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#### SOME RESULTS ON SINGULARITIES OF SOLUTIONS TO AN ANISOTROPIC CRYSTALLINE CURVATURE FLOW

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#### Abstract.

In this paper we study a system of ordinary differential equations, which is concerned with the motion of polygonal curves in the plane moving under anisotropic crystalline curvature flows. The ODEs have a specific feature, which solutions blow up to infinity in a finite time, and its singularity of blow-up is roughly divided into two types by their rates. However, the singularities have not been completely classified. We discuss here known and unknown blow-up rates as follows: First, we mention some known theoretical results; and secondly, we estimate unknown blow-up rates numerically.

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## 1 Introduction

In around 1990, crystalline curvature flows were introduced by J.E. Taylor [9] and S. Angenent and M.E. Gurtin[2]. They established new notion of moving curves in the case where an interfacial energy density, defined on the curves, is not smooth and its Wulff shape is a convex polygon. Such energy is called *crystalline*. Since their pioneer works, so-called crystalline motion has been studied extensively under various kinds of evolution laws by several authors.

In this paper we study the motion of shrinking convex polygonal curves in the plane  $\mathbb{R}^2$ , which is governed by anisotropic crystalline curvature flows. The flows are defined in the following way: Let the Wulff shape be an N-sided convex polygon, and let the sets  $\Theta = \{\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{N-1} < \theta_0 + 2\pi\}$  with  $\vartheta_j = \theta_j - \theta_{j-1}$ , where  $\theta_j$  is the normal angle of the *j*-th edge of the Wulff shape. For a convex N-sided polygon  $\mathcal{P}$  we say that  $\mathcal{P}$  is admissible if the normal angle of the *j*-th edge of  $\mathcal{P}$  is  $\theta_j$  for all  $j \in \mathcal{I} = \{0, 1, \ldots, N-1\}$ . Note that the original notion of the admissibility is defined for piecewise linear curves. On each edge of an admissible convex polygon  $\mathcal{P}$  the crystalline curvature  $\kappa_j = \kappa_j(t)$  is defined by

$$\kappa_j = \frac{\gamma_j}{d_j}, \quad \gamma_j = \tan\frac{\vartheta_j}{2} + \tan\frac{\vartheta_{j+1}}{2} = \frac{\sin\vartheta_j + \sin\vartheta_{j+1} - \sin(\vartheta_j + \vartheta_{j+1})}{\sin\vartheta_j \sin\vartheta_{j+1}}$$

Here  $d_j = d_j(t)$  is the length of the *j*-th edge of  $\mathcal{P}$ . The curvature  $\kappa$  for smooth curves is given as  $\kappa = d\theta/ds$  by using the normal angle  $\theta$  and the arc-length parameter *s*. Then the crystalline curvature  $\kappa_j = \gamma_j/d_j$  is a discrete version of  $\kappa$  in the sense of  $\gamma_j \sim d\theta$  and  $d_j \sim ds$ .

We consider time evolution of a convex polygon  $\mathcal{P}(t)$  in the plane under the evolution law:

$$v_j = a_j \kappa_j^{\alpha},$$

where  $v_j = v_j(t)$  is the inward normal velocity of the *j*-th edge of  $\mathcal{P}(t)$ ,  $\alpha$  is a positive parameter and  $a_j$ 's are positive constants which describe an anisotropy of mobility. Note that the normal angle of each edge does not change under this flow. Therefore, if the initial polygon  $\mathcal{P}(0)$  is admissible, then the admissibility of a solution polygon is preserved as long as the solution polygon exists.

