# ANALYTICAL AND NUMERICAL ASPECTS ON MOTION OF POLYGONAL CURVES WITH CONSTANT AREA SPEED

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## 1   Introduction

The first purpose of this paper is to propose a formulation of general area-preserving motion of polygonal curves by using a system of ODEs. Solution polygonal curves belong to a prescribed polygonal class, which is similar to admissible class used in the so-called crystalline curvature flow. Actually, if the initial curve is a convex polygon, then our polygonal flow is nothing but the crystalline curvature flow. However, if the initial polygon is not convex and does not belong to any admissible class, then the polygonal flow cannot be regarded as a crystalline curvature flow. Because the prescribed polygonal class is determined by the initial polygon and one can take any polygon as the initial data. On the other hand, in the framework of the crystalline curvature flow, the initial polygon should be taken from the admissible class.

The second purpose is to discretize the ODEs implicitly in time keeping a given constant area speed, while the solution polygonal curve exists in the prescribed polygonal class.

The organization of this paper is as follows. In the next section, we will introduce notion of polygonal motion and a polygonal class will be given. In section 3, our problem will be formulated and some examples will be given. In the last section, we will propose a scheme which achieves the second purpose and show convergence of the scheme.

## 2   Polygons and polygonal motions

### 2.1   Polygons

We define a set of polygons in $\mathbb{R}^2$:

$$\mathcal{P} := \{\Gamma;\ \Gamma \text{ is a polygonal Jordan curve in } \mathbb{R}^2\}.$$

For $\Gamma \in \mathcal{P}$, the bounded interior polygonal domain surrounded by $\Gamma$ is denoted by $\Omega$. For simplicity, we consider the case that $\Omega$ is simply connected, but many of the following arguments are valid in other geometrical situations, even in three dimensional case, with some minor changes.

Let $\Gamma \in \mathcal{P}$ be an $N$-polygon. The $N$ vertices of $\Gamma$ are denoted by $\boldsymbol{w}_j \in \mathbb{R}^2$ for $j = 1, 2, \ldots, N$ counterclockwise, where $\boldsymbol{w}_0 = \boldsymbol{w}_N$ and $\boldsymbol{w}_{N+1} = \boldsymbol{w}_1$. Hereafter we use the periodic boundary condition $\mathsf{F}_0 = \mathsf{F}_N$ and $\mathsf{F}_{N+1} = \mathsf{F}_1$ for any quantities defined on $N$-polygon.

The $j$th edge between $\boldsymbol{w}_{j-1}$ and $\boldsymbol{w}_j$ is

$$\Gamma_j = \{(1-\theta)\boldsymbol{w}_{j-1} + \theta\boldsymbol{w}_j;\ 0 < \theta < 1\} \quad (j = 1, 2, \ldots, N),$$

and their lengths are $|\Gamma_j| := |\boldsymbol{w}_j - \boldsymbol{w}_{j-1}|$. We define $\chi_j \in L^\infty(\Gamma)$ as

$$\chi_j(\boldsymbol{x}) := \begin{cases} 1, & \boldsymbol{x} \in \Gamma_j \\ 0, & \boldsymbol{x} \in \Gamma \setminus \Gamma_j \end{cases} \quad (j = 1, 2, \ldots, N),$$

which is the characteristic function of $\Gamma_j$.

The counterclockwise tangential unit vector and the outward unit normal are denoted by $\boldsymbol{t}_j$ and $\boldsymbol{n}_j$, where $\boldsymbol{t}_j = (\boldsymbol{w}_j - \boldsymbol{w}_{j-1})/|\Gamma_j|$ and $\boldsymbol{n}_j$ is defined such as $\det(\boldsymbol{n}_j, \boldsymbol{t}_j) = 1$. The outer angle at the vertex $\boldsymbol{w}_j$ is denoted by $\varphi_j \in (-\pi, \pi) \setminus \{0\}$. We remark that

$$\cos \varphi_j = \boldsymbol{t}_{j+1} \cdot \boldsymbol{t}_j = \boldsymbol{n}_{j+1} \cdot \boldsymbol{n}_j.$$

We define the height of $\Gamma_j$ from the origin $h_j := \boldsymbol{w}_j \cdot \boldsymbol{n}_j$. They satisfy the equalities

$$|\Gamma_j| = a_{j-1} h_{j-1} + b_j h_j + a_j h_{j+1} \quad (j = 1, 2, \ldots, N), \tag{2.1}$$

where $a_j := (\sin \varphi_j)^{-1}$ and $b_j := -\cot \varphi_{j-1} - \cot \varphi_j$. This equality can be checked from the fact that the straight line including $\Gamma_j$ is expressed by the equation $\boldsymbol{n}_j \cdot \boldsymbol{x} = h_j$. The total length of $\Gamma$ is given by

$$|\Gamma| := \sum_{j=1}^{N} |\Gamma_j| = \sum_{j=1}^{N} (a_j + b_j + a_{j-1}) h_j = \sum_{j=1}^{N} \eta_j h_j, \tag{2.2}$$

where $\eta_j := a_j + b_j + a_{j-1} = \tan(\varphi_j/2) + \tan(\varphi_{j-1}/2)$.

The area of interior domain $\Omega$ is denoted by $|\Omega|$, which is given by

$$|\Omega| = \frac{1}{2} \sum_{j=1}^{N} |\Gamma_j| h_j. \tag{2.3}$$

The above symbols are also written as $\Omega = \Omega(\Gamma)$, $\boldsymbol{n}_j = \boldsymbol{n}_j(\Gamma)$ and $h_j = h_j(\Gamma)$ etc., if we need to distinguish from quantities of the other polygons.

## 2.2 Motion of polygons and Lipschitz mappings

We consider a moving polygon $\Gamma(t) \in \mathcal{P}$, where the parameter $t$ (we call $t$ time) belongs to an interval $\mathcal{I} \subset \mathbb{R}$. For $k \in \mathbb{N} \cup \{0\}$, we call a moving polygon $\Gamma(t)$ belongs to $C^k$-class on $\mathcal{I}$, if the number of edges of $\Gamma(t)$ does not change in time and $\boldsymbol{w}_j \in C^k(\mathcal{I}, \mathbb{R}^2)$ for all $j = 1, 2, \ldots, N$.

If $k \geq 1$, we can define the normal velocity at $\boldsymbol{x} \in \Gamma_j(t)$ which is the $j$th edge of $\Gamma(t)$. We suppose $\boldsymbol{x}^* \in \Gamma_j(t^*)$ and $\boldsymbol{x}^* = (1 - \theta) \boldsymbol{w}_{j-1}(t^*) + \theta \boldsymbol{w}_j(t^*)$ for some $\theta \in (0, 1)$, and define $\boldsymbol{x}(\theta, t) := (1 - \theta) \boldsymbol{w}_{j-1}(t) + \theta \boldsymbol{w}_j(t) \in \Gamma(t)$. Then the outward normal velocity of $\Gamma_j(t^*)$ at $\boldsymbol{x}^*$ is defined by

$$V_j(\boldsymbol{x}^*, t^*) := \dot{\boldsymbol{x}}(\theta, t^*) \cdot \boldsymbol{n}_j(t^*) = (1 - \theta) \dot{\boldsymbol{w}}_{j-1}(t^*) \cdot \boldsymbol{n}_j(t^*) + \theta \dot{\boldsymbol{w}}_j(t^*) \cdot \boldsymbol{n}_j(t^*).$$

Here and hereafter, the (partial) derivative of $\mathsf{F}$ with respect to $t$ is denoted by $\dot{\mathsf{F}}$. We remark that $V_j(\cdot, t)$ is a linear function on each $\Gamma_j(t)$. We define the normal velocity of $\Gamma(t)$ by

$$V(\cdot, t) := \sum_{j=1}^{N} V_j(\cdot, t) \chi_j(\cdot, t) \in L^\infty(\Gamma(t)),$$

where $\chi_j(\cdot, t) \in L^\infty(\Gamma(t))$ is the characteristic function of $\Gamma_j(t)$.

