Y+XY^{-1}X)^{-1}XY^{-1}V - (Y+XY^{-1}X)^{-1}VXY^{-1}$$

and

$$(Y+XY^{-1}X)^tV = V(Y+XY^{-1}X).$$

In order to show (3.20) it is sufficient to check

$$\begin{aligned}0 =&Y^{-1}V - Y^{-1}X(Y+XY^{-1}X)^{-1}XY^{-1}V + Y^{-1}X(Y+XY^{-1}X)^{-1}VXY^{-1}\\ &+ (Y+XY^{-1}X)^{-1}VXY^{-1}XY^{-1} - (Y+XY^{-1}X)^{-1}XY^{-1}VXY^{-1} - {}^tVY^{-1}.\end{aligned} \tag{3.31}$$

Then, (3.31) can be checked in the same way as (3.29). □

Remark 3.13. When one of (hence, all) the conditions in Proposition 3.12 holds, $\left(\widetilde{M},\omega_0,J,g\right)$ is a Kähler manifold and J induces a natural holomorphic structure on $\widetilde{L}$ such that $\nabla^{\widetilde{L}}$ is the canonical connection.

3.4. **The Γ-equivariant case.** Suppose that $\pi_0\colon (\widetilde{M},N\omega_0,J)\to\mathbb{R}^n$ with prequantum line bundle $\left(\widetilde{L},\nabla^{\widetilde{L}}\right)^{\otimes N}\to(\widetilde{M},N\omega_0,J)$ is equipped with an action of a group Γ which preserves all the data, and the Γ-actions are described by (2.3) and (2.5) as before. We assume that the Γ-action ρ on $\mathbb{R}^n$ is properly discontinuous and free. Since the Γ-action preserves all the data, the Spin^c Dirac operator D is Γ-equivariant. In particular, Γ acts on $\Gamma\left(\widetilde{L}^{\otimes N}\right)\cap\ker D$.

Lemma 3.14. *Let $s=\sum_{m\in\mathbb{Z}^n}a_m(x)e^{2\pi\sqrt{-1}m\cdot y}$ be a section of $\widetilde{L}^{\otimes N}$. s is Γ-equivariant, i.e., $\widetilde{\widetilde{\rho}}_\gamma\circ s = s\circ\widetilde{\rho}_\gamma$ for all $\gamma\in\Gamma$ if and only if a_m satisfies the following condition*

$$a_{N\rho_\gamma\left(\frac{m}{N}\right)}(\rho_\gamma(x)) = g_\gamma a_m(x)e^{2\pi\sqrt{-1}N\left\{\widetilde{g}_\gamma(x)-\rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)\right\}} \tag{3.32}$$

for all $\gamma\in\Gamma$, $m\in\mathbb{Z}^n$, and $x\in\mathbb{R}^n$. In particular, any Γ-equivariant section of $\widetilde{L}^{\otimes N}$ can be written as follows

$$s(x,y) = \sum_{\left(\gamma,\frac{m}{N}\right)\in\Gamma\times\left(F\cap\frac{1}{N}\mathbb{Z}^n\right)} g_\gamma a_m\left(\rho_{\gamma^{-1}}(x)\right)e^{2\pi\sqrt{-1}N\left\{\widetilde{g}_\gamma\left(\rho_{\gamma^{-1}}(x)\right)-\rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma\left(\rho_{\gamma^{-1}}(x)\right)\right\}}e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot y}. \tag{3.33}$$

Proof. By computing the both sides separately, we have

$$\begin{aligned}\widetilde{\widetilde{\rho}}_\gamma\circ s(x,y) &= g_\gamma e^{2\pi\sqrt{-1}N\left\{\widetilde{g}_\gamma(x)+c_\gamma\cdot{}^tA_\gamma^{-1}y\right\}}\sum_{m\in\mathbb{Z}^n}a_m(x)e^{2\pi\sqrt{-1}m\cdot y}\\ &= g_\gamma\sum_{m\in\mathbb{Z}^n}a_m(x)e^{2\pi\sqrt{-1}N\widetilde{g}_\gamma(x)}e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot{}^tA_\gamma^{-1}y},\end{aligned} \tag{3.34}$$

$$\begin{aligned}s\circ\widetilde{\rho}_\gamma(x,y) &= \sum_{l\in\mathbb{Z}^n}a_l\left(\rho_\gamma(x)\right)e^{2\pi\sqrt{-1}l\cdot\left({}^tA_\gamma^{-1}y+u_\gamma(x)\right)}\\ &= \sum_{m\in\mathbb{Z}^n}a_{N\rho_\gamma\left(\frac{m}{N}\right)}\left(\rho_\gamma(x)\right)e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)}e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot{}^tA_\gamma^{-1}y}.\end{aligned} \tag{3.35}$$

Here, in the last equality, we replace l with $N\rho_\gamma\left(\frac{m}{N}\right)$. Note that the map $\mathbb{Z}^n\ni m\mapsto N\rho_\gamma\left(\frac{m}{N}\right)\in\mathbb{Z}^n$ is bijective. Then, $\widetilde{\widetilde{\rho}}_\gamma\circ s = s\circ\widetilde{\rho}_\gamma$ for all $\gamma\in\Gamma$ implies

$$g_\gamma a_m(x)e^{2\pi\sqrt{-1}N\widetilde{g}_\gamma(x)} = a_{N\rho_\gamma\left(\frac{m}{N}\right)}\left(\rho_\gamma(x)\right)e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)}$$

for all $m\in\mathbb{Z}^n$. In particular, by (3.1) and (3.32), s can be rewritten as follows

$$\begin{aligned}s(x,y) &= \sum_{l\in\mathbb{Z}^n}a_l(x)e^{2\pi\sqrt{-1}l\cdot y}\\ &\overset{(3.1)}{=} \sum_{\left(\gamma,\frac{m}{N}\right)\in\Gamma\times\left(F\cap\frac{1}{N}\mathbb{Z}^n\right)} a_{N\rho_\gamma\left(\frac{m}{N}\right)}(x)e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot y}\\ &\overset{(3.32)}{=} \sum_{\left(\gamma,\frac{m}{N}\right)\in\Gamma\times\left(F\cap\frac{1}{N}\mathbb{Z}^n\right)} g_\gamma a_m\left(\rho_{\gamma^{-1}}(x)\right)e^{2\pi\sqrt{-1}N\left\{\widetilde{g}_\gamma\left(\rho_{\gamma^{-1}}(x)\right)-\rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma\left(\rho_{\gamma^{-1}}(x)\right)\right\}}e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot y}.\end{aligned}$$

□

In the Γ-equivariant case, the condition (3.13) has a symmetry in the following sense.

Lemma 3.15. *The condition* (3.13) *holds for some* $m_0 \in \mathbb{Z}^n$ *with* $\frac{m_0}{N} \in F$ *if and only if for any* $\gamma \in \Gamma$, (3.13) *holds for* $m = N\rho_\gamma\left(\frac{m_0}{N}\right)$. *Moreover, let* a_{m_0} *be a non trivial solution of* (3.12) *for* m_0. *For each* $\gamma \in \Gamma$, *we define* $a_{N\rho_\gamma\left(\frac{m_0}{N}\right)}$ *in such a way that it satisfies* (3.32). *Then,* $a_{N\rho_\gamma\left(\frac{m_0}{N}\right)}$ *is a non trivial solution of* (3.12) *for* $m = N\rho_\gamma\left(\frac{m_0}{N}\right)$.

Proof. Suppose that there exists $m_0 \in \mathbb{Z}^n$ with $\frac{m_0}{N} \in F$ such that (3.13) holds. By Lemma 3.10, (3.12) for m_0 has a non trivial solution a_{m_0}. Then, for each $\gamma \in \Gamma$, define $a_{N\rho_\gamma\left(\frac{m_0}{N}\right)}$ by (3.32). By Lemma 3.10 again, in order to show this lemma, it is sufficient to prove $a_{N\rho_\gamma\left(\frac{m_0}{N}\right)}$ is a solution of (3.12) for $m = N\rho_\gamma\left(\frac{m_0}{N}\right)$. Let us compute the Jacobi matrix of the both sides of (3.32). The left hand side is

$$D\left(a_{N\rho_\gamma\left(\frac{m_0}{N}\right)} \circ \rho_\gamma\right)_x = \left(Da_{N\rho_\gamma\left(\frac{m_0}{N}\right)}\right)_{\rho_\gamma(x)} (D\rho_\gamma)_x = \left(\partial_{x_1} a_{N\rho_\gamma\left(\frac{m_0}{N}\right)}, \dots, \partial_{x_n} a_{N\rho_\gamma\left(\frac{m_0}{N}\right)}\right)_{\rho_\gamma(x)} A_\gamma. \tag{3.36}$$

The right hand side is

$$\begin{aligned} &D\left(g_\gamma a_m(x) e^{2\pi\sqrt{-1}N\left\{\widetilde{g}_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)\right\}}\right)_x \\ &= g_\gamma e^{2\pi\sqrt{-1}N\left\{\widetilde{g}_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)\right\}} (Da_m)_x + g_\gamma a_m(x) D\left(e^{2\pi\sqrt{-1}N\left\{\widetilde{g}_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)\right\}}\right) \\ &\overset{(3.12)}{=} -2\pi\sqrt{-1} g_\gamma e^{2\pi\sqrt{-1}N\left\{\widetilde{g}_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)\right\}} a_m(x)^t \left(\Omega_x(m - Nx)\right) \\ &\quad + 2\pi\sqrt{-1} N g_\gamma a_m(x) e^{2\pi\sqrt{-1}N\left\{\widetilde{g}_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)\right\}} D\left(\widetilde{g}_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)\right) \\ &\overset{(3.32)}{=} -2\pi\sqrt{-1} a_{N\rho_\gamma\left(\frac{m_0}{N}\right)}(\rho_\gamma(x))\, {}^t\left(\Omega_x A_\gamma^{-1}\left(N\rho_\gamma\left(\frac{m}{N}\right) - N\rho_\gamma(x)\right)\right) \\ &\quad + 2\pi\sqrt{-1} N a_{N\rho_\gamma\left(\frac{m_0}{N}\right)}(\rho_\gamma(x))\, D\left(\widetilde{g}_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)\right). \end{aligned} \tag{3.37}$$