From geometric consideration, this flow can be restated as the following system of ordinary differential equations:

(P) 
$$\begin{cases} \frac{d}{dt}v_{j}(t) = \alpha a_{j}^{-1/\alpha}v_{j}^{(\alpha+1)/\alpha}(\Delta_{\theta}v+v)_{j}, & j \in \mathcal{I}, t > 0, \\ v_{N}(t) = v_{0}(t), v_{-1}(t) = v_{N-1}(t), & t \ge 0, \\ v_{j}(0) = a_{j}\kappa_{j}(0)^{\alpha}, & j \in \mathcal{I}. \end{cases}$$

Here  $\kappa_j(0)$  is the crystalline curvature of the *j*-th edge of the initial polygon  $\mathcal{P}(0)$  and the operator  $\Delta_{\theta}$  is defined by

$$(\Delta_{\theta} u)_j = \frac{(\mathbf{D}_+ u)_j - (\mathbf{D}_+ u)_{j-1}}{\gamma_j}, \quad (\mathbf{D}_+ u)_j = \frac{u_{j+1} - u_j}{\sin \vartheta_{j+1}}.$$



Figure 1: Two numerical examples for single point extinction case (left) and degenerate pinching case (right).

A local existence and uniqueness of the solution  $\{v_j\}$  of problem (P) follow from a standard theory of system of ordinary differential equations. Moreover, it is known that for any initial admissible polygon at least the maximum of solution  $\{v_j\}$  blows up to infinity in a finite time, say T. This means that for any initial admissible polygon, the maximal time of preserving the admissibility is finite and this flow develops singularity at t = T.

M.-H. Giga and Y. Giga [3] showed the detailed information on limiting shape at the final time T: if  $\alpha \geq 1$  or there are no parallel pairs of edges, the solution polygon  $\mathcal{P}(t)$  shrinks to a single point, i.e.,  $d_j(t) \to 0$  as  $t \to T$  for all  $j \in \mathcal{I}$ ; and if  $\alpha < 1$  and there exists at least one parallel pair of edges, the solution polygon  $\mathcal{P}(t)$  shrinks to a single point or collapses to a line segment with a positive length. The latter phenomenon is called *degenerate pinching*. B. Andrews [1] gave a sufficient condition of degenerate pinching. In any case, the enclosed area of a solution polygon becomes zero at the final time T. In Figure 1, two numerical examples are shown: The initial polygon  $\mathcal{P}(0)$  is the outmost polygon with N = 6 in each figure, and, from outside to inside, time evolution of  $\mathcal{P}(t)$  are plotted. The left figure indicates single point extinction case with  $\alpha = 2$  and the right figure indicates degenerate pinching case with  $\alpha = 1/2$ .

In section 2, theoretical results will be shown. We shall consider sublinear case  $\alpha < 1$  mainly and discuss a blow-up rate in degenerate pinching case. In section 3, numerical studies will be focused. Recently, C. Hirota and K. Ozawa [5] developed a numerical method of estimating blow-up time and  $(T - t)^{-p}$  type blow-up rate. We will estimate some blow-up rates numerically by using their method.

## 2 Behaviour of blow-up solutions

Here and hereafter, we use the notation  $\sum_{j} u_j$ ,  $u_{\max}$ ,  $u_{\min}$  and  $\dot{u}(t)$  for  $\sum_{j \in \mathcal{I}} u_j$ ,  $\max_{j \in \mathcal{I}} u_j$ ,  $\min_{j \in \mathcal{I}} u_j$  and du(t)/dt, respectively, and  $A \sim B$  means  $c_1A \leq B \leq c_2A$  for generic constants  $c_1 > 0$  and  $c_2 > 0$ .

In this section we will characterize blow-up rate of solutions. As mentioned above, in single point extinction case  $v_j(t)$  blows up for all j since  $d_j(t)$  vanishes at t = T simultaneously for all j (case Figure 1, left), and in degenerate pinching case  $v_{\min}(t)$  is bounded since  $\liminf_{t\to T} d_{\max}(t) > 0$  holds (case Figure 1, right); while  $\lim_{t\to T} v_{\max}(t) = +\infty$  always holds (this is proved from  $\liminf_{t\to T} d_{\min}(t) = 0$ ). In either case,  $v_{\min}(t)$  and  $v_{\max}(t)$  are estimated from above and below by a specific blow-up rate, respectively, as follows:

**Lemma 2.1** Let  $\alpha > 0$ . Then it holds that

$$v_{\min}(t) \le a_{\max}^{1/(\alpha+1)}((\alpha+1)(T-t))^{-\alpha/(\alpha+1)},$$

and that

$$v_{\max}(t) \ge a_{\min}^{1/(\alpha+1)}((\alpha+1)(T-t))^{-\alpha/(\alpha+1)}.$$

In the case  $\alpha = 1$  these estimates were proved in [8]. It is easy to generalize to the case  $\alpha > 0$ . This result implies that the generic lower bound of blow-up rate is  $(T-t)^{-\alpha/(\alpha+1)}$ . Moreover, if  $\alpha > 1$ , then there exists a positive constant C such that  $v_{\max}(t) \leq C(T-t)^{-\alpha/(\alpha+1)}$ , that is, the blow-up rate in the case  $\alpha > 1$  is exactly  $(T-t)^{-\alpha/(\alpha+1)}$  (see [1]). The order  $(T-t)^{-\alpha/(\alpha+1)}$  specifies blow-up rate in the following sense: If  $a_j \equiv 1$  for all  $j \in \mathcal{I}$ , then  $v_j(t) \equiv ((\alpha + 1)(T-t))^{-\alpha/(\alpha+1)}$  is a special solution of (P) and a corresponding solution polygon shrinks to a single point homothetically. This is so-called self-similar solution. Using this order, blow-up solutions can be classified as follows:

**Definition 2.2 (type I and type II blow-up)** Let  $\alpha > 0$ . We say that the solution undergoes a "type I blow-up" if the blow-up rate of the maximum of solution  $\{v_j\}$  is at most the self-similar rate, that is,

$$\sup_{0 < t < T} \max_{j \in \mathcal{I}} v_j(t) (T-t)^{\alpha/(\alpha+1)} < \infty,$$
(2.1)

and that the solution undergoes a "type II blow-up" if (2.1) does not hold.

A type II blow-up is sometimes called *fast blow-up*, because a solution  $\{v_j\}$  undergoes a type II blow-up if and only if

$$\limsup_{t \to T} \max_{j \in \mathcal{I}} v_j(t) (T-t)^{\alpha/(\alpha+1)} = \infty.$$

If  $\alpha > 1$ , the problem (P) has no fast blow-up solutions, that is, a type I blow-up occurs only, and if  $\alpha = 1$ , a type I and a type II blow-up are intermixed [1]; while if  $\alpha \ge 1$ , a solution polygon shrinks to a single point [3], even if a type II blow-up occurs. In general, the next lemma holds:

**Lemma 2.3** Let  $\alpha > 0$ . If a solution  $\{v_j\}$  of (P) undergoes a type I blow-up, then the solution polygon shrinks to a single point.

From this lemma, it holds that if degenerate pinching occurs, then a type II blow-up occurs. If  $\alpha < 1$ , and if the initial admissible polygon  $\mathcal{P}(0)$  has at least one pair of parallel edges and a distance between them is sufficiently small compared with length of the edges, then degenerate pinching occurs [1]. Hence, by this lemma, a type II blow-up solution exists if  $\alpha < 1$ . The next theorem is the lower bound of a type II blow-up rate when degenerate pinching occurs.

**Theorem A** Let  $\alpha < 1$ . If  $\mathcal{P}(0)$  has a pair of parallel edges  $j_0, j_1$  such that  $\theta_{j_0} = 0, \theta_{j_1} = \pi$ and if the  $j_0$ -th and the  $j_1$ -th edges do not disappear as t tends to the final time T, then for all  $j \neq j_0, j_1$  the solutions  $v_j$  blow up to infinity at least the following rate:

 $v_j(t) \ge C(T-t)^{-\alpha}, \quad j \ne j_0, j_1, \quad t \in [0,T),$ 

for a generic constant C > 0.

Under an additional condition on monotonicity of a solution, we obtain the exact type II blow-up rate in degenerate pinching case.