For a $C^k$-class moving $N$-polygon $\Gamma(t)$ $(t \in \mathcal{I})$ with its interior domain $\Omega(t)$, we construct Lipschitz mappings smoothly parametrized by $t$ from a fixed polygonal domain.

We fix $t^* \in \mathcal{I}$ and define $\Gamma^* := \Gamma(t^*)$ and $\Omega^* := \Omega(t^*)$. We also choose another domain $Q$ with $\cup_{t \in \mathcal{I}} \Omega(t) \subset Q$ and fix it. Our aim of this subsection is to construct Lipschitz mappings $\Phi(t) = \Phi(\cdot, t)$ from $\Omega^*$ to $\Omega(t)$ smoothly parametrized by $t$.

**Proposition 2.1** *Under the above condition, there exists $\varepsilon > 0$, $\mathcal{I}^* := \mathcal{I} \cap (t^* - \varepsilon, \ t^* + \varepsilon)$, and $\Phi \in C^k(\mathcal{I}^*, W^{1,\infty}(Q, \mathbb{R}^2))$, and they satisfy the following conditions.*

(1) *$\Phi(t)$ is a bi-Lipschitz transform from $\overline{Q}$ onto itself, i.e. $\Phi(t)$ is bijective from $\overline{Q}$ onto itself and $\Phi(t)$ and $\Phi(t)^{-1}$ are both Lipschitz continuous on $\overline{Q}$.*

(2) *$\Phi(\boldsymbol{x}, t) = \boldsymbol{x}$ for $\boldsymbol{x}$ in a neighborhood of $\partial Q$ for $t \in \mathcal{I}^*$.*

(3) *$\Phi(\overline{\Omega^*}, t) = \overline{\Omega(t)}$ for $t \in \mathcal{I}^*$, and $\Phi(t)$ is an affine map from each edge $\Gamma_j^* := \Gamma_j(t^*)$ onto $\Gamma_j(t)$ with $\Phi(\boldsymbol{w}_j(t^*), t) = \boldsymbol{w}_j(t)$.*

*Proof.* Without loss of generality, we assume that $Q$ is a bounded polygonal domain, too. We consider a triangulation $\mathcal{T}$ of $\Omega^*$ and $Q$. Namely $\mathcal{T}$ is a collection of triangular subatomics of $Q$ with

$$\overline{Q} = \bigcup_{K \in \mathcal{T}} \overline{K}, \ \text{ and } K \cap K' = \emptyset \ \text{ if } K, K' \in \mathcal{T}, \ K \neq K',$$

and $\overline{K} \cap \overline{K'}$ is either the empty set, a common vertex or a common edge of $K$ and $K' \in \mathcal{T}$, and there is a subset $\mathcal{T}_0 \subset \mathcal{T}$ such that

$$\overline{\Omega^*} = \bigcup_{K \in \mathcal{T}_0} \overline{K}, \ \text{ and } \Gamma^* \cap \mathcal{N} = \{\boldsymbol{w}_j^*\}_{j=1}^N,$$

where $\mathcal{N}$ denotes the set of all vertices of triangles in $\mathcal{T}$ and $\boldsymbol{w}_j^* := \boldsymbol{w}_j(t^*)$. We also suppose that there does not exist any $K \in \mathcal{T}$ with $\overline{K} \cap \Gamma^* \neq \emptyset$ and $\overline{K} \cap \partial Q \neq \emptyset$.

We assume that $\Phi(\boldsymbol{x}, t)$ has the following form:

$$\Phi(\boldsymbol{x}, t) = A_K(t) \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \quad (\boldsymbol{x} \in K \in \mathcal{T}), \tag{2.1}$$

where $A_K(t)$ is a $2 \times 3$ matrix depending on $K \in \mathcal{T}$ and $t$. To determine $A_K(t)$, we suppose the condition:

$$\Phi(\boldsymbol{x}, t) = \begin{cases} \boldsymbol{x} & \text{if } \boldsymbol{x} \in \mathcal{N} \setminus \Gamma^*, \\ \boldsymbol{w}_j(t) & \text{if } \boldsymbol{x} = \boldsymbol{w}_j^* \in \mathcal{N} \cap \Gamma^*. \end{cases} \tag{2.2}$$

For sufficiently small $\varepsilon > 0$, for $t \in \mathcal{I}^*$, $\Phi(\boldsymbol{x}, t)$ is uniquely determined by the conditions (2.1) and (2.2). It is also clear that $\Phi(t) = \Phi(\cdot, t) \in W^{1,\infty}(Q, \mathbb{R}^2)$ is bijective from $Q$ onto itself and $\Phi(\boldsymbol{x}, t) = \boldsymbol{x}$ for $\boldsymbol{x} \in K$ if $\overline{K} \cap \Gamma^* = \emptyset$.

Let us fix $K \in \mathcal{T}$ (with $\overline{K} \cap \Gamma^* \neq \emptyset$) and let $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ be the vertices of $K$. Then, from (2.2), $\boldsymbol{y}_l(t) := \Phi(\boldsymbol{x}_l, t)$ satisfies the condition $\boldsymbol{y}_l \in C^k(\mathcal{I}^*, \mathbb{R}^2)$ for $l = 1, 2, 3$. Since

$$\boldsymbol{y}_l = A_K(t) \begin{pmatrix} \boldsymbol{x}_l \\ 1 \end{pmatrix} \quad (l = 1, 2, 3),$$

we have

$$A_K(t) = (\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \boldsymbol{y}_3(t)) \begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ 1 & 1 & 1 \end{pmatrix}^{-1}.$$

From this expression, we obtain $A_K \in C^k(\mathcal{I}^*, \mathbb{R}^{2\times 3})$ and hence $\Phi \in C^k(\mathcal{I}^*, W^{1,\infty}(Q, \mathbb{R}^2))$ follows. ∎

As an application of this proposition, we have the following theorem which is well-known in the case $\Gamma(t)$ is a smooth Jordan curve.