For each $i = 1, \dots, n$, the direct computation shows

$$\begin{aligned} &\partial_{x_i}\left(\widetilde{g}_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)\right) \\ &= (\partial_{x_i} u_\gamma)_x \cdot \left(\rho_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\right) + \left({}^t A_\gamma u_\gamma(x)\right)_i - \left({}^t A_\gamma u_\gamma(0, \dots, 0, x_i, \dots, x_n)\right)_i \\ &\quad - \sum_{j<i} \int_0^{x_j} \left({}^t A_\gamma D u_\gamma\right)_{ji} (0, \dots, 0, x_j, \dots, x_n) dx_j \\ &= (\partial_{x_i} u_\gamma)_x \cdot \left(\rho_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\right) \\ &\quad + \sum_{j<i} \int_0^{x_j} \partial_{x_j}\left({}^t A_\gamma u_\gamma(0, \dots, 0, x_j, \dots, x_n)\right)_i dx_j - \sum_{j<i} \int_0^{x_j} \left({}^t A_\gamma D u_\gamma\right)_{ji} (0, \dots, 0, x_j, \dots, x_n) dx_j \\ &= (\partial_{x_i} u_\gamma)_x \cdot \left(\rho_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\right) \\ &\quad + \sum_{j<i} \int_0^{x_j} \left({}^t A_\gamma D u_\gamma\right)_{ij} (0, \dots, 0, x_j, \dots, x_n) dx_j - \sum_{j<i} \int_0^{x_j} \left({}^t A_\gamma D u_\gamma\right)_{ji} (0, \dots, 0, x_j, \dots, x_n) dx_j \\ &= -(\partial_{x_i} u_\gamma)_x \cdot \left(\rho_\gamma\left(\frac{m}{N}\right) - \rho_\gamma(x)\right). \end{aligned}$$

In the last equality, we used ${}^t\left({}^t A_\gamma D u_\gamma\right) = {}^t A_\gamma D u_\gamma$. Hence, we have

$$D\left(\widetilde{g}_\gamma(x) - \rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(x)\right) = -{}^t\left(\rho_\gamma\left(\frac{m}{N}\right) - \rho_\gamma(x)\right)(D u_\gamma)_x. \tag{3.38}$$

By (3.36), (3.37), and (3.38), we obtain

$${}^t A_\gamma {}^t\left(Da_{N\rho_\gamma\left(\frac{m_0}{N}\right)}\right)_{\rho_\gamma(x)} = -2\pi\sqrt{-1} a_{N\rho_\gamma\left(\frac{m_0}{N}\right)}(\rho_\gamma(x))\left(\Omega_x A_\gamma^{-1} + {}^t(D u_\gamma)_x\right)\left(N\rho_\gamma\left(\frac{m}{N}\right) - N\rho_\gamma(x)\right).$$

On the other hand, by (3.4) and (3.5), we have

$$ {}^t A_\gamma \Omega_{\rho_\gamma(x)} = \Omega_x A_\gamma^{-1} + {}^t \left(Du_\gamma\right)_x . \tag{3.39} $$

This proves the lemma. □

Remark 3.16. By Remark 3.3 and Lemma 3.15, the condition (3.13) holds for all $\frac{m}{N} \in F \cap \frac{1}{N}\mathbb{Z}^n$ if and only if the condition (1), hence all conditions in Proposition 3.12 holds.

4. The integrable case

4.1. **Definition and properties of $\vartheta_{\frac{m}{N}}$.** We use the settting and the notations introduced in the previous section. Let $\frac{m}{N} \in F \cap \frac{1}{N}\mathbb{Z}^n$ be the point for which the condition (3.13) holds, and a_m the non trivial solution of (3.12) of the form (3.14) with $a_m\left(\frac{m}{N}\right) = 1$. For each $\gamma \in \Gamma$, define $a_{N\rho_\gamma\left(\frac{m}{N}\right)}$ in such a way that it satisfies (3.32). As we showed in Lemma 3.15, $a_{N\rho_\gamma\left(\frac{m}{N}\right)}$ is a non trivial solution of (3.12) for $N\rho_\gamma\left(\frac{m}{N}\right)$. Then, we can define the formal Fourier series $\vartheta_{\frac{m}{N}}$ by

$$ \vartheta_{\frac{m}{N}}(x,y) := \sum_{\gamma\in\Gamma} a_{N\rho_\gamma\left(\frac{m}{N}\right)}(x) e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot y}. \tag{4.1} $$

Proposition 4.1. (1) *$\vartheta_{\frac{m}{N}}$ has the following expression*

$$ \vartheta_{\frac{m}{N}}(x,y) = \sum_{\gamma\in\Gamma} g_\gamma e^{2\pi\sqrt{-1}\left[-\sum_{i=1}^n G_m^i\left(\frac{m_1}{N},\dots,\frac{m_{i-1}}{N},\left(\rho_{\gamma^{-1}}(x)\right)_i,\dots,\left(\rho_{\gamma^{-1}}(x)\right)_n\right)+N\left\{\widetilde{g}_\gamma\left(\rho_{\gamma^{-1}}(x)\right)-\rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma\left(\rho_{\gamma^{-1}}(x)\right)+\rho_\gamma\left(\frac{m}{N}\right)\cdot y\right\}\right]}. $$

(2) *$\vartheta_{\frac{m}{N}}$ can be described as $\vartheta_{\frac{m}{N}} = \sum_{\gamma\in\Gamma} \widetilde{\widetilde{\rho}}_\gamma \circ s_m \circ \widetilde{\rho}_{\gamma^{-1}}$, where s_m is the section defined by (3.16).*
(3) *If $Y + XY^{-1}X$ is constant, then, $\vartheta_{\frac{m}{N}}$ converges absolutely and uniformly on any compact set.*

Proof. (1) and (2) are obtained by (3.32), (3.14), (2.5), and (3.16). Let us prove (3). By (2.4) and (3.5), we obtain

$$ {}^t A_{\gamma^{-1}} \left(Y + XY^{-1}X\right)^{-1} A_{\gamma^{-1}} = \left(Y + XY^{-1}X\right)^{-1}. $$

By using this formula together with the assumption, the expression in (1) can be rewritten as

$$ \vartheta_{\frac{m}{N}}(x,y) = \sum_{\gamma\in\Gamma} g_\gamma e^{2\pi\sqrt{-1}\left[\frac{\sqrt{-1}N}{2}\left(x-\rho_\gamma\left(\frac{m}{N}\right)\right)\cdot\left(Y+XY^{-1}X\right)^{-1}\left(x-\rho_\gamma\left(\frac{m}{N}\right)\right)+\text{real part}\right]}. $$

Since $\left(Y + XY^{-1}X\right)^{-1}$ is positive definite, there exists a positive constant $c > 0$ such that $\left(Y + XY^{-1}X\right)^{-1} \geq cI$. Then,

$$ \begin{aligned} &\left| g_\gamma e^{2\pi\sqrt{-1}\left[\frac{\sqrt{-1}N}{2}\left(x-\rho_\gamma\left(\frac{m}{N}\right)\right)\cdot\left(Y+XY^{-1}X\right)^{-1}\left(x-\rho_\gamma\left(\frac{m}{N}\right)\right)+\text{real part}\right]} \right| \\ &= e^{-N\pi\left(x-\rho_\gamma\left(\frac{m}{N}\right)\right)\cdot\left(Y+XY^{-1}X\right)^{-1}\left(x-\rho_\gamma\left(\frac{m}{N}\right)\right)} \\ &\leq e^{-cN\pi\|x-\rho_\gamma\left(\frac{m}{N}\right)\|^2} \\ &= e^{-cN\pi\|x-\frac{l}{N}\|^2} \quad (\text{put } l := N\rho_\gamma\left(\frac{m}{N}\right)) \\ &= \prod_{i=1}^n e^{-cN\pi\left(x_i-\frac{l_i}{N}\right)^2}. \end{aligned} $$

Hence, the series is dominated by $\prod_{i=1}^n \sum_{l_i\in\mathbb{Z}} e^{-cN\pi(\frac{l_i}{N}-x_i)^2}$. Any compact set is contained in a product of closed intervals $I_1 \times \cdots \times I_n$, so it is sufficient to show that $\sum_{l\in\mathbb{Z}} e^{-cN\pi(\frac{l}{N}-x)^2}$ converges uniformly on any closed interval I. Suppose that I is of the form $I := [x_m, x_M]$. Set $l_M := \max\{l \in \mathbb{Z} \mid \frac{l}{N} \in I\}$ and

$l_m := \min\{l \in \mathbb{Z} \mid \frac{l}{N} \in I\}$. On I, $\sum_{-k \le l \le k} e^{-cN\pi(\frac{l}{N}-x)^2}$ can be estimated as

$$\begin{aligned}\sum_{-k\le l\le k} e^{-cN\pi(\frac{l}{N}-x)^2} &= \left(\sum_{-k\le l<l_m} + \sum_{l_m\le l\le l_M} + \sum_{l_M\le l\le k}\right) e^{-cN\pi(\frac{l}{N}-x)^2}\\ &\le \sum_{-k\le l<l_m} e^{\frac{-c\pi}{N}(l-Nx_m)^2} + (l_M - l_n + 1) + \sum_{l_M<l\le k} e^{\frac{-c\pi}{N}(l-Nx_M)^2}\\ &\le \int_{-k}^{l_m} e^{\frac{-c\pi}{N}(\tau-Nx_m)^2}d\tau + (l_M - l_n + 1) + \int_{l_M}^{k} e^{\frac{-c\pi}{N}(\tau-Nx_M)^2}d\tau,\end{aligned}$$

and it is well-known that both $\int_{-k}^{l_m} e^{\frac{-c\pi}{N}(\tau-Nx_m)^2}d\tau$ and $\int_{l_M}^{k} e^{\frac{-c\pi}{N}(\tau-Nx_M)^2}d\tau$ converges as $k \to +\infty$. □

Theorem 4.2. *If $s \in \Gamma\left(\widetilde{L}^{\otimes N}\right)$ is a non trivial Γ-equivariant solution of $0 = Ds$, then, there exists $m \in \mathbb{Z}^n$ with $\frac{m}{N} \in F$ such that the condition (3.13) holds. Conversely, suppose that the condition (3.13) holds for some $\frac{m}{N} \in F \cap \frac{1}{N}\mathbb{Z}^n$ and $\vartheta_{\frac{m}{N}}$ converges absolutely and uniformly. Then, $\vartheta_{\frac{m}{N}}$ is a non trivial Γ-equivariant solution of $0 = Ds$. In particular, if J is integrable and $\vartheta_{\frac{m}{N}}$ converges absolutely and uniformly for any $\frac{m}{N} \in F\cap\frac{1}{N}\mathbb{Z}^n$, then, $\left\{\vartheta_{\frac{m}{N}}\right\}_{\frac{m}{N}\in F\cap\frac{1}{N}\mathbb{Z}^n}$ is a basis of the space of Γ-equivariant holomorphic sections of $\left(\widetilde{L}, \nabla^{\widetilde{L}}\right)^{\otimes N} \to (\widetilde{M}, N\omega_0, J)$.*

Proof. Since $s = \sum_{l\in\mathbb{Z}^n} a_l(x)e^{2\pi\sqrt{-1}l\cdot y}$ is non trivial solution of $0 = Ds$, by Proposition 3.11, there exists $l \in \mathbb{Z}^n$ such that $a_l \neq 0$. On the other hand, as is noticed in Remark 3.3, there exists $\left(\gamma, \frac{m}{N}\right) \in \Gamma \times \left(F \cap \frac{1}{N}\mathbb{Z}^n\right)$ such that $l = N\rho_\gamma\left(\frac{m}{N}\right)$. Since s is Γ-equivariant, by (3.32), $0 \neq a_l = a_{N\rho_\gamma\left(\frac{m}{N}\right)}$ implies $a_m \neq 0$.

