ABŪ KĀMIL'S

"ON THE PENTAGON AND DECAGON"

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Abū Kāmil's
"On the Pentagon and Decagon"

by

Mohammad Yadegari and Martin Levey [1]

Abū Kāmil Shujā^c ibn Aslam ibn Muḥammad ibn Shujā^c (ca. 850-930 A.D.) was known as al-Ḥasib al-Misrī, "the reckoner from Egypt." He is, after al-Khwārizmī (ca. 825), the earliest algebraist of the Islamic Middle Ages whose writings are extant[2]. His work is important in the history of mathematics for a number of reasons.

He was among the early Muslim algebraists whose work in algebra was extensively used by Europeans. It has been established that Leonardo Fibonacci (of Pisa) had access to the treatises of abū Kāmil. Leonardo was aware of "On the Pentagon and Decagon" of abū Kāmil and used it in his Practica Geometriae. There is proof that Leonardo used dozens of abū Kamil's problems in his algebra[3]. From "On the Pentagon and Decagon", Leonardo used seventeen of its twenty problems carrying over the exact number facts[4].

In previous works, Levey has shown that abū Kāmil was much interested in developing a mathematical methodology which combined the more abstract Greek methods with more pragmatic procedures of the Babylonian and Egyptian algebraists. Evidence for this has been established from his Algebra and his Indeterminate Equations[5]. Further proof is in "On the Pentagon and Decagon" to be discussed.

Up to now there have been several translations of "On the Pentagon and Decagon." Because the Arabic text was lost until Levey discovered it in Istanbul about ten years ago, only the Latin and Hebrew texts were known. The first to be translated was the Hebrew into Italian[6]; then the Latin was carried over into German[7]. The Latin and Hebrew were originally translated from the Arabic of abū Kāmil, contrary to other published statements.

The Arabic text of "On the Pentagon and Decagon" with which we are concerned is to be found in Istanbul, in the Kara Mustafa Library, number 379. It has twenty-one lines to the page and is written in a Naskhi hand; it goes from the title page on fol. 67b to fol. 78b. In this original text, there are no vowels, no commas, or other punctuation. Further, the text is completely rhetorical. This, by the way, is not true for the Hebrew and Latin translations; the former shows only a few notational abbreviations while the Latin shows fractional symbols.

In the text, abū Kāmil does not use words for "plus" and "minus" often. Rather, he uses "and" for "plus" and "except" for "minus." It is interesting that abū Kāmil puts a line segment symbol above each geometric line designation as well as a point. For instance, when he refers to point M, he writes point $\overline{M}$. All of abū Kāmil's geometric figures are constructed precisely. In naming an unknown, abū Kāmil calls it "thing." In the text, the square of the thing is called <u>māl</u>. In the same way <u>māl māl</u> is X^4, <u>māl māl māl</u> is X^6, and <u>māl māl māl māl</u> is X^8.

The book on the pentagon and decagon consists of twenty geometric problems. The treatment, however, of these problems is almost entirely algebraic. This algebraic treatment of geometrical problems may be contrasted with Euclid's geometric treatment of algebraic problems in the <u>Elements</u>. Abū Kāmil's method is closer to that of the Babylonian procedures.

The first ten problems deal with finding the sides of inscribed and circumscribed pentagons, decagons, and fifteen-sided regular polygons when given the diameter of the circle, and vice versa. These problems, arranged progressively, lead a reader to believe that abū Kāmil is truly moving toward some method of approximation, here finding the circumference of a circle by the method of exhaustion. This idea does not materialize in the work. Instead, abū Kāmil devotes himself to the determination of geometric details of his figures. The approximative attempt was not taken up again after abū Kāmil for a long time. The time was not yet ripe to work out anything more general for the ideas of exhaustion approximation.

Abū Kāmil used the generalized formula to obtain the roots of a quadratic equation. In his rhetorical method, in the first problem, when he wishes to find the side of an inscribed regular pentagon, he has (in a modernized form):

"... it becomes X^4 plus 3125 equaling 125 X^2. Take $\frac{1}{2}$ of the X^2's; that is 62 $\frac{1}{2}$. Multiply it by itself; it is 3906 $\frac{1}{4}$. Subtract 3125 from it; the difference is 781 $\frac{1}{4}$; Subtract its square root from 62 $\frac{1}{2}$; take the square root of the difference....[8]"

Thus,

$$X^4 + 3125 = 125\, X^2$$

$$X^2 = \frac{125}{2} - \sqrt{\frac{125 \times 125}{4} - \frac{4 \times 3125}{4}}$$

$$X = \left(62\,\frac{1}{2} - \sqrt{781\,\frac{1}{4}}\right)^{\frac{1}{2}}$$

Abū Kāmil, however, did not work with the negative results.

Further, abū Kāmil knew factoring after his fashion. In the same problem, he divided

$$5\, X^2 + \frac{1}{625} X^6 = \frac{1}{5} X^4 \text{ by } X^2$$

$$\text{to get } 5 + \frac{1}{625} X^4 = \frac{1}{5} X^2$$

To similify matters, abū Kāmil chooses constants for his quantities. He often selects the value 10 instead of an unknown constant. It should be remembered that the notation for a constant was not known at that time. Because he chooses definite instead of generalized constants his work looks like a collection of examples in which no general rules or formulas are derived or even noticed. This is illusory for a careful detailed examination and when one writes his rhetorical expressions in modern notation, there is no doubt whatsoever that abū Kāmil was aware of generalized conditions in his problems and of generalized results. Conditions in algebraic methodology were not yet satisfacotry for a completely abstract approach.