**Theorem 2.2** *Let $\Gamma(t) \in \mathcal{P}$ be a $C^1$-class moving $N$-polygon on an interval $t \in \mathcal{I}$ with its interior domain $\Omega(t)$. For all $\phi \in C^1(\mathbb{R}^2 \times \mathcal{I})$, the map $[t \mapsto \int_{\Omega(t)} \phi(\boldsymbol{x}, t)\, d\boldsymbol{x}]$ belongs to $C^1(\mathcal{I})$ and*

$$\frac{d}{dt} \int_{\Omega(t)} \phi(\boldsymbol{x}, t)\, d\boldsymbol{x} = \int_{\Omega(t)} \dot{\phi}(\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\Gamma(t)} \phi(\boldsymbol{x}, t) V(\boldsymbol{x}, t)\, ds$$

*holds.*

*Proof.* We define $f(t) := \int_{\Omega(t)} \phi(\boldsymbol{x}, t)\, d\boldsymbol{x}$. Under the setting of Proposition 2.1, we show $f \in C^1(\mathcal{I}^*)$ and calculate $\dot{f}(t^*)$. For $t \in \mathcal{I}^*$, we have

$$f(t) = \int_{\Omega^*} \phi(\Phi(\boldsymbol{x}, t), t) J(\boldsymbol{x}, t)\, d\boldsymbol{x},$$

where $J(\boldsymbol{x}, t)$ is the Jacobian defined by $J(\boldsymbol{x}, t) := \det(\nabla_{\boldsymbol{x}} \Phi^{\mathrm{T}}(\boldsymbol{x}, t))$. We remark that $J \in C^1(\mathcal{I}^*, L^\infty(Q))$ and $\dot{J}(\boldsymbol{x}, t^*) = \mathrm{div}_{\boldsymbol{x}} \dot{\Phi}(\boldsymbol{x}, t^*)$ since $\Phi \in C^1(\mathcal{I}^*, W^{1,\infty}(Q, \mathbb{R}^2))$ and $\Phi(\boldsymbol{x}, t^*) = \boldsymbol{x}$. Hence, we obtain

$$
\begin{aligned}
\dot{f}(t^*) &= \int_{\Omega^*} (\nabla_{\boldsymbol{x}} \phi(\boldsymbol{x}, t^*) \cdot \dot{\Phi}(\boldsymbol{x}, t^*) + \dot{\phi}(\boldsymbol{x}, t^*))\, d\boldsymbol{x} + \int_{\Omega^*} \phi(\boldsymbol{x}, t^*) \dot{J}(\boldsymbol{x}, t^*)\, d\boldsymbol{x} \\
&= \int_{\Omega^*} \mathrm{div}_{\boldsymbol{x}}(\phi(\boldsymbol{x}, t^*) \dot{\Phi}(\boldsymbol{x}, t^*))\, d\boldsymbol{x} + \int_{\Omega^*} \dot{\phi}(\boldsymbol{x}, t^*)\, d\boldsymbol{x} \\
&= \int_{\Gamma^*} \phi(\boldsymbol{x}, t^*) V(\boldsymbol{x}, t^*)\, ds + \int_{\Omega^*} \dot{\phi}(\boldsymbol{x}, t^*)\, d\boldsymbol{x}
\end{aligned}
$$

∎

In the case where $\phi \equiv 1$, we obtain the following formula for $C^1$-class moving polygon particularly.

$$\frac{d}{dt} |\Omega(t)| = \int_{\Gamma(t)} V(\boldsymbol{x}, t)\, ds, \tag{2.3}$$

where $|\Omega(t)|$ stands for the area of $\Omega(t)$.

We remark that Proposition 2.1 and Theorem 2.2 with their proofs are valid even in the three dimensional case.

## 2.3 Polygonal motion

For two polygons $\Gamma$ and $\Sigma \in \mathcal{P}$, we define an equivalence relation $\Gamma \sim \Sigma$. We say $\Gamma \sim \Sigma$, if their numbers of edges are same (let it be $N$) and $\boldsymbol{n}_j(\Gamma) = \boldsymbol{n}_j(\Sigma)$ for all $j = 1, 2, \ldots, N$ after choosing suitable counterclockwise numbering for $\Gamma$ and $\Sigma$. The equivalence class of $\Gamma \in \mathcal{P}$ is denoted by $\mathcal{P}[\Gamma] := \{\Sigma \in \mathcal{P}; \Sigma \sim \Gamma\}$.

We fix an $N$-polygon $\Gamma^* \in \mathcal{P}$ and let $\mathcal{P}^* := \mathcal{P}[\Gamma^*]$. For $\Gamma$ and $\Sigma$ in $\mathcal{P}^*$, we define the distance between them by

$$d(\Gamma, \Sigma) := \max_{j=1,2,\ldots,N} |h_j(\Gamma) - h_j(\Sigma)|.$$

Then, it is clear that $(\mathcal{P}^*, d)$ becomes a metric space since it is isometrically embedded in $\mathbb{R}^N$ equipped with maximum norm $|\cdot|_\infty$ by the height function $\boldsymbol{h}$: $\mathcal{P}^* \ni \Gamma \mapsto \boldsymbol{h}(\Gamma) = (h_1(\Gamma), h_2(\Gamma), \ldots, h_N(\Gamma))^{\mathrm{T}} \in \mathbb{R}^N$. We have the following proposition.

**Proposition 2.3** *The set $\boldsymbol{h}(\mathcal{P}^*)$ is an open convex subset of $\mathbb{R}^N$.*

*Proof.* It is clear that $\boldsymbol{h}(\mathcal{P}^*)$ is open. For any $\Gamma^0$ and $\Gamma^1 \in \mathcal{P}^*$ and for $\theta \in [0, 1]$, we define

$$\boldsymbol{h}^\theta := (1 - \theta)\boldsymbol{h}(\Gamma^0) + \theta\boldsymbol{h}(\Gamma^1) \in \mathbb{R}^N.$$

From (2.1) and the inequality

$$a_{j-1}h^\theta_{j-1} + b_j h^\theta_j + a_j h^\theta_{j+1} = (1-\theta)|\Gamma^0_j| + \theta|\Gamma^1_j| \geq \min\{|\Gamma^0_j|,\ |\Gamma^1_j|\} > 0 \quad (j = 1, 2, \ldots, N),$$

we can determine the $\theta$-interpolation $\Gamma^\theta \in \mathcal{P}^*$ by the condition $\boldsymbol{h}(\Gamma^\theta) = \boldsymbol{h}^\theta$. ∎

The $\theta$-interpolation of $\Gamma^0 \in \mathcal{P}^*$ and $\Gamma^1 \in \mathcal{P}^*$ is denoted by $(1 - \theta)\Gamma^0 + \theta\Gamma^1 := \Gamma^\theta \in \mathcal{P}^*$. We remark that it satisfies

$$|\Gamma^\theta_j| \geq \min\{|\Gamma^0_j|,\ |\Gamma^1_j|\}.$$

For $\Gamma \in \mathcal{P}^*$ and $\varepsilon > 0$, $\varepsilon$-ball in $\mathcal{P}^* = \mathcal{P}[\Gamma]$ with center $\Gamma$ is denoted by

$$B(\Gamma, \varepsilon) := \{\Sigma \in \mathcal{P}[\Gamma];\ d(\Sigma, \Gamma) < \varepsilon\}.$$

For an open set $\mathcal{O} \subset \mathcal{P}^*$ and $\Gamma \in \mathcal{O}$, we define a positive number $\rho(\Gamma, \mathcal{O}) > 0$ as

$$\rho(\Gamma, \mathcal{O}) := \inf\{|\boldsymbol{a} - \boldsymbol{h}(\Gamma)|_\infty;\ \boldsymbol{a} \in \mathbb{R}^N \setminus \boldsymbol{h}(\mathcal{O})\}.$$

We remark that $\rho(\cdot, \mathcal{O})$ is Lipschitz continuous with Lipschitz constant 1:

$$|\rho(\Gamma, \mathcal{O}) - \rho(\Sigma, \mathcal{O})| \leq d(\Gamma, \Sigma) \quad (\Gamma, \Sigma \in \mathcal{O}).$$