Let us prove the latter. It is trivial that $\left\{\vartheta_{\frac{m}{N}}\right\}_{\frac{m}{N}\in F\cap\frac{1}{N}\mathbb{Z}^n}$ is linear independent. Let s be a Γ-equivariant holomorphic section of $\widetilde{L}^{\otimes N}$. By Lemma 3.14, s can be written as in (3.33). Then,

$$\begin{aligned} s(x,y) &\overset{(3.33)}{=} \sum_{\left(\gamma,\frac{m}{N}\right)\in\Gamma\times\left(F\cap\frac{1}{N}\mathbb{Z}^n\right)} g_\gamma a_m\left(\rho_{\gamma^{-1}}(x)\right) e^{2\pi\sqrt{-1}N\left\{\widetilde{g}_\gamma\left(\rho_{\gamma^{-1}}(x)\right)-\rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma\left(\rho_{\gamma^{-1}}(x)\right)+\rho_\gamma\left(\frac{m}{N}\right)\cdot y\right\}}\\ &\overset{(3.14)}{=} \sum_{\left(\gamma,\frac{m}{N}\right)\in\Gamma\times\left(F\cap\frac{1}{N}\mathbb{Z}^n\right)} g_\gamma a_m\left(\frac{m}{N}\right) e^{2\pi\sqrt{-1}\left[-\sum_{i=1}^n G_m^i\left(\frac{m_1}{N},\dots,\frac{m_{i-1}}{N},\left(\rho_{\gamma^{-1}}(x)\right)_i,\dots,\left(\rho_{\gamma^{-1}}(x)\right)_n\right)\right.}\\ &\qquad\qquad {}^{\left.+N\left\{\widetilde{g}_\gamma\left(\rho_{\gamma^{-1}}(x)\right)-\rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma\left(\rho_{\gamma^{-1}}(x)\right)+\rho_\gamma\left(\frac{m}{N}\right)\cdot y\right\}\right]}\\ &= \sum_{\frac{m}{N}\in F\cap\frac{1}{N}\mathbb{Z}^n} a_m\left(\frac{m}{N}\right)\sum_{\gamma\in\Gamma} g_\gamma e^{2\pi\sqrt{-1}\left[-\sum_{i=1}^n G_m^i\left(\frac{m_1}{N},\dots,\frac{m_{i-1}}{N},\left(\rho_{\gamma^{-1}}(x)\right)_i,\dots,\left(\rho_{\gamma^{-1}}(x)\right)_n\right)\right.}\\ &\qquad\qquad {}^{\left.+N\left\{\widetilde{g}_\gamma\left(\rho_{\gamma^{-1}}(x)\right)-\rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma\left(\rho_{\gamma^{-1}}(x)\right)+\rho_\gamma\left(\frac{m}{N}\right)\cdot y\right\}\right]}\\ &= \sum_{\frac{m}{N}\in F\cap\frac{1}{N}\mathbb{Z}^n} a_m\left(\frac{m}{N}\right)\vartheta_{\frac{m}{N}}(x,y). \end{aligned}$$

This proves the theorem. □

By Corollary 2.25, any Lagrangian fibration $\pi\colon (M,\omega) \to B$ on a connected complete base B with prequantum line bundle $(L,\nabla^L) \to (M,\omega)$ are obtained as the quotient of the action of $\Gamma := \pi_1(B)$. Let J be a compatible almost complex structure on (M,ω) which is invariant along the fiber in the sense of Lemma 3.6 and D^M the associated Spin^c Dirac operator on $(M, N\omega)$ with coefficients in $L^{\otimes N}$. Since the Γ-action preserves all the data, $\Gamma(L^{\otimes N}) \cap \ker D^M$ is identified with $\left(\Gamma\left(\widetilde{L}^{\otimes N}\right) \cap \ker D\right)^\Gamma$. Moreover, $F \cap \frac{1}{N}\mathbb{Z}^n$ is identified with B_{BS} as is noticed in Remark 3.3. Thus, we obtain the following corollary.

Corollary 4.3. *Let $\pi\colon (M,\omega) \to B$ be a Lagrangian fibration on a connected complete base B and $(L,\nabla^L) \to (M,\omega)$ a prequantum line bundle. Let J be a compatible almost complex structure on (M,ω) which is invariant along the fiber in the sense of Lemma 3.6. Assume that J is integrable and $\vartheta_{\frac{m}{N}}$*

converges absolutely and uniformly for each $\frac{m}{N} \in F \cap \frac{1}{N}\mathbb{Z}^n$*. Then,* $\left\{\vartheta_{\frac{m}{N}}\right\}_{\frac{m}{N}\in F\cap\frac{1}{N}\mathbb{Z}^n}$ *gives a basis of the space of holomorphic sections of* $(L, \nabla^L)^{\otimes N} \to (M, N\omega, J)$ *indexed by the Bohr-Sommerfeld points.*

Remark 4.4. When M is compact as well as the assumption of Corollary 4.3, we choose the orientation on M so that $(-1)^{\frac{n(n-1)}{2}}\frac{(N\omega)^n}{n!}$ is a positive volume form, and define the Hermitian inner product of the space of sections of $L^{\otimes N}$ by

$$(s, s')_{L\otimes L} := \int_M \langle s, s'\rangle_{L^{\otimes N}} (-1)^{\frac{n(n-1)}{2}} \frac{(N\omega)^n}{n!},$$

where $\langle\cdot,\ \cdot\rangle_{L^{\otimes N}}$ is the Hermitian metric of $L^{\otimes N}$. Then it is clear that $\left\{\vartheta_{\frac{m}{N}}\right\}_{\frac{m}{N}\in F\cap\frac{1}{N}\mathbb{Z}^n}$ are orthogonal basis.

Example 4.5. For Example 2.30, $Z = X + \sqrt{-1}Y$ can be chosen so that $Y + XY^{-1}X$ is a constant map and XY^{-1} and Y^{-1} satisfy

$$(XY^{-1})_x = (Y + XY^{-1}X)\begin{pmatrix} u_{11}\cdot C^{-1}x & \cdots & u_{1n}\cdot C^{-1}x \\ \vdots & & \vdots \\ u_{n1}\cdot C^{-1}x & \cdots & u_{nn}\cdot C^{-1}x\end{pmatrix},$$

$$\begin{aligned}(Y^{-1})_x = &\begin{pmatrix} u_{11}\cdot C^{-1}x & \cdots & u_{1n}\cdot C^{-1}x \\ \vdots & & \vdots \\ u_{n1}\cdot C^{-1}x & \cdots & u_{nn}\cdot C^{-1}x\end{pmatrix}(Y + XY^{-1}X)\begin{pmatrix} u_{11}\cdot C^{-1}x & \cdots & u_{1n}\cdot C^{-1}x \\ \vdots & & \vdots \\ u_{n1}\cdot C^{-1}x & \cdots & u_{nn}\cdot C^{-1}x\end{pmatrix} \\ &+ Y + XY^{-1}X.\end{aligned}$$

In this case, $Y + XY^{-1}X$ is necessarily I and Ω can be written as

$$\Omega_x = \begin{pmatrix} u_{11}\cdot C^{-1}x & \cdots & u_{1n}\cdot C^{-1}x \\ \vdots & & \vdots \\ u_{n1}\cdot C^{-1}x & \cdots & u_{nn}\cdot C^{-1}x\end{pmatrix} + \sqrt{-1}(Y + XY^{-1}X)^{-1},$$

and the condition (2) in Proposition 3.12 is equivalent to the following condition

$$\left({}^tC^{-1}u_{jk}\right)_i = \left({}^tC^{-1}u_{ik}\right)_j \text{ for all } i, j, k = 1, \ldots, n.$$

Assume this condition as well as the condition $\frac{N}{2}v_i \cdot U_j v_i \in \mathbb{Z}$ for all $i, j = 1, \ldots, n$. Then, for each $\frac{m}{N} \in F \cap \frac{1}{N}\mathbb{Z}^n$, $\vartheta_{\frac{m}{N}}$ is described by

$$\begin{aligned}\vartheta_{\frac{m}{N}}(x, y) = \sum_{\gamma\in\Gamma} g_\gamma \exp 2\pi\sqrt{-1}N &\left[\sum_{i=1}^n\sum_{j>i}\left(\rho_{\gamma^{-1}}(x) - \frac{m}{N}\right)_i\left(\rho_{\gamma^{-1}}(x) - \frac{m}{N}\right)_j\left({}^tC^{-1}u_{ij}\right)\cdot\begin{pmatrix}\frac{m_1}{N}\\ \vdots \\ \frac{m_{i-1}}{N} \\ \frac{1}{2}\left(\rho_{\gamma^{-1}}(x) + \frac{m}{N}\right)_i \\ \left(\rho_{\gamma^{-1}}(x)\right)_{i+1} \\ \vdots \\ \left(\rho_{\gamma^{-1}}(x)\right)_n\end{pmatrix}\right. \\ &+\frac{1}{2}\sum_{i=1}^n\left(\rho_{\gamma^{-1}}(x) - \frac{m}{N}\right)_i^2\left({}^tC^{-1}u_{ii}\right)\cdot\begin{pmatrix}\frac{m_1}{N}\\ \vdots \\ \frac{m_{i-1}}{N} \\ \frac{1}{3}\left(2\rho_{\gamma^{-1}}(x) + \frac{m}{N}\right)_i \\ \left(\rho_{\gamma^{-1}}(x)\right)_{i+1} \\ \vdots \\ \left(\rho_{\gamma^{-1}}(x)\right)_n\end{pmatrix} \\ &+\frac{1}{2}\left(\rho_{\gamma^{-1}}(x) - \frac{m}{N}\right)\cdot\left(\begin{pmatrix} u_{11}\cdot\gamma & \cdots & u_{1n}\cdot\gamma \\ \vdots & & \vdots \\ u_{n1}\cdot\gamma & \cdots & u_{nn}\cdot\gamma\end{pmatrix} + \sqrt{-1}\left(Y + XY^{-1}X\right)^{-1}\right)\left(\rho_{\gamma^{-1}}(x) - \frac{m}{N}\right)\end{aligned}$$

$$-\frac{1}{2}\frac{m}{N}\cdot\begin{pmatrix} u_{11}\cdot\gamma & \cdots & u_{1n}\cdot\gamma \\ \vdots & & \vdots \\ u_{n1}\cdot\gamma & \cdots & u_{nn}\cdot\gamma \end{pmatrix}\frac{m}{N}+\rho_\gamma\left(\frac{m}{N}\right)\cdot y\Bigg].$$

By Proposition 4.1 (3), $\vartheta_{\frac{m}{N}}$ converges absolutely and uniformly on any compact set.