In problem V, in which he wishes to find the diameter of the circumscribed circle about a given regular pentagon, abū Kāmil let the side of the pentagon be 10, a definate value. The diameter of the circle turned out to be $\sqrt{200 + \sqrt{8000}}$[9]. When this problem is worked with the side as "a" instead of 10, then the result comes out to

$$X = \sqrt{2a^2 + \frac{2}{5}\sqrt{5}\ a^2}$$

which can easily be converted to the earlier purely numerical result.

In further clarification, is problem XVII,

"Then, it is clear from what we have described that if you want to know the diameter of a circle that is circumscribed about a regular pentagon it is obvious that you multiply one of its sides by itself, then you double it and keep it. Then,multiply again one of its sides by itself, then the result by itself. Then take $\frac{4}{5}$ of it whose square root you find, then add the result to what you kept, then take the square root of the sum; what is left is the diameter of the circle[10]."

Thus, with a as the side of the regular pentagon and X the diameter of the circumscribed circle, we have $X = \left(2a^2 + \sqrt{\frac{4}{5} a^4}\right)^{1/2}$, the same answer for problem V, or $X = \sqrt{200 + \sqrt{8000}}$ which is easily converted mentally to a more generalized form by a practised mathematician.

In problem IX, in which abū Kāmil wishes to find the diameter of a circle circumscribed about a given regular decagon[11], he choses 10 as the length of the side. If one takes "a" as the side, then we obtain

$X = a + \frac{1}{2} a\sqrt{20}$ as the generalized formula.

The numerical answer of abū Kāmil is $X = 10 + \sqrt{500}$ which is easily converted to the intermediately generalized one, as $X = 10 + \sqrt{5 \times 10^2}$.

In problem XVIII, abū Kāmil refers to this general formula by stating:

"It is clear from what we have described that if you wish to find the diameter of a circle circumscribed about a given regular decagon you should multiply one of its sides by itself, then by 5, then take the root of what is left and add it to one of the sides of the decagon. What is left is the diameter of the circle[12]."

Thus, the diameter is

$$\sqrt{5a^2} + a = a + \frac{1}{2} a\sqrt{20}.$$

It is obvious that the mathematical methodology of abū Kāmil was advanced as seen between the lines. He had an excellent understanding of generalized algebraic relationships in spite of the fact that he had no notation with which to even partially visulaize them as we do today when the formula may come to mind.

Essentially, abū Kāmil's works apply algebra to geometry. In "On the Pentagon and Decagon," abū Kāmil displays a wide range of mathematical knowledge as theorems of Pythagoras and Ptolemy, theorems from the thirteenth book of Euclid and others, equations of the fourth degree solved as second degree equations, and a generally wide knowledge of algebraic processes.

In essence, abū Kāmil took a short mathematical step beyond his predecessor al-Khwārizmī but it was a difficult one toward more effeat abstractive processes.

Finally, it is of interest to set down in modern notation the formulas which abū Kāmil used in an entirely rhetorical fashion. These are for the first eleven problems:

S = side of a regular polygon circumscribed;

s = side of a regular polygon inscribed;

d = diameter ; r= radius

Subscript indicates the type of regular polygon.

I $$s_5 = \sqrt{\frac{5}{8} d^2 - \sqrt{\frac{5}{64} d^4}}$$

II $$s_{10} = \sqrt{\frac{5}{16} d^2} - \frac{d}{4}$$

$$\text{III} \quad S_5 = \sqrt{5d^2 - \sqrt{20d^4}}$$

$$\text{IV} \quad S_{10} = \sqrt{d^2 - \sqrt{\frac{4}{5}d^4}}$$

$$\text{V, VII} \quad d = \sqrt{2s_5^2 + \sqrt{\frac{4}{5}s_5^4}}$$

$$\text{VI, VIII} \quad d = \sqrt{S_5^2 + \sqrt{\frac{4}{5}S_5^4}}$$

$$\text{IX} \quad d = s_{10} + \sqrt{5s_{10}^2}$$

$$\text{X} \quad d = \sqrt{5S_{10}^2 + \sqrt{20\, S_{10}^4}}$$

$$\text{XI} \quad s_{15} = \sqrt{\frac{5}{32}d^2 + \sqrt{\frac{5}{1024}d^4}} + \sqrt{\frac{3}{64}d^2} - \sqrt{\frac{15}{64}d^2}$$

For problem XV, let a be the side of a quadrilateral then[13]:

$$2a + \sqrt{2a^2} - \sqrt{2a^2 + \sqrt{32a^4}} = a\left(2 + \sqrt{2} - \sqrt{2 + 4\sqrt{2}}\right).$$

The Arabic text, although without any notation or numeral figures is fuller in its explanatory passages than the Latin. In thought, it is closer to the Hebrew. The latter is a late translation with many numerals.

Following is a translation of the entire text of "On the Pentagon and Decagon." In order to make it more clear to the reader some basic notation has been employed as X for "thing," X^2 for <u>māl</u>, X^6 for <u>māl māl māl</u>, numerals as are used in place of rhetorical numbers, and punctuation has been added. The translation has been retained in a literal form to yield the full flavor and difficulty of rhetorical mathematical expression of the Arabic early medieval period. The figures have been reproduced exactly as in the text except that the Arabic letters have been replaced by the Latin alphabet.

