Transmission of Indeterminate Equations As Seen in an Istanbul Manuscript of Abū Kāmil

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Abū Kāmil was one of the most creative mathematicians in the medieval Arabic world. It is to his credit that he saw the necessity in his time of fusing the more theoretical Greek approach with the more practical Babylonian algebra. It was in this manner that abū Kāmil proved himself to be a true innovator. As a result of his mathematical procedures, his successors were in a position to forge ahead without profound philosophical difficulties.

Abū Kāmil Shujā' ibn Aslam ibn M. ibn Shujā' (c. 850–930), "the reckoner from Egypt," was the product of a period of intellectual ferment in the Golden Age of the Arabs. After al-Khwārizmī (ca. 825), abū Kāmil is the earliest algebraist of the Islamic period whose writings are still extant. As a result of a different approach to mathematics, abū Kāmil's algebra is much advanced over that of the practical al-Khwārizmī whose roots are almost entirely Babylonian. A comparison of abū Kāmil's *Al-jabr wa'l-muqābala* with the book of the same title by al-Khwārizmī demonstrates the evolution of algebraic method in a fruitful direction.[1] A work of abū Kāmil which contains some indeterminate equations is the *Kitāb al-ṭarā'if fi'l-ḥisāb*. "Book of Rare Things in the Art of Calculation." These problems which show progress are concerned with integral solutions of linear equations.

Another work of abū Kāmil goes more deeply into algebra with solutions for fourth degree equations and for mixed quadratics with irrational coefficients. Only the Latin and Hebrew texts had been known. The Arabic of this text was discovered by Levey in Istanbul in the important Kara Mustafa Library MS 379 as the second treatise.[2]

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[1] This has been shown by M. Levey, *The Algebra of Abū Kāmil* (Madison, 1966). This publication includes the text and translation of abū Kāmil's elementary *Al-jabr wa'l-muqabala* "Algebra."

[2] *Ibid.*, p. 9, Cf. H. Suter, *Bibliotheca Mathematica* **10**, 15–42 (1909–10) for the German translation. The Italian translation is by G. Sacerdote from the Hebrew in *Festschrift z. 80 Geburtstage M. Steinschneiders* (Leipzig, 1896) pp. 169–194. In Arabic, it is called *Kitāb al-muḥammas wa'l-mu'ashshar*.

As its first treatise, the manuscript has the previously studied elementary algebra. Further, the third treatise is also by abū Ḳāmil and is the presently discussed one. It is not titled by abū Kāmil but, to use his terminology, it may be called *Kitāb masā'il allatī hiya ghair maḥdūda* "Book on Indeterminate Problems." Brockelmann does not mention this manuscript.[3] A good Hebrew translation is in the great Munich Cod. Heb. 225. The Arabic manuscript which is complete and in excellent condition has been collated with the Hebrew text. The latter is very close to the Arabic both in language and spirit. The Arabic is written in a naskhi hand and runs from fol. 79a to fol. 111a.

In the introduction to the text, abū Kāmil gives an interesting account of his own work in mathematics as well as a description of indeterminate problems.

"I have completed the explanation of what is difficult in many of its parts for mathematicians of our time and for those of whom we have heard among earlier scholars with regard to the rules of the pentagon and decagon, circumscribed or inscribed. Also, I have found the diameter of that circle circumscribing or inscribing a known pentagon or decagon. I have measured the arc of one part in fifteen or the circumference of a known circle, also the length of the side of a regular pentagon or decagon if their areas are known. I have measured the length of the sides of triangles with known areas if they are found in a regular pentagon or decagon. This is besides other subjects which we have indicated in our book.

"But now, I would call attention to many indeterminate problems which some mathematicians call 'the stream,' I consider it as an outlet for many correct answers based upon a logical method and a simple approach. Some of these problems concern mathematics of topics which are not founded upon a definite basis; others are solved by a sound theory and a simple trick which could be of great use. Thanks be to Allah for the way he has helped us through them. He is the ominscient for all which is within our hearts.