For a compact set $\mathcal{K} \subset \mathcal{O}$, we also define

$$\rho(\mathcal{K}, \mathcal{O}) := \min_{\Gamma \in \mathcal{K}} \rho(\Gamma, \mathcal{O}).$$

Let $a^*_j := a_j[\Gamma^*]$ and $b^*_j := b_j[\Gamma^*]$. Then, from the formula (2.1), we obtain

$$\begin{aligned}
||\Gamma_j| &- |\Sigma_j|| \\
&= |a^*_{j-1}(h_{j-1}[\Gamma] - h_{j-1}[\Sigma]) + b^*_j(h_j[\Gamma] - h_j[\Sigma]) + a^*_j(h_{j+1}[\Gamma] - h_{j+1}[\Sigma])| \\
&\leq C^* d(\Gamma, \Sigma) \quad (j = 1, 2, \ldots, N),
\end{aligned}$$

where we define

$$C^* := \max_{l=1,2,\ldots,N}\{|a^*_{l-1}| + |b^*_l| + |a^*_l|\}. \tag{2.4}$$

For a compact set $\mathcal{K} \subset \mathcal{P}^*$, we define

$$\sigma(\mathcal{K}) := \min\{|\Gamma_j|;\ \Gamma \in \mathcal{K},\ j = 1, 2, \ldots, N\} > 0.$$

We consider a $C^k$-class moving polygon $\Gamma(t) \in \mathcal{P}^*$ $(t \in \mathcal{I})$. We call it **polygonal motion in $\mathcal{P}^*$** in this paper. We remark that a polygonal motion $\Gamma(t) \in \mathcal{P}^*$ $(t \in \mathcal{I})$ belongs to $C^k$-class if and only if $h_j \in C^k(\mathcal{I})$ for $j = 1, 2, \ldots, N$. If $\Gamma(t)$ is

a $C^1$-class polygonal motion, its normal velocity $V_j$ of $\Gamma_j(t)$ is a constant on each $\Gamma_j(t)$ and it is given by $V_j(t) = \dot{h}_j(t)$. The formula (2.3) is written in the form:

$$\frac{d}{dt}|\Omega(t)| = \sum_{j=1}^{N} |\Gamma_j(t)| V_j(t). \tag{2.5}$$

We fix an equivalence class $\mathcal{P}^*$ of polygons and let its $j$th outward unit normal be $\boldsymbol{n}_j$ and outer angle $\varphi_j$. For $\Gamma \in \mathcal{P}^*$, the **polygonal curvature** $\kappa_j$ of $\Gamma_j$ is defined by

$$\kappa_j := \frac{\eta_j}{|\Gamma_j|}.$$

We also define the polygonal curvature of $\Gamma$ by

$$\kappa := \sum_{j=1}^{N} \kappa_j \chi_j \in L^\infty(\Gamma).$$

The reason why this is called "curvature" is shown by the following proposition.

**Proposition 2.4** *Let $\Gamma(t)$ $(t \in \mathcal{I})$ be a $C^1$-class polygonal motion in $\mathcal{P}^*$. Then*

$$\frac{d}{dt}|\Gamma(t)| = \sum_{j=1}^{N} |\Gamma_j(t)| \kappa_j(t) V_j(t) = \int_{\Gamma(t)} \kappa(\boldsymbol{x}, t) V(\boldsymbol{x}, t) \, ds.$$

*Proof.* We obtain

$$\frac{d}{dt}|\Gamma(t)| = \frac{d}{dt} \sum_{j=1}^{N} \eta_j h_j(t) = \sum_{j=1}^{N} \eta_j V_j(t) = \sum_{j=1}^{N} |\Gamma_j(t)| \kappa_j(t) V_j(t),$$

from the formula (2.2). ∎

## 3   Initial value problem of polygonal motion

### 3.1   General polygonal motion problems

We fix an equivalence class of $N$-polygons $\mathcal{P}^*$ as in §2.3. For an open set $\mathcal{O} \subset \mathcal{P}^*$ and $T_* \in (0, \infty]$, let $F$ be a given continuous function from $\mathcal{O} \times [0, T_*)$ to $\mathbb{R}^N$ with the local Lipschitz property: For arbitrary compact set $\mathcal{K} \subset \mathcal{O}$ and $T \in (0, T_*)$, there exists $L(\mathcal{K}, T) > 0$ such that

$$|F(\Gamma, t) - F(\Sigma, t)|_\infty \leq L(\mathcal{K}, T) \, d(\Gamma, \Sigma) \quad (\Gamma, \Sigma \in \mathcal{K}, \ t \in [0, T]). \tag{3.1}$$

Under the condition (3.1), for a compact set $\mathcal{K} \subset \mathcal{O}$ and $T \in (0, T_*)$, we also define

$$M(\mathcal{K}, T) := \max\{|F(\Gamma, t)|_\infty; \ \Gamma \in \mathcal{K}, \ t \in [0, T]\} > 0.$$

We consider the following initial value problem of polygonal motion.

**Problem 3.1** *For a given $N$-polygon $\Gamma^* \in \mathcal{O}$, find a $C^1$-class polygonal motion $\Gamma(t) \in \mathcal{O}$ $(0 \leq t \leq T < T_*)$ such that*

$$\begin{cases} V_j(t) = F_j(\Gamma(t), t) & (t \in [0, T], \ j = 1, 2, \ldots, N) \\ \Gamma(0) = \Gamma^*. \end{cases}$$

Under the Lipschitz condition (3.1), it is clear that there exists a local solution $\Gamma(t)$ in a short time interval $[0, T]$, since Problem 3.1 can be expressed by an initial value problem of an ordinary differential equations for $\boldsymbol{h}(t) = (h_1(t), h_2(t), \ldots, h_N(t))^{\mathrm{T}}$.

We often assume the following condition for $F_j$:

$$\sum_{j=1}^{N} |\Gamma_j| F_j(\Gamma, t) = \mu \quad (\Gamma \in \mathcal{O}, \ t \in [0, T_*)), \tag{3.2}$$

where $\mu$ is a fixed real number. Under the assumption (3.2), from the formula (2.5), any solution $\Gamma(t)$ to Problem 3.1 has the property of the following constant area speed (**CAS** in short):

$$\frac{d}{dt}|\Omega(t)| = \mu.$$

The polygonal flow is regarded as the crystalline curvature flow if the initial polygon $\Gamma^*$ is convex (as mentioned in introduction, if $\Gamma^*$ is not convex, then the polygonal flow is different from the crystalline curvature flow). There are many articles about the crystalline curvature flow and asymptotic behavior of solutions [1, 3, 4, 5, 6, 9, 10, 11, 12], etc., which started from the pioneer works [2] and [7].

## 3.2 Examples of problems of polygonal motion

In this section, we give some basic examples of polygonal motions which are nice polygonal analogues of corresponding smooth moving boundary problems.