Theorem B Assume

(A) 
$$(\Delta_{\theta} v(0) + v(0))_j \ge 0, \quad j \in \mathcal{I}.$$

If a solution has a degenerate pinching singularity, that is, a parallel pair of edges, the  $j_0$ -th and the  $j_1$ -th edges, do not disappear at the final time T, then it holds that

 $v_j(t) \sim (T-t)^{-\alpha}, \quad j \neq j_0, j_1, \quad t \in [0,T).$ 

Lemma 2.3 and Theorem A and B are proved in [6] (with the Wulff shape being regular) and in [7] (in general case).

In our knowledge, the above results and several results in [1] are only known about type II blow-up, and then its rate has not been completely classified, in addition, the existence of a type II blow-up in single point extinction case is still open problem. In the next section, we will study unknown type II blow-up rates numerically.

## 3 Numerical studies

In this section, we estimate blow-up rate numerically for some initial data which does not satisfy the condition in theoretical results in section 2.

In [5], Hirota and Ozawa developed a new numerical estimating method of blow-up time and blow-up rate of solutions to a system of ordinary differential equations. Roughly speaking, their method is based on the following three parts:

1. Arc length transformation technique:

Let us consider the initial value problem for the following system of ordinary differential equations

$$\frac{d}{dt}y_j(t) = f_j(t, y_0, \dots, y_{N-1}), \quad j \in \mathcal{I}.$$

The next transformation is called *arc length transformation*:

$$\frac{d}{ds} \begin{pmatrix} t(s) \\ y_0(s) \\ \vdots \\ y_{N-1}(s) \end{pmatrix} = \frac{1}{\sqrt{1 + \sum_{k=0}^{N-1} f_k^2}} \begin{pmatrix} 1 \\ f_0 \\ \vdots \\ f_{N-1} \end{pmatrix}, \quad t(0) = 0.$$

From this transformation, a solution of a new problem never blows up in a finite time even if the solution of the original problem blows up in a finite time. 2. Generate a linearly convergent sequence to T: Assume that there is only  $(T-t)^{-p}$  type singularity. Here p > 0 and T is a blow-up time of the original problem. We note that blow-up time is given by

$$T = \int_0^\infty \frac{ds}{\sqrt{1 + \sum_{k=0}^{N-1} f_k^2}}$$

Let  $\{s_n\}$  be the geometric sequence given by

$$s_n = s_0 r^n$$
  $(s_0 > 0, r > 1, n = 0, 1, 2, ...),$ 

and let define the sequence  $\{t_n\}$  by

$$t_n = \int_0^{s_n} \frac{ds}{\sqrt{1 + \sum_{k=0}^{N-1} f_k^2}}$$

Then  $\{t_n\}$  converges to T linearly.

- 3. Acceleration by the Aitken  $\Delta^2$  method:
  - The Aitken  $\Delta^2$  method can be applied to linearly convergent sequence in order to accelerate the convergence. Thus, we obtain an approximation of the blow-up time, say  $\tilde{T}$ . Let  $e_n = T t(s_n)$ . Then  $\lim_{n\to\infty} |e_{n+1}/e_n| = r^{-1/p}$ . Using  $\tilde{T}$  instead of T, we can calculate an approximate value of p.

Now we apply this algorithm to the problem (P). For a numerical integrator of ODEs from  $s = s_{n-1}$  to  $s = s_n$ , we use the DOPRI5 code (see [4]) with parameters ITOL=0 and RTOL=ATOL=1.d-15. Computations are performed by using the double precision IEEE arithmetic. Set N = 8,  $\vartheta_j \equiv 2\pi/N$  (the Wulff shape is regular),  $s_n = 16 \cdot 2^n$  ( $s_0 = 16$  and r = 2) and apply the Aitken  $\Delta^2$  method 3 times. The following tables show numerical estimates of blow-up rate  $\tilde{p}$ ,  $|\tilde{p}-\alpha/(\alpha+1)|$ ,  $|\tilde{p}-\alpha|$  and ratio  $d_{\min}/d_{\max}$ . If a solution polygon has degenerate pinching singularity at T, the ratio  $d_{\min}/d_{\max}$  tends to zero. Therefore, if a ratio remains positive, then the solution polygon shrinks to a single point.

