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TOWARDS MODELLING THE MOTION OF RISING AIR BUBBLES IN A HELE-SHAW CELL

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Abstract. The motion of the flattened rising air bubbles is considered when air is injected into water contained in the Hele-Shaw cell which is set perpendicular to the ground. We reduce Navier-Stokes equations to the two dimensional problem by using Hele-Shaw approximation etc., and propose simple model equations.

Key words. Hele-Shaw cell, rising air bubble, surface tension, zigzag, oscillation, modelling

AMS subject classifications. 76D27, 76D45, 35J05, 35R35

1. Introduction. We consider the motion of the flattened rising air bubbles when air is injected into water contained in the narrow gap between two flat, parallel plates, which is so-called the Hele-Shaw cell. In our experiments, the cell is set perpendicular to the ground as in Fig.1.1. The Hele-Shaw cell is an apparatus developed

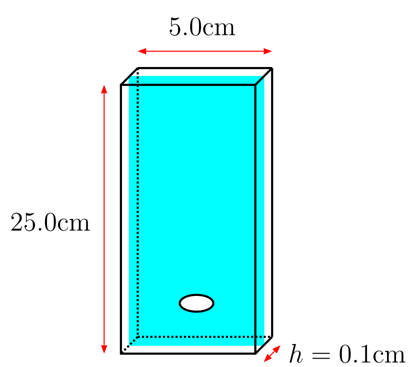

Fig. 1.1. *Hele-Shaw cell is set perpendicular to the ground and is filled with water.*

by H. S. Hele-Shaw in order to visualize the stream line under the two dimensional stationary irrotational flow [2, 3]. Since the Hele-Shaw's beautiful work, there have been several extensive analytical and experimental studies. We will introduce the one of the experiments at Masami Kawaguchi's laboratory [4] as follows. They made the Hele-Shaw cell by using two acrylic resin plates: glued the left, right and bottom sides, and opened the top side. At the center of the bottom, they made a small hole, and the air is injected from a tube which is through the hole. The distance between two plates is $h = 0.1$cm, and the width and the height of the plate is 5.0cm and 25.0cm, respectively, and the cell is set in a thermostat.

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A single rising air bubble with a constant volume undergoes various kinds of behavior depending on the size. FIG.1.2 indicates the situation of a single rising air bubble which are taken by CCD camera and drawn by a software [1]. We can observe both zigzag motion and rectilinear motion depending on the size under the almost unchangeable shape.

