GAKUTO International Series
Mathematical Sciences and Applications Vol. 12(1999)
Advances in numerical mathematics, pp.261-270

CONVERGENCE OF A CRYSTALLINE ALGORITHM FOR THE MOTION OF A CLOSED CONVEX CURVE BY A POWER OF CURVATURE

TAKEO K. USHIJIMA AND SHIGETOSHI YAZAKI*

Abstract. In this paper, we present a numerical scheme for generalized curvature flows. We construct our scheme using crystalline algorithm. Several nice properties of the scheme are explained and its convergence is proved in certain cases. We show some numerical examples in order to strengthen our claims on the cases where a convergence proof is not available.

1. Introduction. Let us consider the motion of a closed immersed curve $C(t) := \{\boldsymbol{x}(t) \in \mathbb{R}^2\}$ which is evolved by the law

$$V := \frac{d\boldsymbol{x}}{dt} \cdot \boldsymbol{n} = g(\boldsymbol{n})|K|^{\alpha-1}K. \tag{1}$$

Here $V, \boldsymbol{n}, K$, and α denote the normal velocity, the unit inward normal vector, the curvature, and the positive parameter, respectively, and g is a positive smooth function of $\boldsymbol{n}$.

When $\alpha = 1$ and $g \equiv 1$, (1) is called the curvature flow or motion by the curvature. There are many papers which study this problem (see [7, 10, 5] and references therein). When $\alpha = 1/3$ and $g \equiv 1$, problem (1) is invariant under the affine transformation ([20]) and it attracts many researchers in recent years (see [4] and references therein). In this case, moreover, (1) is used for the model equation for image processing ([1]).

We call the problem (1) a generalized curvature flow problem (GCF in short). Let us explain some basic properties of GCF. Firstly, GCF is a curve-shortening and area-decreasing

*The second author is partially supported by JSPS Research Fellowships for Young Scientists.

motion. Namely,

(P1) $$\frac{d}{dt}L \leq 0 \quad \text{for} \quad t > 0,$$

and

(P2) $$\frac{d}{dt}A \leq 0 \quad \text{for} \quad t > 0.$$

Here L and A denote the length of the curve and the area enclosed by the curve, respectively. We note that for $\alpha = 1, \eta = 1$, and $g \equiv 1$

(P2)' $$\frac{d}{dt}A = -2\pi \quad \text{for} \quad t > 0.$$

Here η denotes the rotation number of the curve C. Secondly, it is well known that

(P3) solution of GCF develops singularity in finite time, say, T.

We call T the "blow-up time". Thirdly, for $\alpha = 1, \eta = 1$, and $g \equiv 1$ ([10]), and for $\alpha = 1/3, \eta = 1$, and $g \equiv 1$ ([20, 4]), it is proved that

(P4) simple non-convex closed curve becomes convex and eventually shrinks to a point.

Finally, Andrews [2], and independently Matano-Wei [15] extensively studied GCF for simple closed convex curves. They pointed out that the value $\alpha = 1/3$ is critical in the following sense:

(P5) when $\alpha > 1/3$, the curve converges to a unique asymptotic shape as t tends to T, while $\alpha \leq 1/3$, what is called the isoperimetric ratio goes to infinity or the curve converges to an asymptotic shape as t tends to T.

The objective of this paper is to construct a numerical scheme which satisfies discrete versions of the properties above. We construct such a scheme by using so-called crystalline approximation. Also, we report on our on-going research for details of singularity development.

Now let us explain the **crystalline algorithm**. In this approximation, the smooth curve and the curvature are approximated by the piecewise linear curve which is governed by the crystalline motion and the crystalline curvature, respectively. Let the piecewise linear curve Γ consists of the n-vertices $\{\boldsymbol{B}_j\}_{0 \leq j < n}$. Under the **crystalline motion**, the j-th side of Γ moves toward its normal direction $\boldsymbol{n}_j$ with speed v_j. This motion is described by the system of ordinary differential equations for the length d_j of the j-th side:

(2) $$\frac{d}{dt}d_j = -\frac{|v_{j+1}|}{\sin \Delta\theta_{j+1}} + \left(\frac{\cos \Delta\theta_{j+1}}{\sin \Delta\theta_{j+1}} + \frac{\cos \Delta\theta_j}{\sin \Delta\theta_j}\right)|v_j| - \frac{|v_{j-1}|}{\sin \Delta\theta_j}.$$