"Further, we shall explain much that mathematicians have written in their books and what they have achieved regarding the topics of algebra and mensuration. This will help the reader or observer to obtain a good understanding of what is read as a story or to nearly imitate the writer."[4]

The problems which follow are graduated in difficulty since the treatise is meant to be a teaching text. It is, however, well organized as the following order of examples will show. These are all worked out rhetorically in detail. It must be remembered that, although no notation was used, mathematicians were at that time accustomed to the verbal method and had well trained memories.

The first problem reads, "If one says to you that a square having two roots has added to it five dirhams, then it is a root.[5] What is the square in it?"

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[3] C. Brockelmann, *Gesch. der arab. Litteratur* (Leiden, 1937) S, I, p. 390.

[4] MS, fol. 79a.

[5] $x^2 + 5 = \square = y^2$

$y > x$

"... If you put down what is possessed, i.e. a square, which has a root called a thing, and add five dirhams to it, then it is a square plus five dirhams. It is necessary that it appear as a root. It is known that its root is larger than the thing for if its square is alone, then its root equals a thing. Then put its root as a thing and a number; it appears from the number that if it is multiplied by itself, it is less than the dirhams that are added to the square. In this problem, it is five dirhams. Then set it as a thing plus a dirham and multiply it by itself; it appears as a square plus a dirham and two things. Take the square plus five dirhams; take away the square plus one dirham from the square plus five dirhams. There remains two things. You equate the four dirhams to give the thing as two and the square as four. If you add five dirhams to it, then it is nine; its root is then three.

"If you put its root as a thing and two dirhams[6] and then multiply it by itself, it gives a square plus four things plus four dirhams which is equal to a square plus five dirhams. Subtract, then, a square plus five dirhams to give four things which is then equal to a dirham, or the thing equals one-quarter of a dirham which is the root of the square. The square is one-half of [one-eighth of] a dirham. If you add five dirhams, then it is five and one-sixteenth,[7] and its root is two and one-quarter dirhams. If you wish let this square be a thing plus a half-dirham or a thing plus one and a half dirhams, or a thing plus one-third dirham, or whatever number you wish to appear after the thing to which it is added. If you multiply it by itself, it is less than five. Then, for that, gather what occurs of this kind."[8]

The next equation,[9] in modern notation,[10] is

$$x^2 - 10 = \square = y^2$$

It is solved by the author as follows,

$$\text{Let } \sqrt{x^2 - 10} = x - 1;$$
$$\text{then } x^2 - 10 = x^2 - 2x + 1$$
$$2x = 11;\ x = 5\frac{1}{2};\ x^2 = 30\frac{1}{4}$$

Assume $x^2 + 5 = (x + 1)^2$
$\qquad = x^2 + 2x + 1$
$\therefore\ x = 2; x^2 + 5 = 3^2$ or $x^2 = 4$
Only an integral solution is sought by the author.

6 $(x + 2)^2 = x^2 + 4x + 4 = x^2 + 5$
$$4x = 1;\ x = \frac{1}{4};\ x^2 = \frac{1}{2}\cdot\frac{1}{8}$$

7 Generally, $x^2 + 5 = (x + a)^2 = x^2 + 2ax + a^2$
$$\therefore\quad x = \frac{5 - a^2}{2a}$$
Only positive values are considered and the stipulation is that $a^2 < 5$.

8 MS, fols. 79a–79b.

9 MS, fols. 79b–80a.

10 Cf. Heath, *Diophantos of Alexandria* (London, 1938) pp. 166–167; 205–206.

$$x^2 - 10 = 20\frac{1}{4}; \sqrt{20\frac{1}{4}} = 4\frac{1}{2}$$

Brahmagupta, in the early seventh century, in what is considered the golden age of Hindu algebra, stated the algebraic rule for negative numbers and discussed the so-called Pellian equation $x^2 - Dy^2 = 1$ (or $Dy^2 + 1 = x^2$), and $ax + by = c$ a, b, c, as constants and integers. Brahmagupta went a step further to solve $Nx^2 \pm c = y^2$ in positive integers. This would be the more general equation of the cases of $x^2 + 5 = y^2$ and $x^2 - 10 = y^2$. Brahmagupta obtained a single solution in positive integers of the general equation so that he could derive an infinite number of other integral solutions by making use of the integral solutions of $Nx^2 + 1 = y^2$: If (p, q) is a solution of $Nx^2 \pm c = y^2$ and (α, B) is a solution of $Nx^2 + 1 = y^2$, then by the principle of composition,

$$x = p\beta \pm q\alpha \text{ and } y = q\beta \pm Np\alpha$$

will be a solution of the former. This operation may be repeated to obtain many solutions. Abū Kāmil has a simpler solution since N is made equal to one.