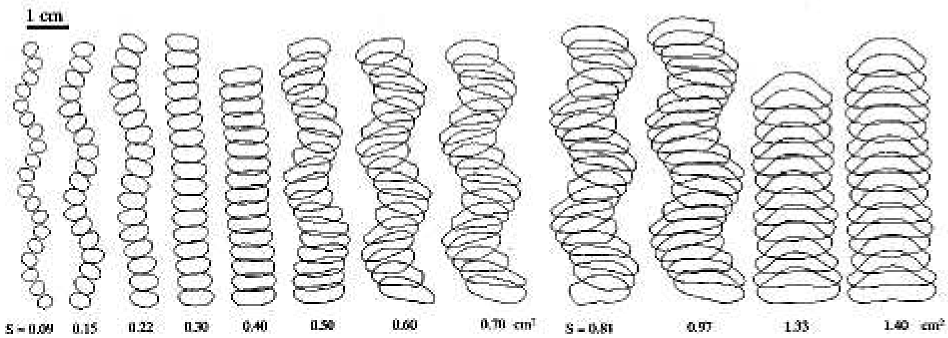

FIG. 1.2. *Reprinted from Gou [1].*

The aim of this research is to investigate principle of behavior of rising air bubble as in FIG.1.2. It is known that in (the three dimensional) water bubbles with a few millimeters do not rise along rectilinear paths. Actually, the fact is far from that: bubbles spiral or zigzag as they rise. Leonardo da Vinci pointed out this phenomenon and drew helically rising bubbles. It is said that his sketch is the first scientific reference to this fact [7, Appendix B]. Behavior of bubbles are familiar in our daily life and their research itself is not particularly novel. We can find so many references of numerical simulations or experiments or analysis, even if we restricted Hele-Shaw cell type problems. See [5] and references therein. However, to the authors' knowledge, principle of behavior of rising air bubble is not clear: What is the exact nature of the zigzag instability? Why does oscillation bifurcate by size? How is the stable shape determined? Can we control the behavior of bubbles? ... the full answer of dynamics of rising bubble is yet unknown. If we control the behavior of bubbles, wide-ranging application will be waiting in physics, chemistry, medicine, and technology [6].

The difficulties arise from the bubbles interaction with its own wake, appearance of Karman vortex street, and deformable surface of bubbles. And mathematical difficulties arise from complex analysis of the fundamental equations of fluids such as Navier-Stokes equations, and movable or deformable surface of bubbles. From this reason, in the first step of our research, we reduce Navier-Stokes equations to the two dimensional problem by using Hele-Shaw approximation etc., and propose a simple –as much as possible– model which can explain the behavior of bubble as in FIG.1.2. At this stage, the study is ongoing, and so we will only report an introduction of simple model equations in the following section.

2. Modelling. We take xyz-coordinate as in FIG.2.1. In the region of water, we assume that motion of particles of water depends on Navier-Stokes equation:

$$\text{(NS)} \qquad \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix} + \nu\nabla^2\boldsymbol{v}.$$

Here ρ is density, ν is kinetic viscosity, and g is acceleration of gravity. Unknown

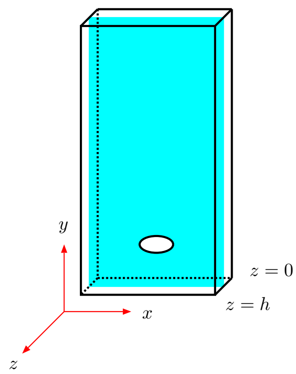

FIG. 2.1. *xyz-coordinate.*

functions are pressure p and velocity $\boldsymbol{v} = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$. We assume the third entry of velocity is zero. Namely, the flow is described in the plane parallel to xy-plane for fixed z. In what follows, we will simplify this equations.

Firstly, we require the following.

Request I: the velocity of water is very slow, and the flow is stationary.

By this request, we operate so-called Stokes approximation for stationary flow, and so the LHS of (NS) is neglected:

$$\text{(SS)} \qquad 0 = -\frac{1}{\rho}\nabla p + \begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix} + \nu \nabla^2 \boldsymbol{v}.$$

By using viscosity $\mu = \rho\nu$, each entry of (SS) is wrote down as follows:

$$\text{(SS}'\text{)} \qquad \begin{cases} p_x - \mu\nabla^2 u = 0, \\ p_y + \rho g - \mu \nabla^2 v = 0, \\ p_z = 0. \end{cases}$$

Then it is clear that the pressure p is a function of x, y, t only: $p = p(x, y, t)$. Here $p_x = \partial p/\partial x$ and so on.

Secondly, we require the following.

Request II: graph of u, v with respect to a variable z draw a parabola satisfying "$u = v = 0$ at $z = 0, h$."

From this request, we have $u = az(z-h)$ and $v = bz(z-h)$ with $a = a(x, y, t)$, $b = b(x, y, t)$. Substitution this for (SS$'$), $a_{xx} + a_{yy} = b_{xx} + b_{yy} = 0$ holds since $p = p(x, y, t)$. Consequently, $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$ is required, and we obtain the following:

$$\text{(SS}''\text{)} \qquad \begin{cases} p_x - \mu u_{zz} = 0, \\ p_y + \rho g - \mu v_{zz} = 0. \end{cases} \quad \Longleftrightarrow \quad \begin{cases} u = \dfrac{p_x}{2\mu} z(z-h), \\ v = \dfrac{p_y + \rho g}{2\mu} z(z-h). \end{cases}$$

Here $u_{zz} = \partial u_z/\partial z$ and so on.