Here $\Delta\theta_j$ denote the angle between the normal vector $\boldsymbol{n}_j$ and the normal vector $\boldsymbol{n}_{j-1}$ (see [3]). When the curve is convex and $\Delta\theta_j \equiv \Delta\theta$ for all j, equation (2) becomes

$$\frac{d}{dt}d_j = -\frac{2(1-\cos\Delta\theta)}{\sin\Delta\theta}(\Delta_\theta v + v)_j, \tag{3}$$

where the operator Δ_θ is

$$(\Delta_\theta v)_j := \frac{v_{j+1} - 2v_j + v_{j-1}}{2(1-\cos\Delta\theta)}. \tag{4}$$

The **crystalline curvature** k_j of the j-th side is the quantity which is defined for each side of the piecewise linear curve as follows:

$$k_j := \chi_j \frac{\gamma_j}{d_j}, \tag{5}$$

where

$$\gamma_j := \frac{1}{\sin\Delta\theta_j} + \frac{1}{\sin\Delta\theta_{j+1}} - \frac{1}{\tan\Delta\theta_j} - \frac{1}{\tan\Delta\theta_{j+1}},$$

and χ_j is the transition number of the j-th side which indicates the convexity of the side, and is defined as

$$\chi_j := \frac{1}{2}(\mathrm{sgn}(\boldsymbol{t}_{j-1} \times \boldsymbol{t}_j) + \mathrm{sgn}(\boldsymbol{t}_j \times \boldsymbol{t}_{j+1})). \tag{6}$$

Here $\boldsymbol{t}_j := (\boldsymbol{B}_{j+1} - \boldsymbol{B}_j)/d_j$ denotes the unit tangent vector of the j-th side. See Figure 1. When the curve is convex and $\Delta\theta_j \equiv \Delta\theta$, the crystalline curvature of the j-th side is given by

$$k_j := \frac{2\tan\frac{\Delta\theta}{2}}{d_j}. \tag{7}$$

We note that the crystalline curvature is a generalization of the usual curvature. In fact, when the piecewise linear curve is convex and $\Delta\theta_j \equiv \Delta\theta$, the crystalline curvature k_j is the reciprocal of the distance from the j-th side to the center of the largest inscribed regular polygon at this side.

By using (5), we approximate the evolution law (1) by

$$v_j = g(\boldsymbol{n}_j)|k_j|^{\alpha-1}k_j. \tag{8}$$

Substituting this into the equation (2), we obtain the system of ordinary differential equations for d_j (or v_j or k_j). We call this way of approximation a crystalline algorithm.

The notion of crystalline motion was originally introduced by Taylor [21] through the study of the motion of surfaces under anisotropic surface energy. Recently, numerous authors study the crystalline motion ([21, 22, 23, 24, 25, 3, 18, 19, 11]). In particular, the convergence

Figure 1: Transition number

of the crystalline algorithm is studied in [6, 8, 9, 12, 26]. In [26], we proved the convergence of the numerical scheme based on crystalline algorithm for the case where $\mathcal{C}$ is convex, $\alpha > 0$ and $g \equiv 1$.

In this paper, we extend the scheme of [26] in the following two directions:

1. the case where $g \not\equiv 1$,

2. the case where $\mathcal{C}$ is not necessarily convex.

We explain 1 in section 2 and 2 in section 3. In both sections, we also exhibit several numerical examples which are computed by our scheme.

We conclude the introduction with remarks on the related numerical works. Kimura [13, 14] construct a numerical scheme for $\alpha = 1$ and $g \equiv 1$, and prove the convergence of his scheme. The main feature of his scheme is that the tracking points are relocated and kept uniformly distributed. It seems that his scheme is suitable for the case $\alpha = 1$ and $g \equiv 1$, because the asymptotic shape just before it disappears is a circle. Mikula [16] and Mikula-Kačur [17] study the numerical scheme for parabolic equations which are related to GCF and prove the convergence of the scheme. In their scheme, they adopt Rothe's approximation with respect to time and the finite element method in space. We note that both Kimura and Mikula *et.al* do not use adaptive time step control, while we do. We would like to emphasize a great advantage of adaptive time steps over a fixed time step when we deal with singular phenomena like extinctions or blow-ups.