4.2. **The case when Z is constant.** Let $\pi\colon (M,\omega)\to B$ be a Lagrangian fibration on a complete n-dimensional B with prequantum line bundle $(L,\nabla^L)\to(M,\omega)$. Then, it is obtained as the quotient of the $\Gamma:=\pi_1(B)$-action on $\pi_0\colon(\widetilde{M},\omega_0)\to\mathbb{R}^n$ with prequantum line bundle $\left(\widetilde{L},\nabla^{\widetilde{L}}\right)\to(\widetilde{M},\omega_0)$. Suppose that the Γ-actions are described by (2.3) and (2.5) as before. Let J be a compatible almost complex structure on (M,ω) and $Z\in C^\infty(\widetilde{M},\mathcal{S}_n)$ be the map corresponding to the pull-back of J to $\widetilde{M}$. A situation in which (2) in Proposition 3.12 holds occurs when Z is a constant map. In this subsection, we discuss this case in detail. Note that in this case, Du_γ is a constant map for each $\gamma\in\Gamma$. It is obtained by (3.4). Moreover, as a special case of the setting in the previous subsection, we can obtain the following theorem.

Theorem 4.6. (1) *For each $\frac{m}{N}\in F\cap\frac{1}{N}\mathbb{Z}^n$, $\vartheta_{\frac{m}{N}}$ can be described as follows*

$$\vartheta_{\frac{m}{N}}(x,y)=\sum_{\gamma\in\Gamma}g_\gamma e^{2\pi\sqrt{-1}N\left[\frac{1}{2}\left\{\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)\cdot(\Omega+{}^tA_\gamma Du_\gamma)\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)-\frac{m}{N}\cdot({}^tA_\gamma Du_\gamma)\frac{m}{N}\right\}-\rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma(0)+\rho_\gamma\left(\frac{m}{N}\right)\cdot y\right]}.$$

(2) *For each $\frac{m}{N}\in F\cap\frac{1}{N}\mathbb{Z}^n$, $\vartheta_{\frac{m}{N}}$ converges absolutely and uniformly on any compact set.*
(3) *J is integrable and $\left\{\vartheta_{\frac{m}{N}}\right\}_{\frac{m}{N}\in F\cap\frac{1}{N}\mathbb{Z}^n}$ gives a basis of the space of holomorphic sections of $(L,\nabla^L)^{\otimes N}\to(M,N\omega,J)$.*

Proof. (1) is obtained from Proposition 4.1 (1). (2) is obtained by the assumption and Proposition 4.1 (3). The first half of (3) is true since J is covariant constant with respect to the associated Levi-Civita connection. The other half is obtained by Corollary 4.3. □

When Z is constant, the associated Riemannian metric of M is flat. So, by Bieberbach's theorem, if M is compact, then, M is finitely covered by the $2n$-dimensional torus T^{2n}, hence, $\vartheta_{\frac{m}{N}}$'s should be obtained from classical theta functions. So, let us see how $\vartheta_{\frac{m}{N}}$'s relate with classical theta functions for Example 2.28 with $C=I$, in which M itself is T^{2n}. First, let us briefly recall classical theta functions. For each $T\in\mathcal{S}_n$ and $a,b\in\mathbb{Q}^n$, the theta function with characteristics is a holomorphic section on the trivial holomorphic line bundle $\mathbb{C}^n\times\mathbb{C}\to\mathbb{C}^n$ which is defined by

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z,T):=\sum_{\gamma\in\mathbb{Z}^n}e^{\pi\sqrt{-1}(\gamma+a)\cdot T(\gamma+a)+2\pi\sqrt{-1}(\gamma+a)\cdot(z+b)}.$$

It is well-known that $\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z,T)$ has the following quasi-periodicity

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z+m,T)=e^{2\pi\sqrt{-1}a\cdot m}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z,T),$$
$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z+Tm,T)=e^{-2\pi\sqrt{-1}b\cdot m}e^{-\pi\sqrt{-1}m\cdot Tm-2\pi\sqrt{-1}m\cdot z}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z,T)$$

for $m\in\mathbb{Z}^n$. For more detail, see [28, Chapter II, §1] and [29, ß2]. Here we need the case when $T=N\Omega$, $a=\frac{m}{N}$, and $b=0$. In this case, define the $\mathbb{Z}^{2n}=\mathbb{Z}^n\times\mathbb{Z}^n$-action on $\mathbb{C}^n\times\mathbb{C}\to\mathbb{C}^n$ by

$$(\gamma,\gamma')\cdot(z,w):=\left(z+N(-\Omega\gamma+\gamma'),e^{-\pi\sqrt{-1}N\gamma\cdot\Omega\gamma+2\pi\sqrt{-1}\gamma\cdot z}w\right)$$

for $(\gamma,\gamma')\in\mathbb{Z}^{2n}$ and $(z,w)\in\mathbb{C}^n\times\mathbb{C}$. Also define the $\mathbb{Z}^{2n}$-action on the trivial complex line bundle $\mathbb{R}^{2n}\times\mathbb{C}\to\mathbb{R}^{2n}$ by

$$(\gamma,\gamma')\cdot(x,y,w):=\left(x+\gamma,y+\gamma',e^{2\pi\sqrt{-1}N\gamma\cdot y}w\right)\tag{4.2}$$

for $(\gamma,\gamma')\in\mathbb{Z}^{2n}$ and $(x,y,w)\in\mathbb{R}^{2n}\times\mathbb{C}$. Note that by taking the quotient of the latter $\mathbb{Z}^n$-action of (4.2), we can recover Example 2.28 with $C=I$ and $g_\gamma=1$. Let $F\colon\mathbb{R}^{2n}\to\mathbb{C}^n$ and $\widetilde{F}\colon\mathbb{R}^{2n}\times\mathbb{C}\to\mathbb{C}^n\times\mathbb{C}$ be the $\mathbb{R}$-linear isomorphism and the bundle isomorphism covering F which are defined by

$$F(x,y):=N\left(-\Omega x+y\right),$$
$$\widetilde{F}(x,y,w):=\left(N\left(-\Omega x+y\right),e^{-\pi\sqrt{-1}Nx\cdot\Omega x}w\right).$$

Then, the direct computation shows the following theorem.

Theorem 4.7. (1) $J_{\sqrt{-1}I}\circ F=F\circ(J_Z)$*, i.e., F is a $\mathbb{C}$-linear isomorphism from $(\mathbb{R}^{2n},J_Z)$ to the standard complex vector space $(\mathbb{C}^n,J_{\sqrt{-1}I})$.*
(2) *$\widetilde{F}$ is equivariant with respect to the $\mathbb{Z}^{2n}$-actions defined above.*
(3) *$\vartheta_{\frac{m}{N}}$ satisfies $\widetilde{F}\circ\vartheta_{\frac{m}{N}}(x,y)=\vartheta\begin{bmatrix}\frac{m}{N}\\0\end{bmatrix}(F(x,y),N\Omega)$, i.e.,*

$$\vartheta_{\frac{m}{N}}(x,y)=e^{\pi\sqrt{-1}Nx\cdot\Omega x}\vartheta\begin{bmatrix}\frac{m}{N}\\0\end{bmatrix}(N(-\Omega x+y),N\Omega).$$

4.3. **Adiabatic-type limit.** In this subsection let us consider a one parameter family $\{(g^t,J^t)\}_{t>0}$ of the Riemannian metrics and the almost complex structures on a Lagrangian fibration so that the fiber shrinks as t goes to ∞, and investigate the behavior of $\vartheta_{\frac{m}{N}}$ defined by (4.1) when t goes to ∞. We use the same notations introduced in the previous sections.

Let $Z=X+\sqrt{-1}Y\in C^\infty\left(\widetilde{M},\mathcal{S}_n\right)$ be the map independent of $y_1,\dots,y_n$. Let $J=J_Z$ be the corresponding compatible almost complex structure on $\left(\widetilde{M},\omega_0\right)$. For each $t>0$, we define the almost complex structure J^t by

$$J^tu:=(-J\partial_y,\,\partial_y)\begin{pmatrix}0&\frac{-1}{t}\\t&0\end{pmatrix}\begin{pmatrix}u_H\\u_V\end{pmatrix}$$

for $u=(-J\partial_y,\,\partial_y)\begin{pmatrix}u_H\\u_V\end{pmatrix}\in T_{(x,y)}\widetilde{M}$. It is easy to see the following lemma.

Lemma 4.8. (1) *For any $t>0$, J^t is compatible with ω_0. The map $Z^t\in C^\infty\left(\widetilde{M},\mathcal{S}_n\right)$ corresponding to J^t is described as*

$$Z^t=\left(\frac{1}{t}X+\sqrt{-1}Y\right)Y^{-1}\left(Y+XY^{-1}X\right)\left(tY+\frac{1}{t}XY^{-1}X\right)^{-1}Y.$$

J^t can be also written as

$$J^t\left((\partial_x,\partial_y)\begin{pmatrix}u_x\\u_y\end{pmatrix}\right)=(\partial_x,\partial_y)\frac{1}{t}\begin{pmatrix}XY^{-1}&-Y-XY^{-1}X\\Y^{-1}\left(t^2Y+XY^{-1}X\right)\left(Y+XY^{-1}X\right)^{-1}&-Y^{-1}X\end{pmatrix}\begin{pmatrix}u_x\\u_y\end{pmatrix}.$$

(2) *For any $t>0$, let g^t be the Riemannian metric corresponding to ω_0 and J^t. Then, for $u=(-J\partial_y,\,\partial_y)\begin{pmatrix}u_H\\u_V\end{pmatrix}$, $v=(-J\partial_y,\,\partial_y)\begin{pmatrix}v_H\\v_V\end{pmatrix}\in T_{(x,y)}\widetilde{M}$, g^t can be written by*

$$\begin{aligned}g^t(u,v)&=\omega_0\left(u,J^tv\right)\\&=t(0,{}^tu_H)\begin{pmatrix}Y^{-1}&-Y^{-1}X\\-XY^{-1}&Y+XY^{-1}X\end{pmatrix}\begin{pmatrix}0\\v_H\end{pmatrix}+\frac{1}{t}(0,{}^tu_V)\begin{pmatrix}Y^{-1}&-Y^{-1}X\\-XY^{-1}&Y+XY^{-1}X\end{pmatrix}\begin{pmatrix}0\\v_V\end{pmatrix}.\end{aligned}$$

Suppose that a group Γ acts on $\pi_0\colon(\widetilde{M},\omega_0)\to\mathbb{R}^n$ and the Γ-actions ρ on $\mathbb{R}^n$ and $\widetilde{\rho}$ on $(\widetilde{M},\omega_0)$ are written as in (2.3).