Here we show three typical numerical examples: Table 1 shows the case of a symmetric anisotropic flow with  $\alpha = 2$  and Figure 2 shows time evolution (from outside to inside) of a solution polygon. From them we see that the solution undergoes a type I blow-up and the solution polygon shrinks to a single point. These numerical observations consist with theoretical results.

Table 2 shows the case of a symmetric anisotropic flow with  $\alpha = 1/2$  and Figure 3 shows time evolution (from outside to inside) of a solution polygon. From them we see that the solution undergoes a type I blow-up and the solution polygon shrinks to a single point. In sublinear case  $\alpha < 1$ , it is not clear that a solution undergoes a type I blow-up or a type II blow-up in single point extinction case. Our numerical computations, including this example, suggest that in every single point extinction case a solution blows up with type I blow-up rate. However, we restricted the form of blow-up rate to  $(T-t)^{-p}$  type, and so, we cannot detect another type blow-up rate, for example  $(T-t)^{-\alpha/(\alpha+1)}(\log \log 1/(T-t))^{-\beta}$ , numerically. At this stage, we conjecture that if there exists such kind of type II blow-up rate, then the blow-up rate of this type II blow-up solution is very closed to  $(T-t)^{-\alpha/(\alpha+1)}$ .

n	$\widetilde{p}$	error(Type I)	error(Type II)	ratio $d_{\min}/d_{\max}$
1	$8.956e{+}01$	8.889e + 01	8.756e + 01	1.143e-01
2	$1.056e{+}00$	3.893e-01	-9.440e-01	1.373e-01
3	2.471e-01	-4.195e-01	-1.753e+00	1.265e-01
4	4.768e-01	-1.899e-01	-1.523e+00	2.056e-01
5	5.736e-01	-9.305e-02	-1.426e+00	2.151e-01
6	6.205 e-01	-4.614e-02	-1.379e+00	2.179e-01
$\overline{7}$	6.437 e-01	-2.297e-02	-1.356e+00	2.189e-01
8	6.552 e-01	-1.145e-02	-1.345e+00	2.193e-01
9	6.609e-01	-5.724e-03	-1.339e+00	2.195e-01
10	6.638e-01	-2.885e-03	-1.336e+00	2.196e-01

Table 1: Convergent behavior of blow-up rate in the case  $\alpha = 2$ .

**Conjecture I** Suppose that  $0 < \alpha < 1$  and a solution polygon shrinks to a single point. If there is a type II blow-up, then the upper bound of blow-up rate is  $(T-t)^{-\alpha/(\alpha+1)-\epsilon}$  for any  $\epsilon > 0$ .

Table 3 shows the case of a symmetric anisotropic flow with  $\alpha = 1/2$  and Figure 4 shows time evolution (from outside to inside) of a solution polygon. From them we see that the blow-up rate is  $(T - t)^{-\alpha}$  and degenerate pinching occurs. In this case the initial data does not satisfy the assumption (A). Our numerical computations, including this example, suggest that in every degenerate pinching case a blow-up rate of solution is exactly  $(T - t)^{-\alpha}$  or very close to  $(T - t)^{-\alpha}$ . If eventually monotonicity occurs, the assertion of Theorem B is valid, that is, the blow-up rate is exactly  $(T - t)^{-\alpha}$ . Here eventually monotonicity means that there exists t' > 0 such that  $\dot{v}_j \ge 0$  for any  $j \in \mathcal{I}$  and  $t \ge t'$ . This is an open problem.

**Conjecture II** Suppose that  $0 < \alpha < 1$  and a solution polygon has degenerate pinching singularity. The upper bound of blow-up rate is  $(T-t)^{-\alpha-\epsilon}$  for any  $\epsilon > 0$ .