TRANSLATION OF

[Fī al-mukhammas wa'l-mu^cashshar]

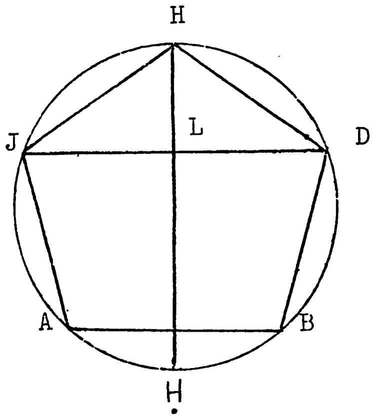
"On the Pentagon and Decagon"

[67b] In the name of Allah, the merciful, the compassionate, abū Kāmil Shujā^c ibn Islam said:

We describe in this book that which algebra[14] encompasses and what Euclid has discussed in his book. We will determine the length of every side of a regular pentagon and decagon whether inscribed or circumscribed. We shall also determine the diameter of the inscribed or circumscribed circle provided that the regular pentagon or regular decagon is known. In addition, we shall determine the length of the side of a regular pentagon or regular decagon if the area is known. Also, we shall find the length of the sides of triangles inscribed in a pentagon or decagon provided the areas of the triangles are known. Other subjects are also touched upon in this book most of the discovery of which was difficult for the mathmaticians of our time. Thus, I hope Allah will ease the path for me to achieve what was not possible for others. Thanks to Allah who has no partner.

[I] We begin by discovering the length of the chord that is $\frac{1}{5}$ of the known circle [circumference] using its diameter. We take the circle $\overline{\text{ABDH}\d{H}}$ whose diameter $\overline{\text{H}\d{H}}$ is 10. $\overline{\text{ABDH}\mathring{\text{H}}}$ is a regular pentagon inscribed in the circle. If we want to know the length of the side of the regular pentagon, [68a] draw $\overline{JLD}$. It cuts off $\frac{2}{5}$ of the circle. We set $\overline{HD}$ equal to X. It is obvious that line $\overline{HL}$ is $\frac{1}{10}X^2$ because $\overline{HD}$ multiplied by itself equals $\overline{H\underset{\cdot}{H}}$ by $\overline{HL}$. Line $\overline{DL}$ equals the square root of X^2 minus $\frac{1}{100}X^4$ and line $\overline{JL}$ equals $\overline{LD}$. Therefore, line $\overline{JD}$ equals the square root of $4X^2$ minus $\frac{2}{5}$ of $\frac{1}{10}X^4$, and it is obvious that the product of $\overline{AB}$ by $\overline{JD}$ plus $\overline{AB}$ by $\overline{AB}$ equals $\overline{JD}$ by $\overline{JD}$. This is because the product of $\overline{AB}$ by $\overline{JD}$ plus $\overline{AJ}$ by $\overline{BD}$ equals the product of $\overline{AD}$ by $\overline{BJ}$ and $\overline{AJ}$ by $\overline{BD}$ equals $\overline{AB}$ by $\overline{AB}$ $\overline{AD}$ by $\overline{BJ}$ equals $\overline{JD}$ by $\overline{JD}$. $\overline{JD}$ by $\overline{JD}$ equals $4X^2$ minus the square root of $\frac{2}{5}$ of $\frac{1}{10}X^4$. Subtract from it the product of $\overline{AB}$ by itself which is X^2; the difference is $3X^2$ minus $\frac{2}{5}$ of $\frac{1}{10}X^4$ equaling the product of $\overline{AB}$ by $\overline{JD}$. Then divide $3X^2$ minus $\frac{2}{5}$ of $\frac{1}{10}X^4$ by line $\overline{AB}$, which is X; the result is line JD, $3X$ minus $\frac{2}{5}$ of $\frac{1}{10}X^3$. We have shown that line JD is the square root of $4X^2$ minus $\frac{2}{5}$ of $\frac{1}{10}X^4$; then we multiply $3X$ minus $\frac{2}{5}$ of $\frac{1}{10}X^3$ by itself and it becomes $9X^2$ plus $\frac{1}{625}X^6$ minus $\frac{6}{25}X^4$ equaling $4X^2$ minus $\frac{2}{5}$ of $\frac{1}{10}X^4$. We rearrrange it and it becomes $\frac{1}{5}X^4$ equaling $5X^2$ plus $\frac{1}{625}X^6$. Then, divide everything by X^2 and it becomes 5 plus $\frac{1}{625}X^4$ equaling $5X^2$. Then, complete your X^4 until it becomes a [whole] X^4 and that

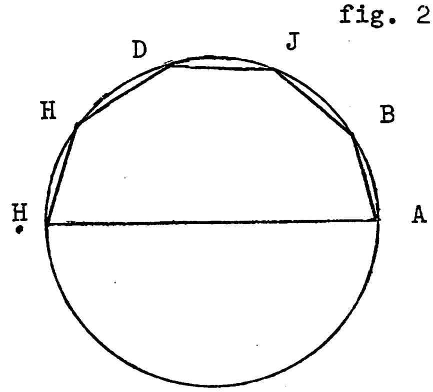
fig. 1



is by multiplying it by 625. Therefore, multiply everything by 625; it becomes X^4 plus 3125 equaling $125x^2$. Take $\frac{1}{2}$ of the X^2's; it is $62\frac{1}{2}$. Multiply it by itself, it is $3906\frac{1}{4}$. [68b] Subtract 3125 from it; the difference is $781\frac{1}{4}$. Subtract its square root from $62\frac{1}{2}$; take the square root of the difference. That is line $\overline{\text{HD}}$ which is one of the sides of the regular pentagon and that is what we wanted to demonstrate[15].