Finally, we require the following.

Request III: take average of u and v in the z-direction.

Then, the two dimensional average velocity vector is expressed by a gradient of pressure and the gravity term:

$$\overline{u} = \frac{1}{h}\int_0^h u\,dz = -\frac{h^2}{12\mu}p_x,\ \ \overline{v} = \frac{1}{h}\int_0^h v\,dz = -\frac{h^2}{12\mu}p_y - \frac{\rho g h^2}{12\mu}.$$

We call the above Hele-Shaw approximation. Note that the usual Hele-Shaw approximation corresponds to the case $g = 0$, and is called Darcy's law.

From the above, we have the two dimensional velocity vector of water:

$$\boldsymbol{v} = \begin{pmatrix}\overline{u}\\ \overline{v}\end{pmatrix} = -\frac{h^2}{12\mu}\nabla p + \frac{\rho g h^2}{12\mu}\begin{pmatrix}0\\ -1\end{pmatrix}.$$

By the assumption of incompressibility $\nabla \cdot \boldsymbol{v} = \overline{u}_x + \overline{v}_y = 0$, we obtain

$$\nabla^2 p = p_{xx} + p_{yy} = 0.$$

Namely, the pressure is harmonic in water region.

Let Ω_1 be water region, Ω_2 bubble region, and Γ the interface between water and bubble (Fig.2.2).

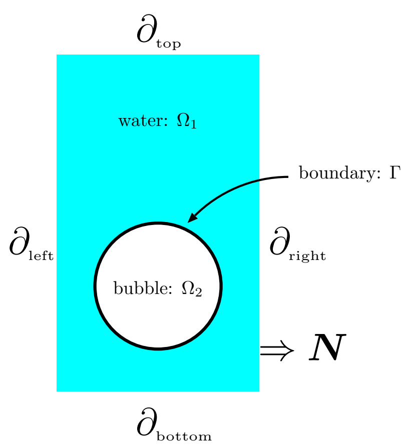

Fig. 2.2. *Water and bubble region, and boundary of the cell.* $\boldsymbol{N}$ *is outward normal vector.*

In the bubble region, we operate Hele-Shaw approximation as in the water region. Then we have the following two dimensional velocity vector:

$$(\mathrm{V}_i) \qquad \boldsymbol{v}_i = -\frac{h^2}{12\mu_i}\nabla p_i + \frac{\rho_i g h^2}{12\mu_i}\begin{pmatrix}0\\ -1\end{pmatrix} \quad \text{in } \Omega_i.$$

Here subscript i indicates unknown function and constants in each region: $i = 1$ water and $i = 2$ bubble. It seems that the area of bubble is almost constant in Fig.1.2, so we assume that two dimensional flattened bubble is incompressible $\nabla \cdot \boldsymbol{v}_2 = 0$. Then we obtain, for $i = 1, 2$

$$(\mathrm{H}_i) \qquad \nabla^2 p_i = 0 \quad \text{in } \Omega_i.$$

Namely, the pressure is harmonic in each region[*1].

We set boundary condition of the cell as follows. Firstly, at the top of the cell ∂_{top}, pressure is equal to the atmospheric pressure p_a:

$$(\mathrm{B_t}) \qquad p_1 = p_a \text{ (atmospheric pressure)} \quad \text{on } \partial_{\text{top}}.$$

Secondly, at the left, right and bottom of the cell $\partial_{\text{left}}, \partial_{\text{right}}, \partial_{\text{bottom}}$, we assume that the entry of velocity of water in the outward normal direction $\boldsymbol{N}$ is zero[*2]:

$$\boldsymbol{v}_1 \cdot \boldsymbol{N} = 0 \quad \text{on } \partial_{\text{left}}, \partial_{\text{right}}, \partial_{\text{bottom}}.$$