Lemma 4.9. *The Γ-action $\widetilde{\rho}$ preserves J^t (hence, g^t) for all $t>0$ if and only if $\widetilde{\rho}$ preserves J.*

For J^t and g^t defined as above, the same arguments in Section 3.3 goes well, just by replacing J, g by J^t, g^t. For each $t>0$, let $\vartheta^t_{\frac{m}{N}}$ be the one defined by (4.1) for J^t and g^t. Let us investigate the behavior of $\vartheta^t_{\frac{m}{N}}$ as t goes to infinity. For $t>0$, Ω^t defined by (3.7) for Z^t can be described as

$$\Omega^t=\left(Y+XY^{-1}X\right)^{-1}\left(X+t\sqrt{-1}Y\right)Y^{-1}. \tag{4.3}$$

Let D^t be the corresponding Spin^c Dirac operator. Then, for a section s of $\widetilde{L}^{\otimes N}$, $D^t s$ can be described as

$$D^t s = -\frac{\sqrt{-1}}{N}\sum_{i=1}^n \partial_{y_i}\otimes_{\mathbb{C}}\left\{\partial_{x_i}s+\sum_{j=1}^n\left(\Omega^t\right)_{ij}\left(\partial_{y_j}s-2\pi\sqrt{-1}Nx_j s\right)\right\}. \tag{4.4}$$

It is clear that

Lemma 4.10. *For any $t>0$, the condition (2) in Proposition 3.12 holds for Ω^t if and only if it holds for $\Omega=\Omega^1$. In particular, J^t is integrable if and only if J is integrable.*

Suppose that $\pi_0\colon(\widetilde{M},N\omega_0,J)\to\mathbb{R}^n$ with prequantum line bundle $\left(\widetilde{L},\nabla^{\widetilde{L}}\right)^{\otimes N}\to(\widetilde{M},N\omega_0,J)$ is equipped with an action of a group Γ which preserves all the data, and the Γ-actions are described by (2.3) and (2.5) as before. We assume that the Γ-action ρ on $\mathbb{R}^n$ is properly discontinuous, free, and cocompact. Let $\pi\colon(M,N\omega)\to B$ and $(L,\nabla^L)^{\otimes N}\to(M,N\omega)$ be the Lagrangian fibration and the prequantum line bundle on it obtained by the quotient of the Γ-action. On M, we consider the orientation so that $(-1)^{\frac{n(n-1)}{2}}\frac{(N\omega)^n}{n!}$ is a positive volume form, and define the L^p-norm of a section s of $L^{\otimes N}$ by

$$\|s\|_{L^p}:=\left(\int_M\langle s,s\rangle_{L^{\otimes N}}^{\frac{p}{2}}(-1)^{\frac{n(n-1)}{2}}\frac{(N\omega)^n}{n!}\right)^{\frac{1}{p}}, \tag{4.5}$$

where $\langle\cdot,\ \cdot\rangle_{L^{\otimes N}}$ is the Hermitian metric of $L^{\otimes N}$ which is induced from the Hermitian metric $\langle\cdot,\ \cdot\rangle_{\widetilde{L}^{\otimes N}}$ of $\widetilde{L}^{\otimes N}$. As noticed in Remark 2.26, there exists a positive constant C such that $\langle\cdot,\ \cdot\rangle_{\widetilde{L}^{\otimes N}}$ can be written as $\langle\cdot,\ \cdot\rangle_{\widetilde{L}^{\otimes N}}=C\langle\cdot,\ \cdot\rangle_{\mathbb{C}}$, where $\langle\cdot,\ \cdot\rangle_{\mathbb{C}}$ is the standard Hermitian inner product on $\mathbb{C}$.

For each $t>0$ and each point $\frac{m}{N}\in F\cap\frac{1}{N}\mathbb{Z}^n$ for which the condition (3.13) holds, the corresponding $\vartheta^t_{\frac{m}{N}}$ is defined by (4.1) for Ω^t. We identify $F\cap\frac{1}{N}\mathbb{Z}^n$ with B_{BS} the set of Bohr-Sommerfeld points of $\pi\colon(M,N\omega)\to B$ with prequantum line bundle $(L,\nabla^L)^{\otimes N}\to(M,N\omega)$ and identify $\vartheta^t_{\frac{m}{N}}$ with the section of $(L,\nabla^L)^{\otimes N}\to(M,N\omega)$ which is induced from $\vartheta^t_{\frac{m}{N}}$. Then, concerning the L^p-norm, we have the following lemma.

Lemma 4.11. *Suppose that $Y+XY^{-1}X$ is constant. Then, the L^p-norm of $\vartheta^t_{\frac{m}{N}}$ can be calculated as follows*

$$\|\vartheta^t_{\frac{m}{N}}\|^p_{L^p}=C\sqrt{\det\left(Y+XY^{-1}X\right)}\left(\frac{N}{pt}\right)^{\frac{n}{2}}.$$

Proof. Let $o(B)$ be the orientation bundle of B which is defined as the quotient bundle of the trivial real line bundle $\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ on the universal cover of B by the Γ-action $\rho'_\gamma(x,r):=(\rho_\gamma(x),(\det A_\gamma)r)$ for $\gamma\in\Gamma$ and $(x,r)\in\mathbb{R}^n\times\mathbb{R}$. Then, we have a push-forward map $\pi_*\colon\Omega^k(M)\to\Omega^{k-n}(B,o(B))$, where $\Omega^\bullet(B,o(B))$ is the de Rham complex twisted by $o(B)$. B has a natural density which we denote by $|dx|$. For densities, see [8, Chapter I, §7]. Then,

$$\begin{aligned}\|\vartheta^t_{\frac{m}{N}}\|^p_{L^p}&=\int_M\langle\vartheta^t_{\frac{m}{N}},\vartheta^t_{\frac{m}{N}}\rangle^{\frac{p}{2}}_{L^{\otimes N}}(-1)^{\frac{n(n-1)}{2}}\frac{(N\omega)^n}{n!}\\&=\int_B\pi_*\left(\langle\vartheta^t_{\frac{m}{N}},\vartheta^t_{\frac{m}{N}}\rangle^{\frac{p}{2}}_{L^{\otimes N}}(-1)^{\frac{n(n-1)}{2}}\frac{(N\omega)^n}{n!}\right)\\&=CN^n\sum_{\gamma\in\Gamma}\int_F e^{-pN\pi t\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)\cdot\left(Y+XY^{-1}X\right)^{-1}\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)}|dx|.\end{aligned}\tag{4.6}$$

By changing the coordinates as $x'=\rho_{\gamma^{-1}}(x)$,

$$\begin{aligned}(4.6)&=CN^n\sum_{\gamma\in\Gamma}\int_{\rho_{\gamma^{-1}}(F)}e^{-pN\pi t\left(x'-\frac{m}{N}\right)\cdot\left(Y+XY^{-1}X\right)^{-1}\left(x'-\frac{m}{N}\right)}|dx'|\\&=CN^n\int_{\mathbb{R}^n}e^{-pN\pi t\left(x'-\frac{m}{N}\right)\cdot\left(Y+XY^{-1}X\right)^{-1}\left(x'-\frac{m}{N}\right)}|dx'|.\end{aligned}\tag{4.7}$$

Since $Y + XY^{-1}X$ is positive definite, symmetric, there exists $P \in O(n)$ such that

$$Y + XY^{-1}X = {}^tP \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P.$$

Then, we define a positive definite symmetric matrix $\sqrt{Y + XY^{-1}X}$ by

$$\sqrt{Y + XY^{-1}X} := {}^tP \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} P, \tag{4.8}$$

and put $\tau := \sqrt{(Y + XY^{-1}X)^{-1}} \left(x' - \frac{m}{N}\right)$. Then,

$$\begin{aligned}
(4.7) &= C\sqrt{\det(Y + XY^{-1}X)} N^n \int_{\mathbb{R}^n} e^{-pN\pi t \|\tau\|^2} |d\tau| \\
&= C\sqrt{\det(Y + XY^{-1}X)} N^n \prod_{i=1}^n \int_{-\infty}^{\infty} e^{-pN\pi t \tau_i^2} d\tau_i \\
&= C\sqrt{\det(Y + XY^{-1}X)} N^n \left(\sqrt{\frac{1}{pNt}}\right)^n.
\end{aligned}$$

□

We define the section $\delta_{\frac{m}{N}}$ of $\left(L, \nabla^L\right)^{\otimes N}|_{\pi^{-1}\left(\frac{m}{N}\right)}$ by

$$\delta_{\frac{m}{N}}(y) := \frac{1}{C} e^{2\pi\sqrt{-1} m\cdot y}. \tag{4.9}$$

By Lemma 3.2, $\delta_{\frac{m}{N}}$ is a covariant constant section of $\left(L, \nabla^L\right)^{\otimes N}|_{\pi^{-1}\left(\frac{m}{N}\right)}$. Let $T^*_\pi M$ be the cotangent bundle along the fiber of π. On $(\wedge^n T^*_\pi M) \otimes \pi^* o(B)^*$, there exists a natural section, i.e., a density along the fiber of π, say $|dy|$, which satisfies $\int_{\pi^{-1}(x)} |dy| = 1$ on each fiber of π. Then, we obtain the following theorem.