[II] If we wish to know the length of a chord that is the side of a circumscribed regular decagon, we draw the circle and let its diameter $\overline{\text{AḤ}}$ be 10. We draw five equal chords in the semicircle; these are $\overline{\text{AB}}$, $\overline{\text{BJ}}$, $\overline{\text{JD}}$, $\overline{\text{DH}}$, $\overline{\text{HḤ}}$. It is clear that $\overline{\text{DḤ}} = \frac{1}{5}$ of the circle of which we have shown its product by itself to be $62\frac{1}{2} - 781\frac{1}{4}$. $\overline{\text{AJ}}$ equals $\overline{\text{DḤ}}$ and the product of $\overline{\text{AJ}}$ by $\overline{\text{DḤ}}$ plus $\overline{\text{JD}}$ by $\overline{\text{AḤ}}$ equals $\overline{\text{AD}}$ by $\overline{\text{JḤ}}$. $\overline{\text{AD}}$ equals $\overline{\text{JḤ}}$ and $\overline{\text{AJ}}$ equals $\overline{\text{DḤ}}$; therefore, the product of $\overline{\text{DḤ}}$ by itself plus $\overline{\text{JD}}$ by $\overline{\text{AḤ}}$ equals $\overline{\text{JḤ}}$ by itself. Line $\overline{\text{JD}}$ is the chord which is $\frac{1}{10}$ of the circle; we let it be X. Multiply it by $\overline{\text{AḤ}}$, the diameter, which is 10; it becomes 10X. Multiply $\overline{\text{DḤ}}$ by itself it equals $62\frac{1}{2}$ minus the square root of $781\frac{1}{4}$. Therefore, the product of $\overline{\text{JḤ}}$ by itself becomes $62\frac{1}{2}$ plus 10X minus the square root of $781\frac{1}{4}$ Add to it the product of $\overline{\text{AJ}}$ by itself

fig. 2



which is $62\frac{1}{2}$ minus the square root of $781\frac{1}{4}$; it becomes 125 plus 10X minus the square root of 3125 equaling $\overline{A\underset{\cdot}{H}}$ [69a] by $\overline{A\underset{\cdot}{H}}$ which is 100. Solving it yields the square root of 3125 minus 25 equaling 10X. Therefore, X equals the square root of $31\frac{1}{4}$ minus $2\frac{1}{2}$; that is line $\overline{JD}$ which is $\frac{1}{10}$ of the circle. That is what we wished to demonstrate[16].

[III] If we wish to find the length of a chord which is the side of an inscribed regular pentagon, then we draw a circle and construct a regular pentagon $\overline{ABJDR}$ on it. It is clear that the chord $\overline{\underset{\cdot}{T}\underset{\cdot}{H}}$ equals $\frac{1}{5}$ of the circle, and we have made it clear that its square equals $62\frac{1}{2}$ minus the square root of $781\frac{1}{4}$. The square of line $\overline{\underset{\cdot}{T}L}$ is $15\frac{5}{8}$ minus the square root of $48\frac{1}{2}$ plus $\frac{1}{4}$ plus $\frac{5}{8}$ of $\frac{1}{8}$[17]. Subtract it from the square of line $\overline{\underset{\cdot}{T}H}$, the radius; it is 25. The difference is the square of line $\overline{LH}$; it is $9\frac{1}{4}$ plus $\frac{1}{8}$ plus the square root of $48\frac{1}{2}$ and $\frac{1}{4}$ and $\frac{5}{8}$ of $\frac{1}{8}$. Then let line $\overline{AB}$ equal X. Its square is X^2 and its ratio with respect to $\overline{KH}^2$ is the same as the ratio of $\overline{\underset{\cdot}{T}\underset{\cdot}{H}}^2$ to $\overline{HL}^2$[18].

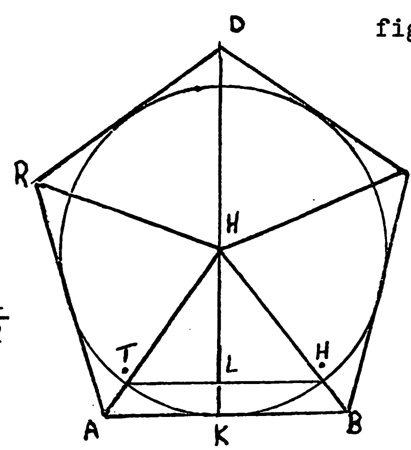


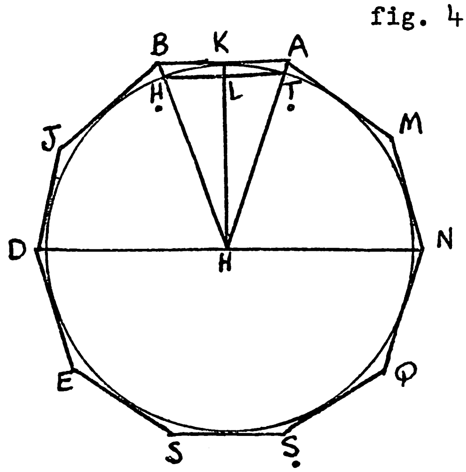
fig. 3

Therefore, multiply X^2 by $9\frac{3}{8}$ plus the square root of $48\frac{1}{2}$ and $\frac{1}{4}$ and $\frac{5}{8}$ of $\frac{1}{8}$; becomes $9\frac{3}{8}$ plus the square root of $48\frac{3}{4}X^4$ and $\frac{5}{8}$ of $\frac{1}{8}X^4$. This equals $\overline{KH}^2$ by $\overline{\underset{\cdot}{T}\underset{\cdot}{H}}^2$ which is $1562\frac{1}{2}$ minus the square root of $488,281\frac{1}{4}$. Then add everything to X^2, <u>i.e.</u>