$$x^2 + c = y^2 = (x + a)^2$$

$$\therefore \quad x = \frac{c - a^2}{2a} \quad \text{where} \quad a^2 < c$$

Al-Karajī (ca. 1010) discusses this type of problem.[11]

$$10 - x^2 = y^2; \ 30 - x^2 = z^2$$

$$\text{Let } x^2 = 10 - x_1^2$$

$$\text{then } 20 + x_1^2 = z^2$$

Take z so that $x_1^2 < 10$; say $z = x_1 + 3$;

$$\text{then one has } 20 + x_1^2 = x_1^2 + 6x_1 + 9$$

$$x_1 = \frac{11}{6}, \ x_1^2 = \frac{121}{36}, \ x^2 = \frac{239}{36}.$$

Specifically, he also works out the same problem of $x^2 + 5y^2$ by letting $y = x + 1$, then solving to get $x = 2$.[12] Leonardo of Pisa[13] (1220) also used the problem $x^2 \pm 5 = y^2$ in his algebra in connection with his number theory development. The strong influence of abū Kāmil upon Leonardo has already been demonstrated conclusively by Levey.[14] Beha-Eddin (ca. 1600) stated one of the seven unsolved problems to be.

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[11] F. Woepcke, *Extrait du Fakhrī... par Alkarkhī* (Paris, 1953) pp. 85, III.

[12] Al-Karajī, p. 84.

[13] B. Boncompagni, *Scritti di Leonardo Pisano* (Roma, 1857–1862), II, p. 253–283; L. Dickson, *History of the Theory of Numbers* (New York, 1952) II, p. 460 for relation to number theory and especially congruent numbers.

[14] M. Levey, *op. cit.*, pp. 6, 217–220.

$$\begin{cases} x^2 + 10 = \square \\ x^2 - 10 = \square \end{cases}$$

This was termed impossible by G.H.F. Nesselmann.[15] Fermat proved that the difference of two biquadrates is never a square.[16] No congruent number can be a square. Leonardo was aware of this but his proof was incomplete.[17]

A similar problem is:

$$x^2 \pm ax = \square$$

In the text, there are $x^2 + 3x = \square$ and $x^2 - 6x = \square$.[18] Diophantos (VI, 12) discusses a right triangle where the general equation $ax^2 + bx = \square$ applies.[19] In abū Kāmil's problems, $a = 1$, making it a simple solution as in the earlier examples. He has a second method:

$$x^2 + ax = \square = k^2x^2 = y^2$$

$$\therefore \quad x = \frac{a}{k^2 - 1};\ y = \frac{ak}{k^2 - 1}$$

$$ax^2 + bx + c = \square$$

Further, abū Kāmil develops solutions for such equations as:

$$x^2 + 10x + 20 = \square$$

$$x^2 + ax + b = (x + y)^2 = x^2 + 2xy + y^2$$

$$x = \frac{y^2 - b}{a - 2y} \text{ where } y < 5 \text{ and } y^2 < 10$$

Bhaskara II in his *Bījagaṇita* (1150)[20] mentions the solution of the general indeterminate equation of the second degree. These equations are not treated in Brahmagupta or in other known Indian works before this time. His rules may be shown in the general solution for the problem, "What number being doubled and added to six times its square becomes capable of yielding a square root?"[21]

$$6x^2 + 2x = y^2$$

Multiply through by six and add one to each side to get:

$$(6x + 1)^2 = 6y^2 + 1$$

Then Bhaskara II states that by the method of square-nature, the roots of $6y^2 + 1$

[15] "Essenz der Rechenkunst von Beha-Eddin" (Berlin, 1843) p. 55.

[16] *Oeuvres de Fermat* (Paris, 1891) I, p. 340.

[17] Dickson, *op. cit.*, II, p. 615.

[18] MS, fol. 80a.

[19] Heath, *op. cit.*, pp. 233–235; Cf. Dickson, *op. cit.*, pp. 176ff. in problems on areas of right triangles.