**Problem 3.2 (polygonal curvature flow)** *For a given $N$-polygon $\Gamma^* \in \mathcal{P}^*$, find a $C^1$-class family of $N$-polygons $\bigcup_{0 \leq t \leq T} \Gamma(t) \subset \mathcal{P}^*$ $(T < T_*)$ satisfying*

$$\begin{cases} V_j(t) = -\kappa_j(t) & (t \in [0, T], \ j = 1, 2, \ldots, N), \\ \Gamma(0) = \Gamma^*. \end{cases}$$

The solution has CAS property with $\mu = -2\sum_{j=1}^{N} \tan(\varphi_j/2)$:

$$\frac{d}{dt}|\Omega(t)| = -\sum_{j=1}^{N} \kappa_j(t)|\Gamma_j(t)| = -\sum_{j=1}^{N} \eta_j = -2\sum_{j=1}^{N} \tan\frac{\varphi_j}{2} = const.$$

**Problem 3.3 (area-preserving polygonal curvature flow)** *For a given $N$-polygon $\Gamma^* \in \mathcal{P}^*$, find a $C^1$-class family of $N$-polygons $\bigcup_{0 \leq t \leq T} \Gamma(t) \subset \mathcal{P}^*$ $(T < T_*)$ satisfying*

$$\begin{cases} V_j(t) = \langle \kappa(\cdot, t) \rangle - \kappa_j(t) & (t \in [0, T], \ j = 1, 2, \ldots, N), \\ \Gamma(0) = \Gamma^*. \end{cases}$$

*Here $\langle \kappa(\cdot, t) \rangle$ is the mean value of $\kappa$ on $\Gamma(t)$:*

$$\langle \kappa(\cdot, t) \rangle = \frac{1}{|\Gamma(t)|} \int_{\Gamma(t)} \kappa(\boldsymbol{x}, t)\, ds = \frac{\sum_{i=1}^{N} \eta_i}{|\Gamma(t)|} = \frac{2\sum_{i=1}^{N} \tan(\varphi_i/2)}{|\Gamma(t)|}.$$

The solution has CAS property with $\mu = 0$:

$$\frac{d}{dt}|\Omega(t)| = \langle \kappa(\cdot, t) \rangle |\Gamma(t)| - \int_{\Gamma(t)} \kappa(\boldsymbol{x}, t) \, ds = 0.$$

In what follows, the mean value of $\mathsf{F}$ on the edge $\Gamma_j$ is denoted by

$$\langle \mathsf{F} \rangle_j := \frac{1}{|\Gamma_j|} \int_{\Gamma_j} \mathsf{F}(\boldsymbol{x}) \, ds.$$

Let $G$ be a bounded Lipschitz domain in $\mathbb{R}^2$. We define

$$\mathcal{O}_G := \{ \Gamma \in \mathcal{P}^*; \ \Omega(\Gamma) \supset \overline{G} \}.$$

**Problem 3.4 (area-preserving polygonal advected flow)** *Let us consider a divergence free vector field* $\boldsymbol{u} \in C^1(\mathbb{R}^2 \setminus G; \ \mathbb{R}^2)$ *with* $\operatorname{div} \boldsymbol{u} = 0$ *in* $\mathbb{R}^2 \setminus G$. *For a given $N$-polygon* $\Gamma^* \in \mathcal{O}_G$, *find a $C^1$-class family of $N$-polygons* $\bigcup_{0 \leq t \leq T} \Gamma(t) \subset \mathcal{O}_G$ *($T < T_*$) satisfying*

$$\begin{cases} V_j(t) = \langle \boldsymbol{u} \cdot \boldsymbol{n} \rangle_j & (t \in [0, T], \ j = 1, 2, \ldots, N), \\ \Gamma(0) = \Gamma^*. \end{cases}$$

The solution has CAS property with $\mu = \displaystyle\int_{\partial G} \boldsymbol{n} \cdot \boldsymbol{u} \, ds$:

$$\frac{d}{dt}|\Omega(t)| = -\int_{\Gamma(t)} \boldsymbol{n}_j \cdot \boldsymbol{u} \, ds = -\int_{\Omega(t) \setminus \overline{G}} \operatorname{div} \boldsymbol{u} \, d\boldsymbol{x} + \int_{\partial G} \boldsymbol{n} \cdot \boldsymbol{u} \, ds = \int_{\partial G} \boldsymbol{n} \cdot \boldsymbol{u} \, ds,$$

where $\boldsymbol{n}_j$ and $\boldsymbol{n}$ are the outward unit normal vector to $\partial(\Omega(t) \setminus \overline{G})$.

**Problem 3.5 (polygonal Hele-Shaw flow)** *For a given $N$-polygon* $\Gamma^* \in \mathcal{O}_G$ *and a function $b$ defined on* $\partial G \times [0, T]$, *find a $C^1$-class family of $N$-polygons* $\bigcup_{0 \leq t \leq T} \Gamma(t) \subset \mathcal{O}_G$ *($T < T_*$) satisfying*

$$\begin{cases} V_j(t) = -\boldsymbol{n}_j \cdot \nabla p(\boldsymbol{x}, t) & (\boldsymbol{x} \in \Gamma_j(t), \ t \in [0, T], \ j = 1, 2, \ldots, N), \\ \Delta p(\boldsymbol{x}, t) = 0 & (\boldsymbol{x} \in \Omega(t) \setminus \overline{G}, \ t \in [0, T]), \\ \langle p(\cdot, t) \rangle_j = \kappa_j(t) & (t \in [0, T], \ j = 1, 2, \ldots, N), \\ \dfrac{\partial p}{\partial \boldsymbol{n}}(\boldsymbol{x}, t) = b(\boldsymbol{x}, t) & (\boldsymbol{x} \in \partial G, \ t \in [0, T]), \\ \Gamma(0) = \Gamma^*. \end{cases}$$

Here $\partial p / \partial \boldsymbol{n} = \nabla_{\boldsymbol{x}} p \cdot \boldsymbol{n}$ and $\boldsymbol{n}_j$ and $\boldsymbol{n}$ are the outward unit normal vector to $\partial(\Omega(t) \setminus \overline{G})$. The solution has a given area speed property:

$$\frac{d}{dt}|\Omega(t)| = -\int_{\Gamma(t)} \boldsymbol{n}_j \cdot \nabla p \, ds = -\int_{\Omega(t) \setminus \overline{G}} \Delta p \, d\boldsymbol{x} + \int_{\partial G} \frac{\partial p}{\partial \boldsymbol{n}}(\boldsymbol{x}, t) \, ds = \int_{\partial G} b(\boldsymbol{x}, t) \, ds.$$

If $b(\boldsymbol{x}, t) \equiv b_0$ for a given constant $b_0$, then the solution has CAS property with $\mu = |\partial G| b_0$.

## 4 Numerical schemes

### 4.1 Notation

In §4, we consider time discretization of Problem 3.1 with the following notation. The discrete time steps are denoted by $0 = t_0 < t_1 < t_2 < \cdots < t_{\bar{m}} \leq T$. The step size which may be nonuniform and their maximum size are defined by

$$\tau_m := t_{m+1} - t_m \quad (m = 0, 1, \cdots, \bar{m} - 1), \quad \tau := \max_{0 \leq m < \bar{m}} \tau_m.$$

Approximate solution of $\Gamma(t_m)$ is denoted by $\Gamma^m \in \mathcal{P}^*$. Quantities of the polygon $\Gamma^m$ are denoted by $h_j^m := h_j(\Gamma^m)$, and $\kappa_j^m := \kappa_j(\Gamma^m)$, etc. We define $e_j^m := h_j(t_m) - h_j^m$ and $\boldsymbol{e}^m := (e_1^m, e_2^m, \ldots, e_N^m)^{\mathrm{T}} \in \mathbb{R}^N$. Then we have $d(\Gamma(t_m), \Gamma^m) = |\boldsymbol{e}^m|_\infty$.