Theorem 4.12. *Suppose that $Y + XY^{-1}X$ is constant. Then, the section $\frac{\vartheta^t_{\frac{m}{N}}}{\|\vartheta^t_{\frac{m}{N}}\|_{L^1}}$ converges to a delta-function section supported on the fiber $\pi^{-1}\left(\frac{m}{N}\right)$ as t goes to ∞ in the following sense: for any section s of $L^{\otimes N}$,*

$$\lim_{t\to\infty} \int_M \left\langle s, \frac{\vartheta^t_{\frac{m}{N}}}{\|\vartheta^t_{\frac{m}{N}}\|_{L^1}} \right\rangle_{L^{\otimes N}} (-1)^{\frac{n(n-1)}{2}} \frac{(N\omega)^n}{n!} = \int_{\pi^{-1}\left(\frac{m}{N}\right)} \left\langle s, \delta_{\frac{m}{N}} \right\rangle_{L^{\otimes N}} |dy|.$$

Proof. We denote by $\widetilde{s}$ the pull-back of s to $\widetilde{L}^{\otimes N} \to \widetilde{M}$. Since $\widetilde{s}$ is Γ-equivariant, the Fourier expansion of $\widetilde{s}$ can be written as in (3.33). Then, by using Proposition 4.1 (1),

$$\begin{aligned}
&\int_M \left\langle s, \frac{\vartheta^t_{\frac{m}{N}}}{\|\vartheta^t_{\frac{m}{N}}\|_{L^1}} \right\rangle_{L^{\otimes N}} (-1)^{\frac{n(n-1)}{2}} \frac{(N\omega)^n}{n!} \\
&= \int_B \pi_* \left(\left\langle s, \frac{\vartheta^t_{\frac{m}{N}}}{\|\vartheta^t_{\frac{m}{N}}\|_{L^1}} \right\rangle_{L^{\otimes N}} (-1)^{\frac{n(n-1)}{2}} \frac{(N\omega)^n}{n!} \right) \\
&= \frac{CN^n}{\|\vartheta^t_{\frac{m}{N}}\|_{L^1}} \sum_{\gamma\in\Gamma} \int_F a_m\left(\rho_{\gamma^{-1}}(x)\right) \overline{e^{-2\pi\sqrt{-1}\sum_{i=1}^n G^i_m\left(\frac{m_1}{N},\dots,\frac{m_{i-1}}{N},\left(\rho_{\gamma^{-1}}(x)\right)_i,\dots,\left(\rho_{\gamma^{-1}}(x)\right)_n\right)}} |dx|.
\end{aligned} \tag{4.10}$$

By putting $x' = \rho_{\gamma^{-1}}(x)$, we have

$$(4.10) = \frac{CN^n}{\|\vartheta^t_{\frac{m}{N}}\|_{L^1}} \sum_{\gamma\in\Gamma} \int_{\rho_{\gamma^{-1}}(F)} a_m(x') \overline{e^{-2\pi\sqrt{-1}\sum_{i=1}^n G^i_m\left(\frac{m_1}{N},\dots,\frac{m_{i-1}}{N},x'_i,\dots,x'_n\right)}} |dx'|$$

$$= \frac{CN^n}{\|\vartheta^t_{\frac{m}{N}}\|_{L^1}} \int_{\mathbb{R}^n} a_m(x') \overline{e^{-2\pi\sqrt{-1}\sum_{i=1}^n G^i_m\left(\frac{m_1}{N},\dots,\frac{m_{i-1}}{N},x'_i,\dots,x'_n\right)}} |dx'|$$

(4.11)

$$= \frac{CN^n}{\|\vartheta^t_{\frac{m}{N}}\|_{L^1}} \int_{\mathbb{R}^n} a_m(x') e^{2\pi\sqrt{-1}\sum_{i=1}^n \operatorname{Re} G^i_m\left(\frac{m_1}{N},\dots,\frac{m_{i-1}}{N},x'_i,\dots,x'_n\right)} e^{-\pi Nt\left(x'-\frac{m}{N}\right)\cdot\left(Y+XY^{-1}X\right)^{-1}\left(x'-\frac{m}{N}\right)} |dx'|.$$

We put $f(x') := a_m(x') e^{2\pi\sqrt{-1}\sum_{i=1}^n \operatorname{Re} G^i_m\left(\frac{m_1}{N},\dots,\frac{m_{i-1}}{N},x'_i,\dots,x'_n\right)}$ and $\tau := \sqrt{(Y+XY^{-1}X)^{-1}}\left(x' - \frac{m}{N}\right)$. By using Lemma 4.11 for $p=1$, (4.11) can be written as follows

$$(4.12) \qquad (4.11) = \frac{CN^n}{\|\vartheta^t_{\frac{m}{N}}\|_{L^1}} \int_{\mathbb{R}^n} f(x') e^{-\pi Nt\left(x'-\frac{m}{N}\right)\cdot\left(Y+XY^{-1}X\right)^{-1}\left(x'-\frac{m}{N}\right)} |dx'|$$

$$= \frac{CN^n}{\|\vartheta^t_{\frac{m}{N}}\|_{L^1}} \sqrt{\det\left(Y+XY^{-1}X\right)} \int_{\mathbb{R}^n} f\left(\sqrt{Y+XY^{-1}X}\tau + \frac{m}{N}\right) e^{-\pi Nt\|\tau\|^2} |d\tau|$$

$$= (Nt)^{\frac{n}{2}} \int_{\mathbb{R}^n} f\left(\sqrt{Y+XY^{-1}X}\tau + \frac{m}{N}\right) e^{-\pi Nt\|\tau\|^2} |d\tau|.$$

It is well-known that

$$\lim_{t\to\infty} (4.12) = f\left(\frac{m}{N}\right) = a_m\left(\frac{m}{N}\right).$$

On the other hand, by using the expression

$$\widetilde{s} = \sum_{\left(\gamma,\frac{m'}{N}\right)\in\Gamma\times\left(F\cap\frac{1}{N}\mathbb{Z}^n\right)} a_{N\rho_\gamma\left(\frac{m'}{N}\right)}(x) e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m'}{N}\right)\cdot y},$$

the right hand side can be computed as

$$\int_{\pi^{-1}\left(\frac{m}{N}\right)} \left\langle s, \delta_{\frac{m}{N}}\right\rangle_{L^{\otimes N}} |dy| = \int_{T^n} \left\langle \widetilde{s}, \delta_{\frac{m}{N}}\right\rangle_{\widetilde{L}^{\otimes N}} |dy|$$

$$= \sum_{\left(\gamma,\frac{m'}{N}\right)\in\Gamma\times\left(F\cap\frac{1}{N}\mathbb{Z}^n\right)} a_{N\rho_\gamma\left(\frac{m'}{N}\right)}\left(\frac{m}{N}\right) \int_{T^n} e^{2\pi\sqrt{-1}\left(N\rho_\gamma\left(\frac{m'}{N}\right)-m\right)\cdot y} |dy|.$$

$\int_{T^n} e^{2\pi\sqrt{-1}\left(N\rho_\gamma\left(\frac{m'}{N}\right)-m\right)\cdot y}|dy|$ vanishes unless $\rho_\gamma\left(\frac{m'}{N}\right) = \frac{m}{N}$. Since both $\frac{m'}{N}$ and $\frac{m}{N}$ lie in the fundamental domain F, this implies $\gamma = e$ and $m' = m$, and in this case, $\int_{T^n} e^{2\pi\sqrt{-1}\left(N\rho_\gamma\left(\frac{m'}{N}\right)-m\right)\cdot y}|dy| = 1$. Thus,

$$\int_{\pi^{-1}\left(\frac{m}{N}\right)} \left\langle s, \delta_{\frac{m}{N}}\right\rangle_{L^{\otimes N}} |dy| = a_m\left(\frac{m}{N}\right).$$

This proves the theorem. □

5. The non-integrable case

We still use the same notations introduced in the previous sections. By Lemma 3.10, the equation (3.12) has no smooth solution for $\frac{m}{N} \in F \cap \frac{1}{N}\mathbb{Z}^n$ such that (3.13) does not holds. For such $\frac{m}{N}$, instead of (3.12), let us consider the following equation which is obtained from (3.12) by replacing Ω by its value $\Omega_{\frac{m}{N}}$ at $\frac{m}{N}$

$$(5.1) \qquad 0 = \begin{pmatrix} \partial_{x_1}\widetilde{a}_m \\ \vdots \\ \partial_{x_n}\widetilde{a}_m \end{pmatrix} + 2\pi\sqrt{-1}\widetilde{a}_m \Omega_{\frac{m}{N}}(m - Nx).$$

The equation (5.1) has a solution of the form

$$\widetilde{a}_m(x) = \widetilde{a}_m\left(\frac{m}{N}\right) e^{\pi\sqrt{-1}N\left(x-\frac{m}{N}\right)\cdot\Omega_{\frac{m}{N}}\left(x-\frac{m}{N}\right)}.$$

We put the initial condition $\widetilde{a}_m\left(\frac{m}{N}\right)=1$ on the above $\widetilde{a}_m$, and for each $\gamma\in\Gamma$, define $\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}$ in such a way that it satisfies (3.32).

Lemma 5.1. $\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}$ *satisfies the following equality*

$$0=\begin{pmatrix}\partial_{x_1}\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}(x)\\ \vdots\\ \partial_{x_n}\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}(x)\end{pmatrix}+2\pi\sqrt{-1}\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}(x)\Omega_x\left(N\rho_\gamma\left(\frac{m}{N}\right)-Nx\right)$$
$$+2\pi\sqrt{-1}\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}(x){}^tA_\gamma^{-1}\left(\Omega_{\frac{m}{N}}-\Omega_{\rho_{\gamma^{-1}}(x)}\right)A_\gamma^{-1}\left(N\rho_\gamma\left(\frac{m}{N}\right)-Nx\right). \tag{5.2}$$

Proof. By the same calculation as in the proof of Lemma 3.15, we have

$${}^tA_\gamma\begin{pmatrix}\partial_{x_1}\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}(\rho_\gamma(x))\\ \vdots\\ \partial_{x_n}\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}(\rho_\gamma(x))\end{pmatrix}=-2\pi\sqrt{-1}\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}(\rho_\gamma(x))\left(\Omega_{\frac{m}{N}}A_\gamma^{-1}+{}^t(Du_\gamma)_x\right)\left(N\rho_\gamma\left(\frac{m}{N}\right)-N\rho_\gamma(x)\right).$$

(5.2) can be obtained from this equation and (3.39). □

By using $\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}$'s, we define $\widetilde{\vartheta}_{\frac{m}{N}}$ in the same manner as $\vartheta_{\frac{m}{N}}$, i.e.,

$$\widetilde{\vartheta}_{\frac{m}{N}}(x,y)=\sum_{\gamma\in\Gamma}\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}(x)e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot y}.$$

$\widetilde{\vartheta}_{\frac{m}{N}}$ converges absolutely and uniformly on any compact set and can be written as

$$\widetilde{\vartheta}_{\frac{m}{N}}=\sum_{\gamma\in\Gamma}\widetilde{\widetilde{\rho}}_\gamma\circ s'_m\circ\widetilde{\rho}_{\gamma^{-1}},$$

where s'_m is the section defined by

$$s'_m(x,y):=e^{\pi\sqrt{-1}N\left(x-\frac{m}{N}\right)\cdot\Omega_{\frac{m}{N}}\left(x-\frac{m}{N}\right)+2\pi\sqrt{-1}m\cdot y}.$$

In particular, when M is compact, it defines a section of $L^{\otimes N}\to M$. Moreover, these two sections with different $\frac{m}{N}$ and $\frac{m'}{N}$ are orthogonal to each other. These can be proved by the same way as Proposition 4.1. In the rest of this section, we assume that M is compact.