2. Crystalline algorithm for convex curves. In this section we explain our numerical scheme based on crystalline algorithm for immersed closed curves which have everywhere positive curvature. We also show the convergence of the scheme. Hereafter, we assume $\Delta\theta_j \equiv \Delta\theta := 2\eta\pi/n$.

We first explain the approximation for initial curve $\mathcal{C}(0)$. This approximation is due to [8], which showed that the accuracy of the approximation is $O(\Delta\theta^2)$.

Procedure 1 (Crystalline approximation for initial curve $\mathcal{C}(0)$) Let $\boldsymbol{x}_j^0$ be the point on $\mathcal{C}(0)$ with the inward normal vector $\boldsymbol{n}_j := -(\cos j\Delta\theta, \sin j\Delta\theta)$. Let Γ_Δ^0 be the union of line segments $|\boldsymbol{B}_{j+1}^0 - \boldsymbol{B}_j^0|$, where $\boldsymbol{B}_j^0$ is the intersection point between the lines tangent to $\mathcal{C}(0)$ at $\boldsymbol{x}_{j-1}^0$ and $\boldsymbol{x}_j^0$ ($\boldsymbol{B}_n^0 = \boldsymbol{B}_0^0$). Namely, the relation between $\boldsymbol{x}_j^0$ and $\boldsymbol{B}_j^0$ is given by

$$\boldsymbol{B}_j^0 := \left(\left(\boldsymbol{x}_{j-1}^0 - \boldsymbol{x}_j^0\right) \cdot \left(\boldsymbol{t}_j + \boldsymbol{n}_j \cot \Delta\theta\right)\right) \boldsymbol{t}_j + \boldsymbol{x}_j^0.$$

Here $\boldsymbol{t}_j := (-\sin j\Delta\theta, \cos j\Delta\theta)$ is the tangent vector at $\boldsymbol{x}_j^0$.

Next, we explain how to determine the piecewise linear curve Γ_Δ^{m+1} from the known curve Γ_Δ^m. We fix a positive number ε.

Procedure 2 (Crystalline algorithm) For given $\{v_j^m\}_{0\le j<n}$, $\{\boldsymbol{x}_j^m\}_{0\le j<n}$ and Γ_Δ^m, we define $\{v_j^{m+1}\}_{0\le j<n}$, $\{\boldsymbol{x}_j^{m+1}\}_{0\le j<n}$ and Γ_Δ^{m+1} as follows:

(a) $\tau_m := \frac{\lambda_\alpha(\varepsilon)\Delta\theta_2}{(v_{\max}^m)^{\alpha+1/\alpha}}$, $\quad \lambda_\alpha(\varepsilon) := \frac{\min_{0\le j\le n} g(\boldsymbol{n}_j)^{1/\alpha}}{2(1+\varepsilon)^{\alpha+1}\max\{1,\alpha\}}$, $\quad \varepsilon > 0$;

(b) $v_j^{m+1} := v_j^m \left(1 - g(\boldsymbol{n}_j)^{-1/\alpha}(v_j^m)^{1/\alpha}(\Delta_\theta v + v)_j^m \tau_m\right)^{-\alpha}$;

(c) $\boldsymbol{x}_j^{m+1} := \boldsymbol{x}_j^m + v_j^m \boldsymbol{n}_j \tau_m$;

(d) the vertices $\boldsymbol{B}_j^{m+1}$:

$$\boldsymbol{B}_j^{m+1} := \left(\left(\boldsymbol{x}_{j-1}^{m+1} - \boldsymbol{x}_j^{m+1}\right)\cdot(\boldsymbol{t}_j + \boldsymbol{n}_j \cot\Delta\theta)\right)\boldsymbol{t}_j + \boldsymbol{x}_j^{m+1}. \tag{9}$$

Piecewise linear curve Γ_Δ^{m+1} is given as the union of the line segments $|\boldsymbol{B}_{j+1}^{m+1} - \boldsymbol{B}_j^{m+1}|$. Here we set $\Delta\theta_2 := 2(1-\cos\Delta\theta) = \Delta\theta^2 + \mathrm{O}(\Delta\theta^4)$, and $t_m := \sum_{i=0}^{m-1}\tau_i$.