The discrete normal velocity $V_j^m$, which is an approximation of $V_j(t_m) = \dot{h}_j(t_m)$, is defined by

$$V_j^m := \frac{h_j^{m+1} - h_j^m}{\tau_m} \quad (m = 0, 1, \ldots, \bar{m} - 1).$$

Corresponding to the formula (2.5), the following formula holds.

$$\frac{|\Omega^{m+1}| - |\Omega^m|}{\tau_m} = \sum_{j=1}^{N} \frac{|\Gamma_j^m| + |\Gamma_j^{m+1}|}{2} V_j^m. \tag{4.1}$$

This has a form of sum of areas of $N$ trapezoids and is actually derived from (2.3) as follows:

$$
\begin{aligned}
|\Omega^{m+1}| - |\Omega^m| &= \frac{1}{2} \sum_{j=1}^{N} \left( |\Gamma_j^{m+1}| h_j^{m+1} - |\Gamma_j^m| h_j^m \right) \\
&= \frac{1}{2} \sum_{j=1}^{N} \left\{ (|\Gamma_j^{m+1}| + |\Gamma_j^m|)(h_j^{m+1} - h_j^m) + |\Gamma_j^{m+1}| h_j^m - |\Gamma_j^m| h_j^{m+1} \right\} \\
&= \frac{\tau_m}{2} \sum_{j=1}^{N} (|\Gamma_j^{m+1}| + |\Gamma_j^m|) V_j^m + \frac{1}{2} \sum_{j=1}^{N} \left( |\Gamma_j^{m+1}| h_j^m - |\Gamma_j^m| h_j^{m+1} \right),
\end{aligned}
$$

where the last sum is equal to zero due to the equality (2.1).

In the following subsections, we suppose that there exists a unique solution $\Gamma(t)$ for $0 \leq t \leq T < T_*$ to Problem 3.1 under the condition (3.1), and that discrete time steps $0 = t_0 < t_1 < t_2 < \cdots < t_{\bar{m}} \leq T$ are given a priori such as the uniform time stepping $t_m = m\tau$. It is, however, possible to apply any a posteriori adaptive time step control scheme.

### 4.2 Euler scheme

We consider the following Euler scheme to discretized Problem 3.1.

**Problem 4.1** *For a given $N$-polygon $\Gamma_* \in \mathcal{O}$ and time steps $0 = t_0 < t_1 < t_2 < \cdots < t_{\bar{m}} \leq T$, find polygons $\Gamma^m \in \mathcal{O}$ $(m = 1, 2, \ldots, \bar{m})$ such that*

$$
\begin{cases}
V_j^m = F_j(\Gamma^m, t_m) \quad (m = 0, 1, 2, \ldots, \bar{m} - 1, \ j = 1, 2, \ldots, N), \\
\Gamma^0 = \Gamma_*.
\end{cases}
$$

**Theorem 4.2** *We suppose that $\{\Gamma(t)\}_{0 \le t \le T}$ be a $C^1$-class solution of Problem 3.1. There exists $\delta^* > 0$ and $\tau^* > 0$, $C_1 > 0$ and $C_2 > 0$ such that, if $d(\Gamma(0), \Gamma^0) \le \delta^*$ and $\tau \le \tau^*$, then $\Gamma^m \in \mathcal{O}$ $(m = 1, 2, \ldots, \bar{m})$ are inductively determined and*

$$\max_{0 \le m \le \bar{m}} d(\Gamma(t_m), \Gamma^m) \le C_1 \tau^k + C_2 d(\Gamma(0), \Gamma^0),$$

*holds.*

The proof is similar to the one of Theorem 4.7.

### 4.3 Second order implicit scheme

We consider the following implicit scheme for Problem 3.1.

**Problem 4.3** *For a given $N$-polygon $\Gamma_* \in \mathcal{O}$ and time steps $0 = t_0 < t_1 < t_2 < \cdots < t_{\bar{m}} \le T$, find polygons $\Gamma^m \in \mathcal{O}$ $(m = 1, 2, \ldots, \bar{m})$ such that*

$$\begin{cases} V_j^m = F_j(\Gamma^{m+1/2}, t_{m+1/2}) & (m = 0, 1, 2, \ldots, \bar{m} - 1, \ j = 1, 2, \ldots, N), \\ \Gamma^0 = \Gamma_*, \end{cases}$$

*where $\Gamma^{m+1/2}$ and $t_{m+1/2}$ are the $1/2$-interpolations:*

$$\Gamma^{m+1/2} := \frac{\Gamma^m + \Gamma^{m+1}}{2} \in \mathcal{P}^*, \quad t_{m+1/2} := \frac{t_m + t_{m+1}}{2} = t_m + \frac{\tau_m}{2}.$$

This is a generalized version of [8] for area-preserving crystalline curvature flow.

**Proposition 4.4** *We suppose the constant speed condition (3.2). Let $\Gamma^m \in \mathcal{O}$ $(m = 1, 2, \ldots, \bar{m})$ be a solution of Problem 4.3. Then it satisfies*

$$|\Omega^{m+1}| = |\Omega^m| + \mu \tau_m \quad (m = 0, 1, \ldots, \bar{m} - 1).$$

*In other words, $|\Omega^m| = |\Omega(t_m)|$ holds if the exact solution $\Omega(t)$ of Problem 3.1 exists.*

*Proof.* Since $|\Gamma_j^{m+1/2}| = (|\Gamma_j^m| + |\Gamma_j^{m+1}|)/2$, we have

$$\frac{|\Omega^{m+1}| - |\Omega^m|}{\tau_m} = \sum_{j=1}^{N} |\Gamma_j^{m+1/2}| F_j(\Gamma^{m+1/2}, t_{m+1/2}) = \mu,$$

from the formula (4.1). ∎

Since Problem 4.3 is an implicit scheme, it is not clear whether $\Gamma^{m+1} \in \mathcal{O}$ can be determined uniquely from the previous polygon $\Gamma^m \in \mathcal{O}$, the time $t_m$, and the time step size $\tau_m$. Another question is how to solve the equations

$$\boldsymbol{h}^{m+1} = \boldsymbol{h}^m + \tau_m F\left(\frac{\Gamma^m + \Gamma^{m+1}}{2}, t_{m+1/2}\right), \tag{4.2}$$

to obtain (approximation of) $\Gamma^{m+1}$ numerically.

We fix $\hat{\Gamma} \in \mathcal{O}$ and $\hat{t} \in [0, T)$, which correspond to $\Gamma^m$ and $t_{m+1/2}$, respectively. Let $\mathcal{K}$ be a compact convex set in $\mathcal{P}^*$ with $\hat{\Gamma} \in \mathcal{K} \subset \mathcal{O}$. For $\Sigma \in \mathcal{K}$ and $\hat{\tau} \in (0, \rho(\hat{\Gamma}, \mathcal{O}) M(\mathcal{K}, T)^{-1})$, we can define $\tilde{\Sigma} \in \mathcal{O}$ by

$$\boldsymbol{h}(\tilde{\Sigma}) = \boldsymbol{h}(\hat{\Gamma}) + \hat{\tau} F\left(\frac{\Sigma + \hat{\Gamma}}{2}, \hat{t}\right).$$

In other words, we can define $\Lambda(\Sigma) := \Lambda(\Sigma; \hat{\Gamma}, \hat{t}, \hat{\tau}) := \tilde{\Sigma}$ which is a map from $\mathcal{K}$ to $\mathcal{O}$. We have the following lemma.