Next let us consider the one parameter family of J^t and g^t defined in Section 4.3. Then, corresponding to J^t and g^t, we can obtain $\widetilde{\vartheta}^t_{\frac{m}{N}}$, which can be explicitly describe as

$$\widetilde{\vartheta}^t_{\frac{m}{N}}(x,y)=\sum_{\gamma\in\Gamma}g_\gamma e^{2\pi\sqrt{-1}N\left[\frac{1}{2}\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)\cdot\Omega^t_{\frac{m}{N}}\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)+\widetilde{g}_\gamma\left(\rho_{\gamma^{-1}}(x)\right)-\rho_\gamma\left(\frac{m}{N}\right)\cdot u_\gamma\left(\rho_{\gamma^{-1}}(x)\right)\right]}e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot y},$$

where $\Omega^t_{\frac{m}{N}}$ is the value of Ω^t given in (4.3) at $\frac{m}{N}$. Then, $\widetilde{\vartheta}^t_{\frac{m}{N}}$ has the following property. The proof is same as Theorem 4.12.

Theorem 5.2. *For each* $\frac{m}{N}\in F\cap\frac{1}{N}\mathbb{Z}^n$, *the section* $\dfrac{\widetilde{\vartheta}^t_{\frac{m}{N}}}{\|\widetilde{\vartheta}^t_{\frac{m}{N}}\|_{L^1}}$ *converges to a delta-function section supported on the fiber* $\pi^{-1}\left(\frac{m}{N}\right)$ *as t goes to* ∞ *in the following sense: for any section s of* $L^{\otimes N}$,

$$\lim_{t\to\infty}\int_M\left\langle s,\frac{\widetilde{\vartheta}^t_{\frac{m}{N}}}{\|\widetilde{\vartheta}^t_{\frac{m}{N}}\|_{L^1}}\right\rangle_{L^{\otimes N}}(-1)^{\frac{n(n-1)}{2}}\frac{(N\omega)^n}{n!}=\int_{\pi^{-1}\left(\frac{m}{N}\right)}\left\langle s,\delta_{\frac{m}{N}}\right\rangle_{L^{\otimes N}}|dy|.$$

$\widetilde{\vartheta}^t_{\frac{m}{N}}$ is not a solution of $0=D^ts$, but we can show that $\widetilde{\vartheta}^t_{\frac{m}{N}}$ approximates the solution of this equation in the following sense:

Theorem 5.3.

$$\lim_{t\to\infty}\|D^t\widetilde{\vartheta}^t_{\frac{m}{N}}\|_{L^2((TM,J^t)\otimes_{\mathbb{C}}L)}=0.$$

Before proving the theorem, we have to make sure the meaning of L^2-norm in the left hand side. $D^t\widetilde{\vartheta}^t_{\frac{m}{N}}$ is a section of $(TM, J^t)\otimes_{\mathbb{C}} L$, and $(TM, J^t)\otimes_{\mathbb{C}} L$ admits a Hermitian metric $\langle\cdot,\ \cdot\rangle_{(TM,J^t)\otimes_{\mathbb{C}}L}$ induced by the one parameter version of (3.10) of (TM, J^t) and the Hermitian metric of L. In terms of this Hermitian metric, the L^2-norm is defined as

$$\|D^t\widetilde{\vartheta}^t_{\frac{m}{N}}\|^2_{L^2((TM,J^t)\otimes_{\mathbb{C}}L)} := \int_M \langle D^t\widetilde{\vartheta}^t_{\frac{m}{N}},\ D^t\widetilde{\vartheta}^t_{\frac{m}{N}}\rangle_{(TM,J^t)\otimes_{\mathbb{C}}L}(-1)^{\frac{n(n-1)}{2}}\frac{(N\omega)^n}{n!}.$$

Proof. For $n = 1$, it is clear that the condition 3.13 automatically holds for all $m \in \mathbb{Z}$. Thus, it is sufficient to prove the theorem for $n \geq 2$. By the definition of $\widetilde{\vartheta}^t_{\frac{m}{N}}$ and (5.2), $D^t\widetilde{\vartheta}^t_{\frac{m}{N}}$ can be written as

$$\begin{aligned}
&D^t\widetilde{\vartheta}^t_{\frac{m}{N}}\\
&= -\frac{\sqrt{-1}}{N}\sum_{i=1}^n \partial_{y_i}\otimes_{\mathbb{C}}\left\{\partial_{x_i}\widetilde{\vartheta}^t_{\frac{m}{N}} + \sum_{j=1}^n \left(\Omega^t_x\right)_{ij}\left(\partial_{y_j}\widetilde{\vartheta}^t_{\frac{m}{N}} - 2\pi\sqrt{-1}Nx_j\widetilde{\vartheta}^t_{\frac{m}{N}}\right)\right\}\\
&= -\frac{\sqrt{-1}}{N}\sum_{i=1}^n \partial_{y_i}\otimes_{\mathbb{C}}\sum_{\gamma\in\Gamma}\left\{\partial_{x_i}\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}(x) + 2\pi\sqrt{-1}\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}(x)\left(\Omega^t_x\left(N\rho_\gamma\left(\frac{m}{N}\right) - Nx\right)\right)_i\right\}e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot y}\\
&= -2\pi\sum_{i=1}^n \partial_{y_i}\otimes_{\mathbb{C}}\sum_{\gamma\in\Gamma}\widetilde{a}_{N\rho_\gamma\left(\frac{m}{N}\right)}(x)\left({}^tA_\gamma^{-1}\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{\rho_{\gamma^{-1}}(x)}\right)\left(\frac{m}{N} - \rho_{\gamma^{-1}}(x)\right)\right)_i e^{2\pi\sqrt{-1}N\rho_\gamma\left(\frac{m}{N}\right)\cdot y}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\langle D^t\widetilde{\vartheta}^t_{\frac{m}{N}},\ D^t\widetilde{\vartheta}^t_{\frac{m}{N}}\rangle_{(TM,J^t)\otimes_{\mathbb{C}}L}\\
&= (2\pi)^2\sum_{\gamma_1,\gamma_2\in\Gamma}\sum_{i_1,i_2}\langle\widetilde{a}_{N\rho_{\gamma_1}\left(\frac{m}{N}\right)}(x)e^{2\pi\sqrt{-1}N\rho_{\gamma_1}\left(\frac{m}{N}\right)\cdot y}, \widetilde{a}_{N\rho_{\gamma_2}\left(\frac{m}{N}\right)}(x)e^{2\pi\sqrt{-1}N\rho_{\gamma_2}\left(\frac{m}{N}\right)\cdot y}\rangle_{L^{\otimes N}}Ng^t\left(\partial_{y_{i_1}}, \partial_{y_{i_2}}\right)\\
&\times\left({}^tA_{\gamma_1}^{-1}\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{\rho_{\gamma_1^{-1}}(x)}\right)\left(\frac{m}{N} - \rho_{\gamma_1^{-1}}(x)\right)\right)_{i_1}\overline{\left({}^tA_{\gamma_2}^{-1}\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{\rho_{\gamma_2^{-1}}(x)}\right)\left(\frac{m}{N} - \rho_{\gamma_2^{-1}}(x)\right)\right)_{i_2}}\\
&= (2\pi)^2\frac{N}{t}\sum_{\gamma_1,\gamma_2\in\Gamma}\langle\widetilde{a}_{N\rho_{\gamma_1}\left(\frac{m}{N}\right)}(x)e^{2\pi\sqrt{-1}N\rho_{\gamma_1}\left(\frac{m}{N}\right)\cdot y}, \widetilde{a}_{N\rho_{\gamma_2}\left(\frac{m}{N}\right)}(x)e^{2\pi\sqrt{-1}N\rho_{\gamma_2}\left(\frac{m}{N}\right)\cdot y}\rangle_{L^{\otimes N}}\\
&\times\left({}^tA_{\gamma_1}^{-1}\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{\rho_{\gamma_1^{-1}}(x)}\right)\left(\frac{m}{N} - \rho_{\gamma_1^{-1}}(x)\right)\right)\cdot\left(Y + XY^{-1}X\right)_x\overline{\left({}^tA_{\gamma_2}^{-1}\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{\rho_{\gamma_2^{-1}}(x)}\right)\left(\frac{m}{N} - \rho_{\gamma_2^{-1}}(x)\right)\right)}.
\end{aligned}$$

For each $x \in F$ and $u \in \mathbb{C}^n$, define the norm of u with respect to $(Y + XY^{-1}X)_x$ by

$$\|u\|^2_{(Y+XY^{-1}X)_x} := u\cdot(Y + XY^{-1}X)_x\overline{u}.$$

By (3.5), for each $\gamma \in \Gamma$, $\|u\|^2_{(Y+XY^{-1}X)_x}$ satisfies

$$\|{}^tA_\gamma u\|^2_{(Y+XY^{-1}X)_x} = \|u\|^2_{(Y+XY^{-1}X)_{\rho_\gamma(x)}}.$$