multiply everything by $\frac{1}{2}$ and $\frac{1}{25}$ minus the square root of $\frac{20}{625}$. Therefore multiply $9\frac{3}{8}X^2$ minus the square root of $48\frac{3}{4}X^4$ and $\frac{5}{8}$ of $\frac{1}{8}X^4$ by $\frac{1}{5}$ and $\frac{1}{25}$ minus the square root of $\frac{20}{625}$, and it becomes X^2. Now, multiply $1,562\frac{1}{2}$ minus the square root of $488,281\frac{1}{4}$ by $\frac{1}{5}$ and $\frac{1}{25}$ minus the square root of $\frac{20}{625}$; it is 375 plus the square root of 15,625 minus the square root of 28,125 minus the square root of 78,125 to give 500 minus the square root of 200,000. Then it becomes X^2 equal to 500 times the square root of 200,000. Its root is line $\overline{AB}$ which is one of the sides of the regular pentagon. That is what we wished to demonstrate[19].

[IV] [70a] If you are told to find the length of a side of a circumscribed regular decagon, given the circle, we construct a circle and let its diameter be 10. We construct on it a regular decagon; call it $\overline{\text{ABJDESṢQNM}}$. We need to know $\overline{AB}$. It is obvious that $\overline{\text{ṬḤ}}$ is the square root of $31\frac{1}{4}$ minus $2\frac{1}{2}$. Line $\overline{\text{ṬL}}$ is $\frac{1}{2}$ of it. It is the square root of $7\frac{1}{2}$ and $\frac{1}{4}$ and $\frac{1}{2}$ of $\frac{1}{8}$ minus $1\frac{1}{4}$. Therefore, its square is $9\frac{1}{4}$ and $\frac{1}{8}$ minus the square root of $48\frac{1}{2}$ and $\frac{1}{4}$ and $\frac{5}{8}$ of $\frac{1}{8}$. Then we subtract it from the square of $\overline{\text{ṬḤ}}$ which is 25; the difference is the square of $\overline{HL}$ which is $15\frac{5}{8}$ plus the square root of $48\frac{1}{2}$ and $\frac{1}{4}$ and $\frac{5}{8}$ of $\frac{1}{8}$. $\overline{\text{ṬḤ}}^2$ is $37\frac{1}{2}$ minus the

fig. 4



square root of $781\frac{1}{4}$. Then we let line $\overline{AB}$ be X. Its square is X^2. Therefore, X^2 with respect to $\overline{TH}^2$ which is $37\frac{1}{2}$ minus the square root of $781\frac{1}{4}$, is the same as $\overline{KH}^2$, which is 25, to $\overline{HL}^2$ which is $15\frac{5}{8}$ plus the square root of $48\frac{1}{2}$ and $\frac{1}{4}$ and $\frac{5}{8}$ of $\frac{1}{8}$. Then, multiply it [i.e. the last term] by $\overline{AB}^2$ which is X^2; it becomes $15\frac{5}{8}\ X^2$ plus the square root of $48\frac{3}{4}\ X^4$ and $\frac{5}{8}$ by $\frac{1}{8}\ X^4$ equaling $\overline{KH}^2$ by $\overline{TH}^2$ which is $937\frac{1}{2}$ minus the square root of $488{,}281\frac{1}{4}$. Then add everything to X^2; that is, multiply everything by $\frac{2}{5}$ of $\frac{1}{5}$ minus the square root of $\frac{4}{5}$ of $\frac{1}{625}$. It becomes X^2 equal to [70b] 75 plus the square root of 625 minus the square root of 3125 minus the square root of 1125. That is 100 minus the square root of 8000, and that is $\overline{AB}^2$ which is the side of the regular decagon. That is what we wished to demonstrate[20].

[V] We want to know the length of the diameter of a circle which circumscribes a given regular pentagon. Let the regular pentagon be $\overline{ABJDH}$ and let each of the sides be 10. Let the diameter be $\overline{KD}$. Join $\overline{HJ}$ and call it X. It is clear that $\overline{HJ}$ by $\overline{AB}$ plus $\overline{AB}$ by itself equals $\overline{HJ}$ by itself. Therefore, the product of $\overline{HJ}$ which is X by $\overline{AB}$ becomes 10X. And $\overline{AB}$ by itself is 100; and $\overline{HJ}$ by itself is X^2. Then it becomes equal to

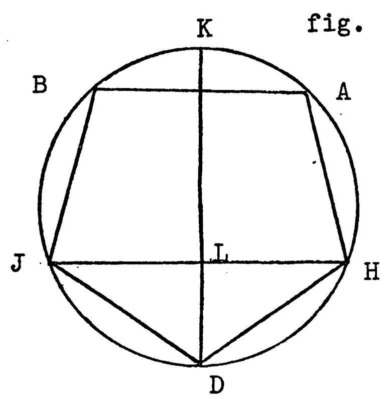
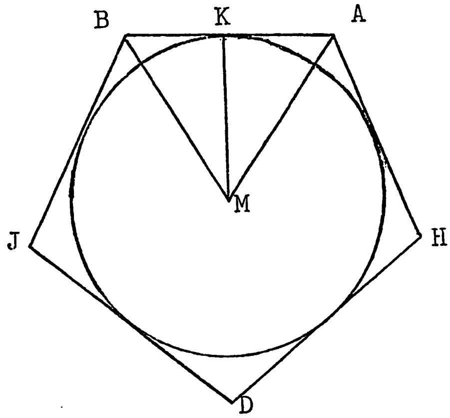