Let us expound this procedure. We consider the points $\{\boldsymbol{x}_j^m\}$ which are evolved by

$$(D_\tau \boldsymbol{x})_j^m := \frac{\boldsymbol{x}_j^{m+1} - \boldsymbol{x}_j^m}{\tau_m} = v_j^m \boldsymbol{n}_j, \tag{10}$$

where τ_m denotes the time increment at the time t_m. This relation is the Euler discretization of the evolution law

$$\frac{d\boldsymbol{x}_j(t)}{dt} = v_j(t)\boldsymbol{n}_j,$$

which is the crystalline approximation of the equation (1). We determine the vertices $\{\boldsymbol{B}_j^{m+1}\}$ from the points $\{\boldsymbol{x}_j^{m+1}\}$ by the relation (9). Then we can derive the evolutionary relations of the length of the sides $d_j^m := |\boldsymbol{B}_{j+1}^m - \boldsymbol{B}_j^m|$ and the normal velocity v_j^m. This evolutionary relation is consistent with the one which is derived from the system of ordinary differential equations (3).

Remark 1 Because of the adaptive time step control, the following properties hold:

1. no sides vanishes in finite step, and the number of self-intersection does not increase;
2. a discrete analogue of super- and sub-solution method works.

Remark 2 Properties (P1) and (P2) hold. Also (P3) holds (see Theorem 1 below).

Remark 3 We can construct a certain implicit scheme which satisfies properties (P1), (P2), and a discrete analogue of (P2)'.

Remark 4 In our scheme, we can compute v_j^m by two different ways. One is based on the evolutionary relation (b) of Procedure 2.2. The other is computed from the length d_j^m. If there is no numerical error, these two values agree. Hence we can estimate the numerical error by the difference of these two values. We stop the computation when this difference becomes larger than a prescribed threshold value.

We can prove the blow-up and convergence results [26].

Theorem 1 *Summation of the time step τ_m converges, namely, $T_\Delta := \sum_{m=0}^{\infty} \tau_m < \infty$. Hence we have $\lim_{m\to\infty} \max_{0\le j<n} v_j^m = \infty$.*

Theorem 2 *Fix a parameter $\alpha > 0$ and a winding number $\eta \ge 1$. Let $\mathcal{C}(t)$ be a solution of (1) and let T_* the blow-up time of this solution. Fix a positive number ε. Let Γ_Δ^m be the solution of our scheme. Then for any $T \in [0, T_*)$, there exist finite constant $C = C(T, \min_{\theta\in\mathbb{R}/2\eta\pi\mathbb{Z}} V(\theta,0), \|V\|_T)$ and a positive integer $n_0 = n_0(T, \|V\|_T, \min_{\theta\in\mathbb{R}/2\eta\pi\mathbb{Z}} V(\theta,0), \varepsilon)$ such that the following error estimates hold:*

(a) $\max_{0\le j<n} \left|v_j^m - V(j\Delta\theta, t_m)\right| \le C(\Delta\theta)^2$ *for all* $n \ge n_0, \quad 0 \le t_m \le T,$

(b) $\mathcal{D}_{\mathcal{H}}(\Gamma_\Delta^m, \mathcal{C}(t_m)) \le C(\Delta\theta)^2$ *for all* $n \ge n_0, \quad 0 \le t_m \le T,$

(c) $|L_\Delta^m - L(t_m)| \le C(\Delta\theta)^2$ *for all* $n \ge n_0, \quad 0 \le t_m \le T,$

(d) $|A_\Delta^m - A(t_m)| \le C(\Delta\theta)^2$ *for all* $n \ge n_0, \quad 0 \le t_m \le T,$

where $\mathcal{D}_{\mathcal{H}}$ is the Hausdorff distance and L_Δ^m (resp. $L(t_m)$) is the length of Γ_Δ^m (resp. the arc-length of $\mathcal{C}(t_m)$), and (if $\eta = 1$,) A_Δ^m (resp. $A(t_m)$) is the area of the region enclosed by Γ_Δ^m (resp. $\mathcal{C}(t_m)$).