**Lemma 4.5** *Let $\varepsilon \in (0, \rho(\hat{\Gamma}, \mathcal{O}))$ and $\lambda \in (0,1)$ be fixed. Suppose that $\hat{\tau}$ satisfies the condition*

$$0 < \hat{\tau} \le \min\left\{T - \hat{t}, \ \frac{\varepsilon}{M(\hat{\mathcal{K}}, T)}, \ \frac{2\lambda}{L(\hat{\mathcal{K}}, T)}\right\},$$

*where $\hat{\mathcal{K}} := \overline{B(\hat{\Gamma}, \varepsilon)}$. Then $\Lambda$ maps $\hat{\mathcal{K}}$ into $\hat{\mathcal{K}}$ and*

$$d(\Lambda(\Sigma^1), \Lambda(\Sigma^2)) \le \lambda d(\Sigma^1, \Sigma^2) \quad (\Sigma^1, \Sigma^2 \in \hat{\mathcal{K}}). \tag{4.3}$$

*Namely, $\Lambda$ is a contraction mapping on $\hat{\mathcal{K}}$ and there exists a unique fixed point of $\Lambda$ in $\hat{\mathcal{K}}$.*

*Proof.* It is enough to show (4.3), which is proved as follows:

$$d(\Lambda(\Sigma^1), \Lambda(\Sigma^2))$$

$$= |\boldsymbol{h}(\Lambda(\Sigma^1)) - \boldsymbol{h}(\Lambda(\Sigma^2))|_\infty = \hat{\tau}\left|F\left(\frac{\Sigma^1 + \hat{\Gamma}}{2}, \ \hat{t}\right) - F\left(\frac{\Sigma^2 + \hat{\Gamma}}{2}, \hat{t}\right)\right|_\infty$$

$$\le \hat{\tau}L(\hat{\mathcal{K}}, T)d\left(\frac{\Sigma^1 + \hat{\Gamma}}{2}, \frac{\Sigma^2 + \hat{\Gamma}}{2}\right)$$

$$= \hat{\tau}L(\hat{\mathcal{K}}, T)\left|\frac{\boldsymbol{h}(\Sigma^1) + \boldsymbol{h}(\hat{\Gamma})}{2} - \frac{\boldsymbol{h}(\Sigma^2) + \boldsymbol{h}(\hat{\Gamma})}{2}\right|_\infty$$

$$= \frac{\hat{\tau}}{2}L(\hat{\mathcal{K}}, T)d\left(\Sigma^1, \Sigma^2\right) \le \lambda d\left(\Sigma^1, \Sigma^2\right).$$

∎

We immediately have the following theorem, which gives us efficient numerical scheme to obtain $\Gamma^{m+1}$.

**Theorem 4.6** *Let $\mathcal{K}$ be a compact set in $\mathcal{O}$ and let $\varepsilon \in (0, \rho(\mathcal{K}, \mathcal{O}))$. We define*

$$\mathcal{K}_\varepsilon := \overline{\bigcup_{\Sigma \in \mathcal{K}} B(\Sigma, \varepsilon)}.$$

*For fixed $m$ ($< \bar{m}$) in Problem 4.3, we assume that $\Gamma^m \in \mathcal{K}$ and*

$$\tau_m \le \min\left\{\frac{\varepsilon}{M(\mathcal{K}_\varepsilon, T)}, \frac{2\lambda}{L(\mathcal{K}_\varepsilon, T)}\right\},$$

*where $\lambda \in (0,1)$. Then there exists uniquely $\Gamma^{m+1} \in \overline{B(\Gamma^m, \varepsilon)}$ satisfying (4.2).*

*Furthermore, $\Gamma^{m+1}$ is the fixed point of the contraction $\Lambda_m := \Lambda(\cdot\,; \Gamma^m, t_m, \tau_m)$ in $\overline{B(\Gamma^m, \varepsilon)}$, and is given by the limit of $\Lambda_m^\nu(\Gamma^m)$ as $\nu \to \infty$ with the following estimate*

$$d(\Gamma^{m+1}, \Lambda_m^\nu(\Gamma^m)) \le \lambda^\nu d(\Gamma^{m+1}, \Gamma^m) \quad (\nu \in \mathbb{N}).$$

**Theorem 4.7** *We suppose that $\{\Gamma(t)\}_{0 \le t \le T}$ be a $C^{k+1}$-class solution of Problem 3.1 for $k = 0, 1$, or $2$. There exists $\delta^* > 0$, $\tau^* > 0$, $C > 0$ and a non-decreasing function $\omega(a) > 0$ with*

$$\omega(a) = \begin{cases} o(a^k) & \text{if } k = 0 \text{ or } 1, \\ O(a^2) & \text{if } k = 2, \end{cases} \quad \text{as } a \downarrow 0, \tag{4.4}$$

*such that, if $d(\Gamma^*, \Gamma^0) \leq \delta^*$ and $\tau \leq \tau^*$, then $\Gamma^m \in \mathcal{O}$ $(m = 1, 2, \ldots, \bar{m})$ are inductively determined and*

$$\max_{0 \leq m \leq \bar{m}} d(\Gamma(t_m), \Gamma^m) \leq \omega(\tau) + Cd(\Gamma(0), \Gamma^0),$$

*holds.*

*Proof.* We put $\hat{\rho} := \rho(\{\Gamma(t); \ 0 \leq t \leq T\}, \mathcal{O})$, and fix $\delta \in (0, \hat{\rho})$ and $\varepsilon \in (0, \hat{\rho} - \delta)$. We define

$$\mathcal{K} := \overline{\bigcup_{0 \leq t \leq T} B(\Gamma(t), \delta)}, \quad \mathcal{K}_\varepsilon := \overline{\bigcup_{\Sigma \in \mathcal{K}} B(\Sigma, \varepsilon)},$$

$$L := L(\mathcal{K}_\varepsilon, T), \quad R(a) := e^{\frac{L}{2}} \left(1 - \frac{aL}{2}\right)^{-1/a} \quad (0 < a < 2/L),$$

$$p_m := |e^m|_\infty + \frac{\omega(\tau)}{L} \quad (m = 0, 1, 2, \ldots, \bar{m}),$$

where a non-decreasing function $\omega(a)$ $(0 < a < T)$, which satisfies (4.4), will be defined in (4.8) later. Since $R(\cdot)$ is an increasing function and $\lim_{a \downarrow 0} R(a) = e^L$, there exists $\delta^* > 0$ and $\tau^* > 0$ such that

$$R(\tau^*) \left(\delta^* + \frac{\omega(\tau^*)}{L}\right) \leq \delta, \quad \tau^* < \min\left(\frac{\varepsilon}{M(\mathcal{K}_\varepsilon, T)}, \frac{2}{L}\right).$$

For $m = 0, 1, 2, \ldots, \bar{m} - 1$, we will prove the following inductive conditions:

$$\Gamma^m \in \mathcal{K}, \quad p_m \leq R(\tau)^{t_m} p_0 \quad \Rightarrow \quad {}^{\exists}\Gamma^{m+1} \in \mathcal{K}, \quad p_{m+1} \leq R(\tau)^{t_{m+1}} p_0. \tag{4.5}$$

The conditions $\Gamma^0 \in \mathcal{K}$ and $p_0 \leq R(\tau)^0 p_0$ for the case $m = 0$ are obviously satisfied.