fig. 5

$10X$ plus 100. Therefore, X equals 5 plus the square root of 125; that is $\overline{HJ}$ whose half is $2\frac{1}{2}$ plus the square root of $31\frac{1}{4}$. That is line $\overline{HL}$. Then, multiply it by itself; it becomes $37\frac{1}{2}$ plus the square root of $781\frac{1}{4}$. Subtract it from [the product of] $\overline{HD}$ by $\overline{HD}$ which is 100. The difference is $62\frac{1}{2}$ minus the square root of $781\frac{1}{4}$. That is the square of $\overline{DL}$. Now let the diameter of the circle $\overline{KLD}$ be X. Its square is X^2. Therefore, X^2 with respect to $\overline{DH}^2$, which is 100, is the same as $\overline{DH}^2$, which is 100, to $\overline{DL}^2$ which is $62\frac{1}{2}$ minus the square root of $781\frac{1}{4}$. Multiply it [71a] by X^2; it becomes $62\frac{1}{2}X^2$ minus the square root of $781\frac{1}{4}X^4$ to equal $\overline{DH}^2$ which is 10,000. Then, add everything to X^2. That is, multiply everything by $\frac{1}{5}$ of $\frac{1}{10}$ plus the square root of $1\frac{1}{4}$ parts of $\frac{1}{15,625}$; then, it becomes X^2 to equal 200 plus the square root of 8,000. It has become clear now that the square of the diameter $\overline{KLD}$ is 200 plus the square root of 8,000. That is what we wished to find[21].

[VI] If we wish to know the length of the diameter of a circle that is contained in a circumscribed regular pentagon, we let the pentagon be $\overline{ABJDH}$ and let each of its sides equal 10. Let M be the center and the radius be $\overline{KM}$. Let $\overline{AM}$ be the radius of a circle that contains the regular pentagon. We have shown the square of the diameter of the latter circle to be 200 plus the square root of 8,000. Therefore, the square of $\overline{AM}$, which is the radius, is 50 plus the square root of 500. Subtract from it the square of $\overline{AK}$ which is 25; the difference is $\overline{KM}^2$ which is half of the inscribed circle. It is 25 plus the square root of 500. Therefore, the square of the diameter is 100 plus the square root of 8,000. That is what we wished to find[22].

fig. 6



If you wish you may multiply one of the sides of this pentagon by itself; subtract it from the square of the diameter of the circumscribed circle, and then take the square root of the difference. That is the diameter of the inscribed circle. This applies to every regular polygon. Understand that[23].

[VII] If we wish to know the diameter of a circumscribed circle of a regular pentagon [71b] whose side is 10 in a manner different from what we have described previously. We describe, what Euclid discussed, that the ratio of the chords is the same as the ratio of the diameter[24]. We have shown that a regular pentagon inscribed in a circle whose diameter is 10 has a side whose square is $62\frac{1}{2}$ minus the square root of $781\frac{1}{4}$. Therefore, 100 to $62\frac{1}{2}$ minus the square root of

$781\frac{1}{4}$ is the same as X^2, which is the square of the unknown diameter, to 100. Then, we multiply 100 by 100, it is 10,000. Then, we multiply X^2 by $62\frac{1}{2}$ minus the square root of $781\frac{1}{4}$; and it is $62\frac{1}{2}X^2$ minus the square root of $781\frac{1}{4}$ X^4 which equals 10,000. Add everything to X^2. That is, multiply everything by $\frac{1}{5}$ of $\frac{1}{10}$ plus the square root of $1\frac{1}{4}$ of $\frac{1}{15,625}$; it becomes X^2 to equal 200 plus the square root of 8,000. That is the square of the [diameter of the] circumscribed circle[25].

[VIII] In the same way, if we want to find the diameter of a circle inscribed in a regular pentagon whose side is 10, we have shown that a regular pentagon that is circumscribed to a circle with a diameter of 10, has [72a] a side whose square of its length is 500 minus the square root of 200,000. Then, we say that 100, which is the square of the side of the pentagon, with respect to 500 minus the square root of 200,000 is the same as X^2, which is the square of the unknown diameter; to 100 which is the square of the known diameter. Multiply 100 by 100 to equal 10,000. Then, multiply X^2 by 500 minus the square root of 200,000; it equals $500X^2$ minus the square root of $200,000X^4$. This equals 10,000. Then add everything you have to X^2. That is, multiply it by $\frac{1}{10}$ of $\frac{1}{10}$ plus the square root of $\frac{4}{5}$ of $\frac{1}{10,000}$; it becomes X^2 to equal 100 plus the square root of 8,000

That is the square of the diameter of the inscribed circle[26].

[IX] Also, if we want to construct an inscribed regular decagon whose side is 10, we have shown that the side of an inscribed regular decagon in a circle with a diameter of 10 is the square root of $31\frac{1}{4}$ minus $2\frac{1}{2}$. Then, let the diameter of the unknown circle be X. Then, we say that the ratio of X to 10 is the same as 10 to the square root of $31\frac{1}{4}$ minus $2\frac{1}{2}$. Multiply 10 by 10; that is 100. Then, multiply X by the square root of $31\frac{1}{4}$ minus $2\frac{1}{2}$; it is the square root of $31\frac{1}{4}X^2$ minus $2\frac{1}{2}X$ which equals 100. We add $2\frac{1}{2}$ to 100; we get 100 plus $2\frac{1}{2}X$ to equal the square root of $31\frac{1}{4}X^2$. Then, multiply 100 plus $2\frac{1}{2}X$ by itself; it becomes 10,000 plus $6\frac{1}{4}X^2$ plus 500X to equal $31\frac{1}{4}X^2$. We solve it to get X as 10 plus the square root of 500; that is the diameter[27].