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#### References

- B. Andrews, Singularities in crystalline curvature flows, Asian J. Math. 6 (2002) 101–122.
- [2] S. Angenent and M.E. Gurtin, Multiphase thermomechanics with interfacial structure, 2. Evolution of an isothermal interface, Arch. Rational Mech. Anal. 108 (1989) 323–391.

n	$\widetilde{p}$	error(Type I)	error(Type II)	ratio $d_{\min}/d_{\max}$
1	5.021e-01	1.688e-01	2.129e-03	4.837e-01
2	4.162e-01	8.287 e-02	-8.380e-02	5.080e-01
3	3.739e-01	4.056e-02	-1.261e-01	5.584 e-01
4	3.530e-01	1.970e-02	-1.470e-01	6.183 e- 01
5	3.428e-01	9.497 e-03	-1.572e-01	6.807 e-01
6	3.379e-01	4.553e-03	-1.621e-01	7.404e-01
7	3.355e-01	2.172e-03	-1.645e-01	7.935e-01
8	3.344e-01	1.030e-03	-1.656e-01	8.385e-01
9	3.338e-01	4.844e-04	-1.662e-01	8.752e-01
10	3.336e-01	2.232e-04	-1.664e-01	9.044 e-01

Table 2: Convergent behavior of blow-up rate in the case  $\alpha = 1/2$ .

n	$\widetilde{p}$	error(Type I)	error(Type II)	ratio $d_{\min}/d_{\max}$
1	7.569e-01	4.235e-01	2.569e-01	2.672e-01
2	5.808e-01	2.475e-01	8.081e-02	1.410e-01
3	5.141e-01	1.808e-01	1.409e-02	7.606e-02
4	4.696e-01	1.363e-01	-3.036e-02	3.766e-02
5	4.639e-01	1.306e-01	-3.610e-02	1.673 e-02
6	4.766e-01	1.433e-01	-2.337e-02	6.360e-03
$\overline{7}$	4.865e-01	1.532e-01	-1.351e-02	2.074e-03
8	4.926e-01	1.593e-01	-7.379e-03	6.055 e- 04
9	4.962e-01	1.628e-01	-3.834e-03	1.648e-04
10	4.982e-01	1.649e-01	-1.800e-03	4.307 e-05

Table 3: Convergent behavior of blow-up rate in the case  $\alpha = 1/2$ .



Figure 2: Time evolution of a solution polygon in the case  $\alpha = 2$ : The solution polygon shrinks to a single point.



Figure 3: Time evolution of a solution polygon in the case  $\alpha = 1/2$ : The solution polygon shrinks to a single point.



Figure 4: Time evolution of a solution polygon in the case  $\alpha = 1/2$ : The solution polygon has degenerate pinching singularity.

- [3] M.-H. Giga and Y. Giga, Crystalline and level set flow convergence of a crystalline algorithm for a general anisotropic curvature flow in the plane, Free boundary problems: theory and applications, I (Chiba, 1999), GAKUTO Internat. Ser. Math. Sci. Appli., 13, Gakkōtosho, Tokyo, (2000) 64–79.
- [4] E. Hairer, S.P. Nørsett and G. Wanner, Solving Ordinary Differential Equations I, 2nd rev. ed., Springer, Berlin, 2000. (Fortran source of DOPRI5 is available at the web site http://www.unige.ch/ math/folks/hairer/software.html)
- [5] C. Hirota and K. Ozawa, A method of estimating the blow-up time and blow-up rate of the solution of the system of ordinary differential equations – An application to the blow-up problems of partial differential equations –, Transactions of the Japan Society for Industrial and Applied Mathematics, 14 (2004) 13–38.
- [6] T. Ishiwata and S. Yazaki, On the blow-up rate for fast blow-up solutions arising in an anisotropic crystalline motion, J. Comp. App. Math. 159 (2003) 55–64.
- [7] T. Ishiwata and S. Yazaki, A fast blow-up solution and degenerate pinching arising in an anisotropic crystalline motion, preprint.
- [8] A. Stancu, Asymptotic behavior of solutions to a crystalline flow, Hokkaido Math. J. 27 (1998) 303–320.
- [9] J. E. Taylor, Constructions and conjectures in crystalline nondifferential geometry, Proceedings of the Conference on Differential Geometry, Rio de Janeiro, Pitman Monographs Surveys Pure Appl. Math. 52 (1991) 321–336, Pitman London.