Let us assume the conditions $\Gamma^m \in \mathcal{K}$ and $p_m \leq R(\tau)^{t_m} p_0$ for a fixed $m$. Then, from Theorem 4.6, there exists $\Gamma^{m+1}$ uniquely in $\overline{B(\Gamma^m, \varepsilon)} \subset \mathcal{K}_\varepsilon$, and we have

$$\boldsymbol{e}^{m+1} - \boldsymbol{e}^m = \boldsymbol{h}(t_{m+1}) - \boldsymbol{h}(t_m) - \tau_m V^m = \tau_m \left\{\boldsymbol{\xi}^m + \left(V(t_{m+1/2}) - V^m\right)\right\}, \tag{4.6}$$

$$\boldsymbol{\xi}^m := \frac{\boldsymbol{h}(t_{m+1}) - \boldsymbol{h}(t_m)}{\tau_m} - \dot{\boldsymbol{h}}(t_{m+1/2}),$$

where $V(t) := (V_1(t), V_2(t), \ldots, V_N(t))^{\mathrm{T}}$ and $V^m := (V_1^m, V_2^m, \ldots, V_N^m)^{\mathrm{T}}$.

The last term of (4.6) is estimated as follows. Since $\Gamma(t_{m+1/2}) \in \mathcal{K} \subset \mathcal{K}_\varepsilon$ and $\Gamma^{m+1/2} \in \overline{B(\Gamma^m, \varepsilon)} \subset \mathcal{K}_\varepsilon$, we have

$$\left|V(t_{m+1/2}) - V^m\right|_\infty$$

$$= \left|F(\Gamma(t_{m+1/2}), t_{m+1/2}) - F(\Gamma^{m+1/2}, t_{m+1/2})\right|_\infty \leq Ld(\Gamma(t_{m+1/2}), \Gamma^{m+1/2})$$

$$= L\left|\boldsymbol{h}(t_{m+1/2}) - \frac{\boldsymbol{h}^m + \boldsymbol{h}^{m+1}}{2}\right|_\infty = L\left|\frac{1}{2}(\boldsymbol{e}^m + \boldsymbol{e}^{m+1}) - \boldsymbol{\zeta}^m\right|_\infty, \tag{4.7}$$

where

$$\boldsymbol{\zeta}^m := \frac{\boldsymbol{h}(t_m) + \boldsymbol{h}(t_{m+1})}{2} - \boldsymbol{h}(t_{m+1/2}).$$

Combining (4.6) and (4.7), we obtain

$$|e^{m+1}|_\infty \le |e^m|_\infty + \tau_m |\boldsymbol{\xi}^m|_\infty + \tau_m L \left| \frac{1}{2}(e^m + e^{m+1}) - \boldsymbol{\zeta}^m \right|_\infty$$

$$\le |e^m|_\infty + \frac{\tau_m L}{2}\left(|e^{m+1}|_\infty + |e^m|_\infty\right) + \tau_m(|\boldsymbol{\xi}^m|_\infty + L|\boldsymbol{\zeta}^m|_\infty).$$

By the Taylor expansion, we can obtain an non-decreasing function $\omega(a)$ $(0 < a < T)$ which satisfies the condition (4.4) and the inequality

$$|\boldsymbol{\xi}^m|_\infty + L|\boldsymbol{\zeta}^m|_\infty \le \omega(\tau). \tag{4.8}$$

Hence, we have

$$\left(1 - \frac{\tau_m L}{2}\right)|e^{m+1}|_\infty \le \left(1 + \frac{\tau_m L}{2}\right)|e^m|_\infty + \tau_m \omega(\tau),$$

and this inequality is equivalent to

$$\left(1 - \frac{\tau_m L}{2}\right) p_{m+1} \le \left(1 + \frac{\tau_m L}{2}\right) p_m.$$

From the inequalities

$$\left(1 - \frac{\tau_m L}{2}\right) \ge \left(1 - \frac{\tau L}{2}\right)^{\tau_m/\tau} \quad \text{and} \quad \left(1 + \frac{\tau_m L}{2}\right) \le e^{\tau_m L/2},$$

we obtain

$$p_{m+1} \le \left(1 - \frac{\tau_m L}{2}\right)^{-1}\left(1 + \frac{\tau_m L}{2}\right) p_m \le R(\tau)^{\tau_m}(R(\tau)^{t_m} p_0) \le R(\tau)^{t_{m+1}} p_0.$$

The condition $\Gamma^{m+1} \in \mathcal{K}$ follows from this estimate as

$$|e^{m+1}|_\infty \le p_{m+1} \le R(\tau^*)^{t_{m+1}} p_0 \le R(\tau^*)^T \left(\delta^* + \frac{\omega(\tau^*)}{L}\right) \le \delta.$$

Hence, we have proved (4.5), which leads us to the estimate:

$$|e^m|_\infty \le R(\tau^*)^T \left(|e^0|_\infty + \frac{\omega(\tau)}{L}\right) - \frac{\omega(\tau)}{L} \le R(\tau^*)^T |e^0|_\infty + \frac{R(\tau^*)^T - 1}{L}\omega(\tau).$$

The assertion of the theorem is obtained by putting $C := R(\tau^*)^T$ and denoting the above term $L^{-1}(R(\tau^*)^T - 1)\omega(\tau)$ again by $\omega(\tau)$. ∎

# References

[1] B. Andrews, *Singularities in crystalline curvature flows*, Asian J. Math. **6** (2002) 101–122.

[2] S. Angenent and M. E. Gurtin, *Multiphase thermomechanics with interfacial structure, 2. Evolution of an isothermal interface*, Arch. Rational Mech. Anal. **108** (1989) 323–391.

[3] M.-H. Giga and Y. Giga, *Crystalline and level set flow – convergence of a crystalline algorithm for a general anisotropic curvature flow in the plane*, Free boundary problems: theory and applications, I (Chiba, 1999), GAKUTO Internat. Ser. Math. Sci. Appl., Gakkōtosho, Tokyo **13** (2000) 64–79.

[4] K. Ishii and H. M. Soner , *Regularity and convergence of crystalline motion*, SIAM J. Math. Anal. **30** (1999) 19–37.

[5] T. Ishiwata , T. K. Ushijima , H. Yagisita and S. Yazaki, *Two examples of nonconvex self-similar solution curves for a crystalline curvature flow*, Proc. Japan Academy **80**, Ser. A, No. 8 (2004), 151–154.

[6] T. Ishiwata and S. Yazaki, *On the blow-up rate for fast blow-up solutions arising in an anisotropic crystalline motion*, J. Comp. App. Math. **159** (2003), 55–64.

[7] J. E. Taylor, *Motion of curves by crystalline curvature, including triple junctions and boundary points*, Diff. Geom.: partial diff. eqs. on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., **54** (1993), Part I, 417–438, AMS, Providencd, RI.

[8] T. K. Ushijima and S. Yazaki, *Convergence of a crystalline approximation for an area-preserving motion*, Journal of Computational and Applied Mathematics **166** (2004), 427–452.

[9] S. Yazaki, *Asymptotic behavior of solutions to an expanding motion by a negative power of crystalline curvature*, Adv. Math. Sci. Appl. **12** (2002), 227–243.

[10] S. Yazaki, *On an area-preserving crystalline motion*, Calc. Var. **14** (2002), 85–105.

[11] S. Yazaki, *Motion of nonadmissible convex polygons by crystalline curvature*, Publications of Research Institute for Mathematical Sciences **43** (2007), 155–170.

[12] S. Yazaki, *An area-preserving motion by crystalline curvature*, To appear in Kybernetika (2007).

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