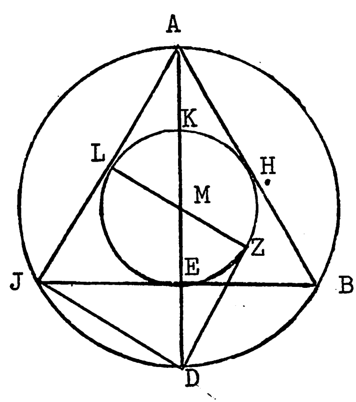
[X] Again, if we want to construct a circle in a regular decagon, we have shown that a circle whose diameter is 10 is inscribed in a regular decagon whose side is 100 [72b] minus the square root of 8,000. Then we may say that the ratio of 100 minus the square root of 8,000 to 100 is the same as the ratio of 100 to X^2. Then, multiply 100 by 100; it is 10,000. Then, multiply 100 minus the square root of 8,000 by X^2; it is $100X^2$ minus the square root of 8,000 X^4

equal to 10,000. Then, add everything to X^2, [i.e. multiply it by $\frac{1}{2}$ of $\frac{1}{10}$ plus the square root of $\frac{4}{5}$ of $\frac{1}{400}$] it becomes X^2 equal to 500 plus the square root of 200,000, which is the square of the diameter of the inscribed circle[28].

If you wish, multiply the diameter of the circle circumscribing this regular decagon by itself. We have shown it to be 100 plus the square root of 500. Therefore, it becomes 600 plus the square root of 200,000. Subtract from it 100, which is the product of the side by itself; the difference is 500 plus the square root of 200,000. Take the root of the difference; that is the diameter of the inscribed circle in the given decagon.

In every given regular polygon that is inscribed and also circumscribed, the product of one of its sides by itself added to the product of the diameter of the inscribed circle by itself equals the product of the diamter of the circumscribed circle by itself. For example, we construct an equilateral triangle $\overline{ABJ}$. We draw the inscribed circle, $\overline{\text{HZL}}$. On it we draw the circumscribed circle whose diameter is $\overline{AD}$. The diameter of the other circle is $\overline{KE}$. Then, I say that the product of $\overline{AJ}$ by itself plus $\overline{KE}$ by itself is the same as the product of $\overline{AD}$ by itself. The proof of that is that we construct on $\overline{AJ}$ from point $\overline{L}$, the point on which the circle

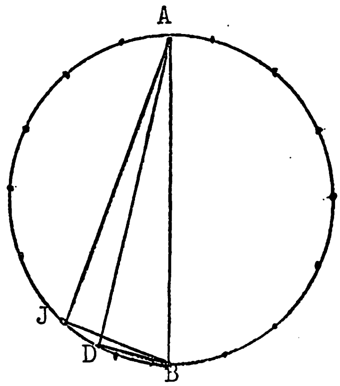
fig. 7



passes, a perpendicular line. It is obvious that it passes through the center of the circles. It meets the circle $\overline{\underset{\cdot}{H}ZL}$. That line is $\overline{LZ}$. Therefore, line $\overline{LZ}$ is the diameter of the circle $\overline{\underset{\cdot}{H}ZL}$. It is equal to $\overline{KE}$. We join point $\overline{Z}$ to point $\overline{D}$ by line $\overline{ZD}$. And point $\overline{D}$ [73a] to point $\overline{J}$ by line $\overline{JD}$. Therefore, line $\overline{AM}$ is the same as line $\overline{MD}$, and line $\overline{ML}$ is the same as $\overline{MZ}$. Therefore, both lines $\overline{AM}$, $\overline{ML}$ are equal to both lines $\overline{DM}$, $\overline{MZ}$ and the angle $\overline{AML}$ is equal to the angle $\overline{DMZ}$. Therefore, the base $\overline{DZ}$ is the same as the base $\overline{AL}$ and $\overline{AL}$ is the same as $\overline{LJ}$ and $\overline{LJ}$ is the same as $\overline{DZ}$, and triangle $\overline{AML}$ is the same as triangle $\overline{DMZ}$ and their angles are the same. Therefore, angle $\overline{ALM}$ is the same as angle $\overline{AMD}$; then line $\overline{ZD}$ is parallel to line $\overline{LJ}$. And we have shown it to be equal to it; and the lines joining them , $\overline{ZL}$, $\overline{DJ}$, are parallel and equal. The product of $\overline{AJ}$ by itself plus $\overline{JD}$ by itself is the same as $\overline{AD}$ by itself because angle $\overline{AJD}$ is a right angle, and $\overline{DJ}$ is equal to $\overline{LZ}$, and $\overline{LZ}$ equals $\overline{KE}$. Therefore, $\overline{AJ}$ by itself plus $\overline{KE}$ by itself equals $\overline{AD}$ by itself and that is what we wished to demonstrate[29].