Here we set $\|V\|_0 := \sum_{k=0}^{2} \sup_{\theta\in\mathbb{R}/2\eta\pi\mathbb{Z}} \left|\frac{\partial^k}{\partial\theta^k} V(\theta,0)\right|$ and
$\|V\|_T := \sum_{k=0}^{4} \sup_{0\le t\le T, \theta\in\mathbb{R}/2\eta\pi\mathbb{Z}} \left|\frac{\partial^k}{\partial\theta^k} V(\theta,t)\right| + \sum_{k=1}^{2} \sup_{0\le t\le T, \theta\in\mathbb{T}_\eta} \left|\frac{\partial^k}{\partial t^k} V(\theta,t)\right|$.

This result can be proved by a discrete version of the comparison lemma. For the smooth solution $K(\theta,t)$ of GCF, we define

$$(K^{\pm})_j^m := K(j\Delta\theta, t_m) \pm Ae^{Bt_m}\Delta\theta_2.$$

If we choose A, B, and $\Delta\theta$ properly, we can prove K^+ (resp. K^-) is a super-solution (resp. sub-solution). Hence by the comparison principle, we obtain

$$(K^-)_j^m \le k_j^m \le (K^+)_j^m.$$

It implies the convergence of the curvature. Using this convergence result, we can prove the rest of the theorem. See [26] for further detail.

In the end of this section, we show several examples. Figure 2 lets us verify that our scheme enjoys the property (P5). Figure 3 shows the effect of anisotropy g.

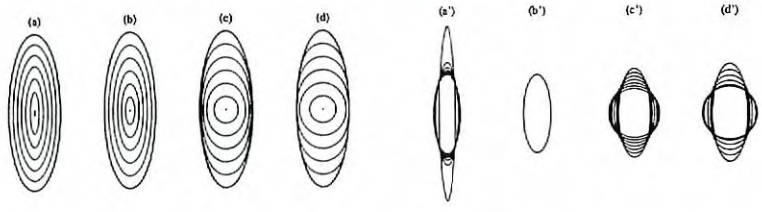

Figure 2: (a),(a') $\alpha = \frac{1}{4}$, (b),(b') $\alpha = \frac{1}{3}$, (c),(c') $\alpha = 1$ and (d),(d') $\alpha = 2$.

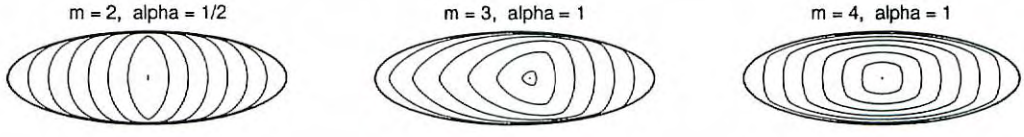

Figure 3: $V = gK^{\alpha}$, $g = 1 + 0.9\cos(m\theta)$.

3. Crystalline algorithm for non-convex curves. In this section, we construct a numerical scheme which satisfies the property (P4). We consider the case where the curve $\mathcal{C}(t) \subset \mathbb{R}^2$ is an embedded, non-convex, closed curve and $g \equiv 1$, for simplicity[†]. We note that the set of normal direction of this curve is $\Theta := [0, 2\pi)$. We define the normal angle set: $\Theta_n := \left\{ \theta_j := j\Delta\theta \;\middle|\; j = 0, 1, ..., n-1, \quad \Delta\theta := \dfrac{2\pi}{n}, \quad n \geq 4 \right\}$, which is a discretization of Θ.

We approximate the smooth non-convex initial curve by n-smooth polygon[‡]. Then, for evolution, we use the system of ordinary differential equations (2) and the definition of crystalline curvature (5). For $\alpha = 1$, Ishii-Soner [12] shows that if a side of the solution for equations (2) and (5) disappears before extinction, then the transition number of this side must be zero. They claim that all sides whose transition number is zero disappear before extinction. However, we do not know a proof of this property for a general α. We discretize (2) explicitly in time and obtain our scheme. In our scheme, we control the time step carefully to realize the property (P4). See Figure 4.

For this scheme we have not obtained any theoretical convergence result yet. So we examine the reliability of the scheme numerically. For this purpose, we check that the

[†]We can easily extend the algorithm to the case where the rotation number η is greater than 1 and $g \not\equiv 1$.