[20] B. Datta and A. N. Singh, *History of Hindu Mathematics* (Bombay, 1962), II, pp. 88, 90, 139.

[21] *Ibid.*, II, pp. 184ff.

are: the lesser 2 and the greater 5, or the lesser 20 and the greater 49. The greater root is equated with the square root of the first side to get the value of x as 2/3 or 8.

Starting with the general equation, $ax^2 + bx + c = y^2$, complete the square to get:

$$(ax + \frac{1}{2}b)^2 = ay^2 + \frac{1}{4}(b^2 - 4ac).$$

Let $z = ax + \frac{1}{2}b$ and $k = \frac{1}{4}(b^2 - 4ac)$.

Then $ay^2 + k = z^2$

If $y = t$, $z = m$ are found empirically as solutions; then another solution is

$$\begin{cases} y = tq \pm mp \\ z = mq \pm atp \end{cases}$$

where $ap^2 + 1 = q^2$

Hence, a solution of $ax^2 + bx + c = y^2$ is

$$x = \frac{-b}{2a} + \frac{1}{a}(mq \pm artp)$$

If $m = ar + \frac{b}{2}$ (i.e., $x = r$ when $z = m$),

then $x = \frac{1}{2a}(bq - b) \pm qr \pm tp$

where $ap^2 + 1 = q^2$ and $ar^2 + br + c = t^2$.[22]

Fermat's Equation

The Pellian equation is related to the general equation $ax^2 + bx + c = \square$. Diophantos and abū Kāmil both discussed types of the Pellian equation; the former treated of the $ax^2 + b = \square$ type and the latter $x^2 + b = \square$.

Fermat's equation, or incorrectly called the Pellian equation, has an infinite number of solutions according to Fermat (1657). This was proved by Lord Brouncker and J. Wallis, and later an improved method was given by L. Euler (1765) for obtaining solutions. Euler was a giant in indeterminate analysis.

Double Equations of the First Degree

Simultaneous equations in the abū Kāmil text range from simple cases to fairly complex quadratics with indeterminate solutions. Similarly, Diophantos (II, 11) solves analogous examples as:

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[22] *Ibid.*, p. 185.

$$\begin{cases} x+2=y^2 \\ x+3=z^2 \end{cases}$$

He takes the difference between these double equations and resolves it into two factors as 4 and 1/4. Then, the square of half the difference between these factors is taken and equated to the lesser expression. Or, the square of half the sum is equated to the greater.[23] Number 27 of abū Kāmil may be compared with this process. This is briefly stated by the author and will be quoted.

"If one says there is a square which has a root, and if you subtract it from ten of its roots minus eight dirhams, it will have a root. Then, if you take hold of this problem, multiply half of the roots by itself. Then subtract from what is gathered of the dirhams. You divide what remains into two parts for every one root, so then the problem comes out to what you wish of the answer. If you do not divide what remains into two parts for every one root, then the problem cannot be solved. When this is so, multiply half of the roots by itself. The total is like the dirhams or less; then the problem comes out. Multiply half of the roots by itself, and in this problem five becomes twenty-five. Subtract from it the eight dirhams; then it is seventeen. Divide it into two parts so that each part has a root; then they are sixteen and one. Then you say that you wish a square plus sixteen dirhams equals ten roots minus eight dirhams, or a square plus a dirham equals ten roots minus eight dirhams. Then the square you desire comes out to thirty-six. If you wish sixteen, then take a square plus a dirham equal to ten roots minus eight dirhams to give the desired square as eighty-one, and if you wish, one."[24]

The Bakhashālī treatise (ca. 10th cent.) is probably one of the more important in Indian mathematics to give solutions for double equations of the first degree.[25] This text solves:

$$\begin{cases} x+a=y^2 \\ x-b=z^2 \end{cases}$$

Its solution is $x=\left\{\frac{1}{2}\left[\frac{a+b}{m}-m\right]\right\}^2+b$

where m is any integer. In the text, m is taken as 2.

Mahāvīrā (9th cent.) was well acquainted with these operations. Brahmagupta (628) improved upon this to solve the general case:

$$\begin{cases} x\pm a=y^2 \\ x\pm b=z^2 \end{cases}$$

Al-Karajī gives the problem:[26]

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[23] Heath, *op. cit.*, pp. 146–147.