Since boundary of the cell is parallel to axis, this boundary condition is equivalent to the following:

$$(\mathrm{B_{lrb}}) \qquad \frac{\partial p_1}{\partial x} = 0 \quad \text{on } \partial_{\text{left}}, \partial_{\text{right}}; \quad \frac{\partial p_1}{\partial y} = -\rho_1 g \quad \text{on } \partial_{\text{bottom}}.$$

Now let us consider conditions on the boundary Γ. Firstly, we consider law of conservation of mass on Γ. Let normal vector $\boldsymbol{n}$, on Γ, be toward from water to bubble (FIG.2.3). Normal velocity of water ($i = 1$) and bubble ($i = 2$) is $V_i = \boldsymbol{v}_i \cdot \boldsymbol{n}$

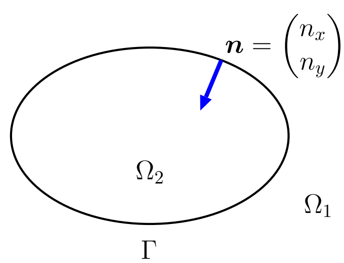

FIG. 2.3. *Normal vector $\boldsymbol{n}$ on Γ.*

($i = 1, 2$). Let $\widehat{V}_i$ be a relative velocity of Γ ($i = 1, 2$). Then law of conservation of

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[*1]Of course, air is compressible in general, so this approximation seems to be inappropriate one. From this point of view, we can require another approximation, i.e., the pressure is a constant in the bubble region:

$$(\mathrm{H_2}') \qquad p_2 \equiv p_* \ (const.) \quad \text{in } \Omega_2.$$

We will mention this approximation at the end of this section.

[*2]Since we assume water being viscous fluid, condition of adhesiveness $\boldsymbol{v}_1 = \boldsymbol{0}$ can be considerable. This condition follows that at each boundary $\partial_{\text{left}}, \partial_{\text{right}}, \partial_{\text{bottom}}$, $\dfrac{\partial p_1}{\partial x} = 0$ and $\dfrac{\partial p_1}{\partial y} = -\rho_1 g$ are required. However these conditions are over-determined. So we propose the next simple condition instead of ($\mathrm{B_{lrb}}$):

$$(\mathrm{B_{lrb}}') \qquad p_1 = p_a + \rho_1 g (y_{\text{top}} - y) \quad \text{on } \partial_{\text{left}}, \partial_{\text{right}}, \partial_{\text{bottom}}.$$

Here y_{top} is y-coordinate of ∂_{top}.

mass is $\rho_1 \widehat{V}_1 = \rho_2 \widehat{V}_2$. Since air will not mix with water, $\rho_1 \widehat{V}_1 = \rho_2 \widehat{V}_2 = 0$ holds, and then we have the following condition:

$$\widehat{V}_1 = \widehat{V}_2 = 0 \quad \text{on } \Gamma.$$

Then we obtain velocity V of Γ:

$$(\Gamma_{\mathrm{a}}) \qquad V = V_1 = V_2 \quad \text{on } \Gamma.$$

Secondly, we consider momentum conservation law on Γ. Let T_i be a stress tensor. Momentum vector is given by $\rho_i \boldsymbol{v}_i \widehat{V}_i - T_i \boldsymbol{n}$. Then we have the following condition:

$$\left[\left(\rho_i \boldsymbol{v}_i \widehat{V}_i - T_i \boldsymbol{n}\right) \cdot \boldsymbol{n}\right]_{i=1}^{i=2} = \sigma\kappa \quad \text{on } \Gamma.$$

Here $\sigma > 0$ is a coefficient of surface tension and κ is a curvature of $\boldsymbol{n}$-direction[*3].