[‡]A polygon Γ is called n-smooth if and only if $\langle 1 \rangle$ Γ encloses a simple connected compact region, $\langle 2 \rangle$ for each side its normal angle θ_j is a element of Θ_n, and $\langle 3 \rangle$ for each j we have $|\theta_{j+1} - \theta_j| = \Delta\theta$ (see [12]).

property (P2)' holds for $\alpha = 1$. See Figure 5. This figure shows that the area is decreasing with ratio $\sim -2n \tan \frac{\Delta\theta}{2}$. It would be safe to conclude that the scheme is reliable in this sense. In Figure 6, we exhibit the evolution which starts from a spiral-shaped polygon.

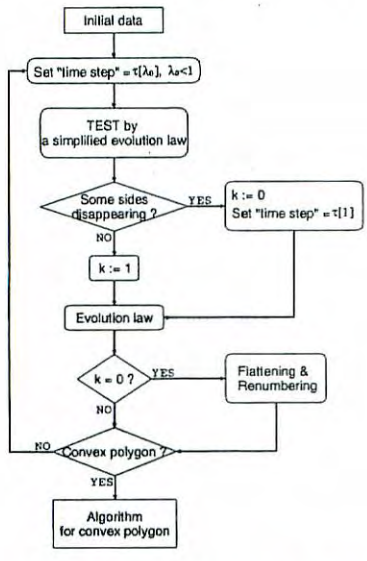

Figure 4: Flow chart of the algorithm.

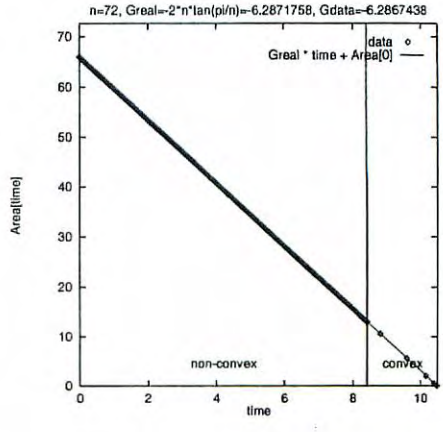

Figure 5: Graph of area versus time for non-convex initial curve ($\alpha = 1$, 72-smooth, $|\text{Greal} - \text{Gdata}| = 0.0004320$).

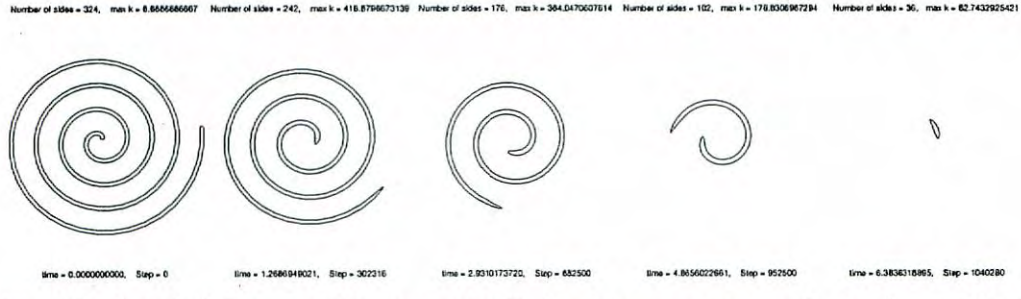

Figure 6: A spiral-shaped polygon eventually becomes a convex polygon ($\alpha = 1/4$, 36-smooth).

Acknowledgment. We would like to thank Professor Hisashi Okamoto for kind encouragement and for reading the manuscript.

References

[1] L. Alvarez, F. Guichard, P. L. Lions, and J. M. Morel. Axioms and fundamental equations of image processing. *Arch. Rational Mech. Anal.*, 123:199–257, 1993.

[2] B. Andrews. Evolving convex curves. *Calc. Var.*, 7:315–371, 1998.

[3] S. Angenent and M. E. Gurtin. Multiphase thermodynamics with interfacial structure 2. evolution of an isothermal interface. *Arch. Rational Mech. Anal.*, 108:323–391, 1989.

[4] S. Angenent, G. Sapiro, and A. Tannenbaum. On the affine heat equation for non-convex curves. *J. Amer. Math. Soc.*, 11:601–634, 1998.