In terms of this norm, we obtain

$$\begin{aligned}
&\|D^t\widetilde{\vartheta}^t_{\frac{m}{N}}\|^2_{L^2((TM,J^t)\otimes_{\mathbb{C}}L)}\\
&= (2\pi)^2\frac{CN^{n+1}}{t}\sum_{\gamma\in\Gamma}\int_F e^{-2\pi Nt\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)\cdot\left(Y+XY^{-1}X\right)^{-1}_{\frac{m}{N}}\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)}\\
&\times\left({}^tA_\gamma^{-1}\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{\rho_{\gamma^{-1}}(x)}\right)\left(\frac{m}{N} - \rho_{\gamma^{-1}}(x)\right)\right)\cdot\left(Y + XY^{-1}X\right)_x\overline{\left({}^tA_\gamma^{-1}\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{\rho_{\gamma^{-1}}(x)}\right)\left(\frac{m}{N} - \rho_{\gamma^{-1}}(x)\right)\right)}|dx|\\
&= (2\pi)^2\frac{CN^{n+1}}{t}\sum_{\gamma\in\Gamma}\int_F e^{-2\pi Nt\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)\cdot\left(Y+XY^{-1}X\right)^{-1}_{\frac{m}{N}}\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)}\\
&\times\|{}^tA_{\gamma^{-1}}\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{\rho_{\gamma^{-1}}(x)}\right)\left(\frac{m}{N} - \rho_{\gamma^{-1}}(x)\right)\|^2_{(Y+XY^{-1}X)_x}|dx|\\
&= (2\pi)^2\frac{CN^{n+1}}{t}\sum_{\gamma\in\Gamma}\int_F e^{-2\pi Nt\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)\cdot\left(Y+XY^{-1}X\right)^{-1}_{\frac{m}{N}}\left(\rho_{\gamma^{-1}}(x)-\frac{m}{N}\right)}
\end{aligned}$$

$$\times \|\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{\rho_{\gamma^{-1}}(x)}\right)\left(\frac{m}{N} - \rho_{\gamma^{-1}}(x)\right)\|^2_{(Y+XY^{-1}X)_{\rho_{\gamma^{-1}}(x)}}|dx|$$
$$= (2\pi)^2 \frac{CN^{n+1}}{t} \sum_{\gamma\in\Gamma} \int_{\rho_{\gamma^{-1}}(F)} e^{-2\pi N t\left(x' - \frac{m}{N}\right)\cdot\left(Y+XY^{-1}X\right)^{-1}_{\frac{m}{N}}\left(x' - \frac{m}{N}\right)}$$
$$\times \|\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{x'}\right)\left(\frac{m}{N} - x'\right)\|^2_{(Y+XY^{-1}X)_{x'}}|dx'| \quad (\because x' := \rho_{\gamma^{-1}}(x))$$
$$= (2\pi)^2 \frac{CN^{n+1}}{t} \int_{\mathbb{R}^n} \|\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{x'}\right)\left(\frac{m}{N} - x'\right)\|^2_{(Y+XY^{-1}X)_{x'}} e^{-2\pi N t\left(x' - \frac{m}{N}\right)\cdot\left(Y+XY^{-1}X\right)^{-1}_{\frac{m}{N}}\left(x' - \frac{m}{N}\right)}|dx'|.$$

Since Ω^t can be described as (4.3),

$$\|\left(\Omega^t_{\frac{m}{N}} - \Omega^t_{x'}\right)\left(\frac{m}{N} - x'\right)\|^2_{(Y+XY^{-1}X)_{x'}}$$
$$= \|\left(\mathrm{Re}\left(\Omega_{\frac{m}{N}} - \Omega_{x'}\right)\right)\left(\frac{m}{N} - x'\right)\|^2_{(Y+XY^{-1}X)_{x'}} + t^2\|\left(\mathrm{Im}\left(\Omega_{\frac{m}{N}} - \Omega_{x'}\right)\right)\left(\frac{m}{N} - x'\right)\|^2_{(Y+XY^{-1}X)_{x'}}.$$

We put

$$R(x') := \|\left(\mathrm{Re}\left(\Omega_{\frac{m}{N}} - \Omega_{x'}\right)\right)\left(\frac{m}{N} - x'\right)\|^2_{(Y+XY^{-1}X)_{x'}},$$
$$I(x') := \|\left(\mathrm{Im}\left(\Omega_{\frac{m}{N}} - \Omega_{x'}\right)\right)\left(\frac{m}{N} - x'\right)\|^2_{(Y+XY^{-1}X)_{x'}}.$$

By changing coordinates as $\tau := \sqrt{(Y+XY^{-1}X)^{-1}_{\frac{m}{N}}}\left(x' - \frac{m}{N}\right)$, $\|D^t\widetilde{\vartheta}^t_{\frac{m}{N}}\|^2_{L^2((TM,J^t)\otimes_{\mathbb{C}}L)}$ can be written by

$$\|D^t\widetilde{\vartheta}^t_{\frac{m}{N}}\|^2_{L^2((TM,J^t)\otimes_{\mathbb{C}}L)} = 2^{2-\frac{n}{2}}\pi^2 CN^{\frac{n}{2}+1}\sqrt{\det(Y+XY^{-1}X)_{\frac{m}{N}}}$$
$$\times\left\{t^{-1-\frac{n}{2}}\int_{\mathbb{R}^n} R\left(\sqrt{(Y+XY^{-1}X)_{\frac{m}{N}}}\tau + \frac{m}{N}\right)(2Nt)^{\frac{n}{2}} e^{-2\pi N t\|\tau\|^2}|d\tau|\right.$$
$$\left.+t^{1-\frac{n}{2}}\int_{\mathbb{R}^n} I\left(\sqrt{(Y+XY^{-1}X)_{\frac{m}{N}}}\tau + \frac{m}{N}\right)(2Nt)^{\frac{n}{2}} e^{-2\pi N t\|\tau\|^2}|d\tau|\right\}.$$

It is well-known that

$$\lim_{t\to\infty}\int_{\mathbb{R}^n} R\left(\sqrt{(Y+XY^{-1}X)_{\frac{m}{N}}}\tau + \frac{m}{N}\right)(2Nt)^{\frac{n}{2}} e^{-2\pi N t\|\tau\|^2}|d\tau| = R\left(\frac{m}{N}\right) = 0,$$
$$\lim_{t\to\infty}\int_{\mathbb{R}^n} I\left(\sqrt{(Y+XY^{-1}X)_{\frac{m}{N}}}\tau + \frac{m}{N}\right)(2Nt)^{\frac{n}{2}} e^{-2\pi N t\|\tau\|^2}|d\tau| = I\left(\frac{m}{N}\right) = 0.$$

Since $n \geq 2$, this proves Theorem 5.2. □

Example 5.4. For Example 2.29, let us consider the compatible almost complex structure associated with

$$Z := \begin{pmatrix} 0 & 0 \\ 0 & x_1 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} \frac{1}{x_1^2+1} & 0 \\ 0 & 1 \end{pmatrix}.$$

The corresponding Ω is

$$\Omega_x = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & x_1 + \sqrt{-1} \end{pmatrix}.$$

This Z does not satisfies (2) in Proposition 3.12, nor the condition 3.13 for any $m \in \mathbb{Z}^2$. In fact, for any $m \in \mathbb{Z}^2$, $\left(\left(\partial_{x_1}\Omega\right)(m - Nx)\right)_2 = m_2 - Nx_2$ while $\left(\left(\partial_{x_2}\Omega\right)(m - Nx)\right)_1 = 0$. In this case, $\widetilde{\vartheta}^t_{\frac{m}{N}}$ can be written as

$$\widetilde{\vartheta}^t_{\frac{m}{N}}(x,y) = \sum_{\gamma\in\mathbb{Z}^2} g_\gamma e^{2\pi\sqrt{-1}N\left[\frac{1}{2}\left\{t\sqrt{-1}(x_1-\gamma_1-\frac{m_1}{N})^2 + \left(\frac{m_1}{N}+t\sqrt{-1}\right)(x_2-\gamma_2-\frac{m_2}{N})^2\right\} + (x_2-\gamma_2)\left\{\frac{1}{2}\gamma_1(x_2+\gamma_2) - \left(\frac{m_2}{N}+\gamma_2\right)\gamma_2\right\}\right]} e^{2\pi\sqrt{-1}(m+N\gamma)\cdot y}.$$

Example 5.5. In the case where $n = 2$ of Example 2.31, we can take the compatible almost complex structure associated with

$$Z := \frac{1}{x_2^2+1}\begin{pmatrix} \lambda^2 x_2^3 & \lambda x_2^2 \\ \lambda x_2^2 & x_2 \end{pmatrix} + \frac{\sqrt{-1}}{x_2^2+1}\begin{pmatrix} (1+\lambda^2)x_2^2+1 & \lambda x_2 \\ \lambda x_2 & 1 \end{pmatrix}.$$

The corresponding Ω is

$$\Omega_x = \begin{pmatrix} \sqrt{-1} & -\sqrt{-1}\lambda x_2 \\ -\sqrt{-1}\lambda x_2 & x_2 + \sqrt{-1}(\lambda^2 x_2^2 + 1) \end{pmatrix}.$$

In this case, $\partial_{x_2}\Omega_{12} = -\sqrt{-1}\lambda$ and $\partial_{x_1}\Omega_{22} = 0$. So, Z satisfies (2) in Proposition 3.12 if and only if $\lambda = 0$, which is the special case of Example 4.5. Equivalently, Z does not satisfy the condition 3.13 for any $m \in \mathbb{Z}^2$ unless $\lambda = 0$. In fact, for any $m \in \mathbb{Z}^2$, $((\partial_{x_1}\Omega)(m - Nx))_2 = 0$ while $((\partial_{x_2}\Omega)(m - Nx))_1 = -\sqrt{-1}\lambda(m_2 - Nx_2)$. In this case, $\widetilde{\vartheta}^t_{\frac{m}{N}}$ can be written as

$$\begin{aligned}\widetilde{\vartheta}^t_{\frac{m}{N}}(x,y) = \sum_{\gamma\in\Gamma} g_\gamma e^{2\pi\sqrt{-1}N\left[\frac{t\sqrt{-1}}{2}\left\{x_1-\gamma_1-\gamma_2\lambda(x_2-\gamma_2)-\frac{m_1}{N}\right\}^2 - t\sqrt{-1}\lambda\frac{m_2}{N}\left\{x_1-\gamma_1-\gamma_2\lambda(x_2-\gamma_2)-\frac{m_1}{N}\right\}\left(x_2-\gamma_2-\frac{m_2}{N}\right)\right.} \\ {}^{\left. +\frac{1}{2}\left\{\frac{m_2}{N}+t\sqrt{-1}\left(\lambda^2\frac{m_2^2}{N^2}+1\right)\right\}\left(x_2-\gamma_2-\frac{m_2}{N}\right)^2+\frac{1}{2}\gamma_2(x_2-\gamma_2)(x_2+\gamma_2)-\left(\frac{m_2}{N}+\gamma_2\right)\gamma_2(x_2-\gamma_2)\right]} \\ \times e^{2\pi\sqrt{-1}\{(m_1+\gamma_2\lambda m_2+N\gamma_1)y_1+(m_2+N\gamma_2)y_2\}}.\end{aligned}$$

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Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashimita, Tama-ku, Kawasaki, 214-8571, Japan

Email address: takahiko@meiji.ac.jp