Now, stress tensor T is given by

$$T = -pE + \mu \begin{pmatrix} 2\overline{u}_x & \overline{u}_y + \overline{v}_x \\ \overline{u}_y + \overline{v}_x & 2\overline{v}_y \end{pmatrix},$$

then we substitute (V_i) for this and obtain the following[*4]:

$$T_i = -p_i E - \frac{h^2}{6} \operatorname{Hess}(p_i) \quad (i = 1, 2).$$

From the above, we have the following condition:

$$(\Gamma_{\mathrm{b}}) \qquad \left[p_i + \frac{h^2}{6} \operatorname{Hess}(p_i) \boldsymbol{n} \cdot \boldsymbol{n}\right]_{i=1}^{i=2} = \sigma\kappa \quad \text{on } \Gamma.$$

In the case where viscous stress is neglected, this condition will be so-called Laplace's relation:

$$(\Gamma_{\mathrm{b}}{}') \qquad p_2 - p_1 = \sigma\kappa \quad \text{on } \Gamma.$$

Summarizing the above, we obtain the following model. Unknown functions are pressures $p_i(x, y, t)$ $(i = 1, 2)$, and moving boundary Γ:

Model 1: $$(\mathrm{H}_1), (\mathrm{H}_2), (\mathrm{B}_{\mathrm{t}}), \left\{\begin{matrix} (\mathrm{B}_{\mathrm{lrb}}) \\ \text{or} \\ (\mathrm{B}_{\mathrm{lrb}}{}') \end{matrix}\right\}, (\Gamma_{\mathrm{a}}), \left\{\begin{matrix} (\Gamma_{\mathrm{b}}) \\ \text{or} \\ (\Gamma_{\mathrm{b}}{}') \end{matrix}\right\}.$$

In the case where we choose $(\mathrm{H}_2{}')$, we put $P = p_* - p_1$ and $P_a = p_* - p_a$, and we obtain the following model.

Model 2: $$\begin{cases} (\widetilde{\mathrm{H}}_1) & \nabla^2 P = 0 \quad \text{in } \Omega_1, \\ (\widetilde{\mathrm{B}}_{\mathrm{t}}) & P = P_a \quad \text{on } \partial_{\text{top}}, \\ (\widetilde{\mathrm{B}}_{\mathrm{lrb}}) & \dfrac{\partial P}{\partial x} = 0 \quad \text{on } \partial_{\text{left}}, \partial_{\text{right}}; \quad \dfrac{\partial P}{\partial y} = \rho_1 g \quad \text{on } \partial_{\text{bottom}}, \\ (\widetilde{\Gamma}_{\mathrm{a}}) & V = \dfrac{h^2}{12\mu_i} \dfrac{\partial P}{\partial \boldsymbol{n}} - \dfrac{\rho_i g h^2}{12\mu_i} n_y \quad \text{on } \Gamma, \\ (\widetilde{\Gamma}_{\mathrm{b}}) & P + \dfrac{h^2}{6} \operatorname{Hess}(P) \boldsymbol{n} \cdot \boldsymbol{n} = \sigma\kappa \quad \text{on } \Gamma. \end{cases}$$

[*3] We take positive sign if bubble is a circle.

[*4] $\operatorname{Hess}(p) = \begin{pmatrix} p_{xx} & p_{xy} \\ p_{yx} & p_{yy} \end{pmatrix}$ is Hessian.

Instead of $(\widetilde{\mathrm{B}}_{\mathrm{lrb}})$ or $(\widetilde{\Gamma}_{\mathrm{b}})$, the following conditions are also available:

$$(\widetilde{\mathrm{B}}_{\mathrm{lrb}}{}') \quad P = P_a - \rho_1 g(y_{\mathrm{top}} - y) \quad \text{on } \partial_{\mathrm{left}}, \partial_{\mathrm{right}}, \partial_{\mathrm{bottom}}.$$

$$(\widetilde{\Gamma}_{\mathrm{b}}{}') \quad P = \sigma\kappa \quad \text{on } \Gamma.$$

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