[24] MS, fol. 90b, $10x-8-x^2=y^2$ or, $y^2+x^2-10x+25=17=4^2+1^2$ which admits of an infinite number of solutions.

[25] G. R. Kaye, *The Bakhshālī Manuscript* (Calcutta, 1927) p. 149, pp. 42–43.

[26] F. Woepcke, *op. cit.*, p. 86.

$$\begin{cases} x + 10 = y^2 \\ x + 15 = z^2 \end{cases}$$

Nārāyana (1357) gives the solution for the same type of problem:

$$\begin{cases} x + a = y^2 \\ x + b = z^2 \end{cases} \quad \text{where } a > b:$$

Bhaskara II treated the general case:[27]

$$\begin{cases} ax + c = y^2 \\ bx + d = z^2 \end{cases}$$

In the problem

$$\begin{cases} 3x + 1 = y^2 \\ 5x + 1 = z^2 \end{cases}$$

let $y = 3u + 1$; then, in the first equation,

$$x = 3u^2 + 2u\,.$$

Substitute in the second equation to get $15u^2 + 10u + 1 = z^2$ which may be solved.

Double Equations of the Second Degree

Abū Kāmil gives as a type of double equation:

$$\begin{cases} x^2 + 3x + 1 = y^2 \\ x^2 - 3x + 2 = z^2 \end{cases}$$

It was well known to the Indians who solved many much more difficult problems of the second degree. Bhaskara II solved such types as:

$$\begin{cases} ax^2 + by^2 + e = u^2 \\ cx^2 + dy + f = v^2 \end{cases}$$

and

$$\begin{cases} a^2x^2 + bxy + c^2y^2 = u^2 \\ dx^2 + exy + fy^2 + g = v^2 \end{cases}$$

He also discussed problems of double equations of higher degrees. In abū Kāmil's work, there are no equations higher than the second degree.

At the end of the abū Kāmil text are mainly linear multiple equations of a simple type. These resemble equations found in inheritance problems of other Arabic works. On fol. 104a, there is a relatively unimportant indeterminate problem which would appear as:

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[27] Datta, *op. cit.*, 11, pp. 261ff.

$$\begin{cases} 20x + 15y + 10z = 18 \\ x + y + z = 1 \end{cases}$$

These reduce to

$$10x + 5y = 8\,.$$

It is necessary at this point to select two positive fractions for x and y such that $x + y < 1$ because of the second equation above.

A somewhat similar problem is on fol. 104b where "A *raṭl* [coin] is valued at five dirhams, another *raṭl* is four dirhams, and ten *raṭls* is one dirham. From all these, obtain a *raṭl* valued at two dirhams. This question has more than a single answer."

At the time of abū Kāmil, Arabic work in indeterminate equations did not come up to the standard set by Hindu mathematicians. It is certain, however, from abū Kāmil's work that the Arabs knew much of the Indian algebra in this area as well as some solutions by Diophantos. Al-Karajī (ca. 1010) improved upon abū Kāmil's treatise not only in those problems discussed by the latter but also in treating algebraic equations of higher than the second degree.[28] Abū Kāmil may have laid the groundwork but it was al-Karajī who repeated many of the formers' examples and elaborated much upon them.[29] Leonardo Fibonacci, in bringing this work to Europe later, repeated many of the same examples. Thus, there is a discernible outline of the growth in knowledge regarding indeterminate equations up to the thirteenth century A.D. The story is still a discontinuous one, however, with much more remaining to be unfolded particularly in the manner in which indeterminate problems affected the development of number theory.

[28] *Woepcke, op. cit.*, pp. 129ff.

[29] Levey, *op. cit.*, p. 6 for the debt of al-Karajī to abū Kāmil. Cf. also A. P. Juschkewitsch, *Geschichte der Mathematik im Mittelalter* (Basel, 1964) pp. 220ff. For indeterminate equations in Chinese mathematics, see J. Needham, *Science and Civilization in China* (Cambridge, 1959) III, pp. 119ff; D. E. Smith and Y. Mikami. *A History of Japanese Mathematics* (Chicago, 1914) pp. 192, 196, 233, 246.