[5] S. Angenent and J. J. L. Velázquez. Asymptotic shape of cusp singularities in curve shortening. *Duke Math. J.*, 77:71–110, 1995.

[6] T. Fukui and Y. Giga. Motion of a graph by nonsmooth weighted curvature. In V. Lakshmikantham, editor, *World Congress of Nonlinear Analysis '92*, volume I, pages 47–56. Walter de Gruyter, Berlin, 1996.

[7] M. Gage and R. S. Hamilton. The heat equation shrinking convex plane curves. *J. Diff. Geom.*, 23:69–96, 1986.

[8] P. M. Girão. Convergence of a crystalline algorithm for the motion of a simple closed convex curve by weighted curvature. *SIAM J. Numer. Anal.*, 32:886–899, 1995.

[9] P. M. Girão and R. V. Kohn. Convergence of a crystalline algorithm for the heat equation in one dimension and for the motion of a graph by weighted curvature. *Numer. Math.*, 67:41–70, 1994.

[10] M. A. Grayson. The heat equation shrinks embedded plane curves to round points. *J. Diff. Geom.*, 26:285–314, 1987.

[11] H. Imai, N. Ishimura, and T. K. Ushijima. Motion of spirals by crystalline curvature. *Math. Model. Num. Anal.*, to appear.

[12] K. Ishii and H. M. Soner. Regularity and convergence of crystalline motion. *SIAM J. Math. Anal.*, 30:19–37, 1999.

[13] M. Kimura. Accurate numerical scheme for the flow by curvature. *Appl. Math. Lett.*, 7:69–73, 1994.

[14] M. Kimura. Numerical analysis for moving boundary problems using the boundary tracking method. *Japan J. Indust. Appl. Math.*, 14:373–398, 1997.

[15] H. Matano and J. Wei. On anisotropic curvature flow equations. preprint, 1997.

[16] K. Mikula. Solution of nonlinear curvature driven evolution of plane convex curves. *Appl. Num. Math.*, 23:347–360, 1997.

[17] K. Mikula and J. Kačur. Evolution of convex curves describing anisotropic motions of phase interfaces. *SIAM J. Sci. Comput.*, 17:1302–1327, 1996.

[18] P. Rybka. A quasi-steady approximation to an integro-differential model of interface motion. *Applicable Anal.*, 56:19–34, 1995.

[19] P. Rybka. A crystalline motion: uniqueness and geometric properties. *SIAM J. Appl. Math.*, 57:53–72, 1997.

[20] G. Sapiro and A. Tannenbaum. On affine plane curve evolution. *J. Funct. Anal.*, 119:79–120, 1994.

[21] J. E. Taylor. Crystalline variational problem. *Bull. Amer. Math. Soc.*, 84:568–588, 1978.

[22] J. E. Taylor. Constructions and conjectures in crystalline nondifferential geometry. In B. Lawson and K. Tenenblat, editors, *Differential Geometry (A symposium in honour of Manfredo do Carmo)*, pages 321–336. Longman, Essex, 1991.

[23] J. E. Taylor. Motion of curves by crystalline curvature, including triple junctions and boundary points. *Proc. Symp. Pure. Math.*, 54:417–438, 1993.

[24] J. E. Taylor. Surface motion due to crystalline surface energy gradient flows. In B. Chow, R. Gulliver, S. Levy, J. Sulliva, and A. K. Peters, editors, *Elliptic and Parabolic Methods in Geometry*, pages 145–162. Massachusetts, 1996.

[25] J. E. Taylor, J. W. Cahn, and C. A. Handwerker. Geometric models of crystal growth. *Acta Metall. Mater.*, 40:1443–1474, 1992.

[26] T. K. Ushijima and S. Yazaki. Convergence of a crystalline algorithm for the motion of a closed convex curve by a power of curvature $v = k^{\alpha}$. *SIAM J. Num. Anal.*, to appear.

Takeo K. Ushijima
Meme Media Laboratory (VBL), Faculty of Engineering, Hokkaido University,
Kita 13, Nishi 8, Kita-ku, Sapporo 060-8628, JAPAN

Shigetoshi Yazaki
Graduate School of Mathematical Sciences, University of Tokyo,
Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, JAPAN