[XI] If there is a regular polygon having 15 angles inscribed in a circle whose diameter is 10 and we want to find its side, then we let the circle be $\overline{AJB}$ and its diameter be $\overline{AB}$. We inscribe in it a decagon and there would be $1\frac{1}{2}$ chords which are $\overline{BD}$. We have shown that it

fig. 8



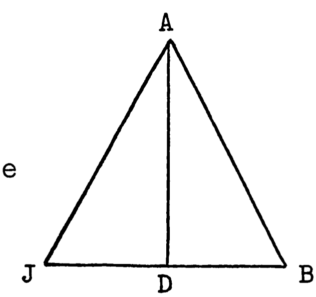
is the square root of $31\frac{1}{4}$ minus $2\frac{1}{2}$, and the chord $\overline{AD}$, all of it, which is $\frac{2}{5}$ of the circle, is $62\frac{1}{2}$ plus the square root of $781\frac{1}{4}$, of which its root is taken. We draw the chord in it which is $\frac{1}{6}$; that is $\overline{BJ}$ and it is $2\frac{1}{2}$ chords. It is equal to 5. Line $\overline{AJ}$, all of it, which is $\frac{1}{3}$, is the square root of 75. It becomes the square root of $2343\frac{1}{2}$ and $\frac{1}{4}$ minus the square root of $468\frac{1}{2}$ and $\frac{1}{4}$. Then, subtract it from the product of $\overline{AD}$ by $\overline{BJ}$ which is [73b] $1562\frac{1}{2}$ plus the square root of $488,281\frac{1}{4}$ whose root is taken; the difference is $1562\frac{1}{2}$ plus the square root of $488,281\frac{1}{4}$ with its root taken plus the square root of $68\frac{1}{2}$ and $\frac{1}{4}$ minus the square root of $2343\frac{1}{2}$ and $\frac{1}{4}$. This is equal to the product of $\overline{AB}$ by $\overline{JD}$. Then, we divide by line $\overline{AB}$, which is 10; we get line $\overline{JD}$, $15\frac{5}{8}$ plus the square root of $48\frac{3}{4}$ and $\frac{5}{8}$ of $\frac{1}{8}$, all of it with is root taken plus the square root of $4\frac{1}{2}$ and $\frac{1}{8}$ and $\frac{1}{2}$ of $\frac{1}{8}$ minus the square root of $23\frac{3}{8}$ and $\frac{1}{2}$ of $\frac{1}{8}$. That is what we wished to demonstrate [30].

We approximate the chord $\overline{JD}$ in order that you understand how you take the square root of $48\frac{3}{4}$ and $\frac{5}{8}$ of $\frac{1}{8}$ by approximation. You find it to be 6, 56 minutes, and 15 seconds, and 46 thirds. Then add it to 15 and $\frac{5}{8}$;

it becomes 22, 36 minutes, 45 seconds, and 46 thirds. Take its square root. You find it, approximately, to be 4 and 45 minutes, 19 seconds, and 1 third. Add to it the square root of 4 and $\frac{5}{8}$ and $\frac{1}{2}$ of $\frac{1}{8}$, by approximation, which is 2, 9 minutes, 54 seconds, and 12 thirds. It becomes 6, 55 minutes, 13 seconds, and 13 thirds. Subtract from it the square root of 23 and $\frac{3}{8}$ and $\frac{1}{2}$ of $\frac{1}{8}$, by approximation, which is 4, 50 minutes, 28 seconds, and 25 thirds. The difference is 2, 4 minutes, 44 seconds, and 48 thirds. That is chord $\overline{JD}$ approximately and that is what we wanted to show.

[XII] [74a] If you are given an equilaterial triangle with [the sum of] its area and the altitude as 10, what is its altitude? We let the triangle be triangle $\overline{ABJ}$ and its altitude be $\overline{AD}$. Then, if we want to know what $\overline{AD}$, the alititude, is, we let it be X. Therefore, line $\overline{DJ}$ is the square root of $\frac{1}{3}X^2$. The side of the triangle $\overline{ABJ}$ is the square root of X^2 and $\frac{1}{3}$, the area of the triangle is the square root of $\frac{1}{3}X^4$, and its altitude is X. Therefore, X plus the square root of $\frac{1}{3}X^4$ equals 10. Then, complete the square root of $\frac{1}{3}$ X^4 until it becomes a square root of X^4 which is X^2; you complete this by multiplying it by the square root of 3. It becomes X^2 plus the square root of 3 X^2 to equal the square root of 300. Therefore, $\frac{1}{2}$ of the square root of $3X^2$ is the square root of $\frac{3}{4}$.

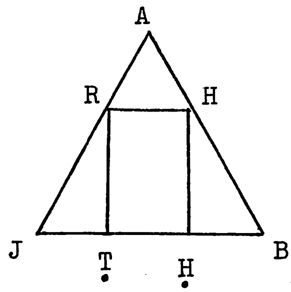
fig. 9



Then, multiply it by itself; it becomes $\frac{3}{4}$. Then add it to the square root of 300; it becomes $\frac{3}{4}$ plus the square root of 300. Take its square root and subtract from it the square root of $\frac{3}{4}$; the difference is line AD which is the altitude. That is what we wished to demonstrate[31].

[XIII] If you are given equilateral triangle whose side is 10, and in it is a rectangle[32] whose area is 10, what is the base of the rectangle? An example of that is that we let the triangle be ABJ and the quadrilateral in it be rectangle HRḤṬ. Then, if we want to know line HḤ which is the base of the rectangle we let it be X. Therefore, line BḤ is the square root of $\frac{1}{3}X^2$; also, line ṬJ is the square root of $\frac{1}{3}X^2$. There remains ṬḤ which is 10 minus the square root of X^2 and $\frac{1}{3}X^2$. Then, multiply it by line HḤ which is X; it becomes 10X [74b] minus the square root of X^4 and $\frac{1}{3}X^4$ to equal 10. Then perform the algebraic procedure for the X's by the square root of X^4 and $\frac{1}{3}X^4$ and add its equivalence to the dirhams[33] so that it becomes 10 plus the square root of X^4 and $\frac{1}{3}X^4$ to equal 10X. Then add everything to the square root of X^4; it is X^2. You do that by multiplying it by the square root of $\frac{3}{4}$. You multiply everything you have

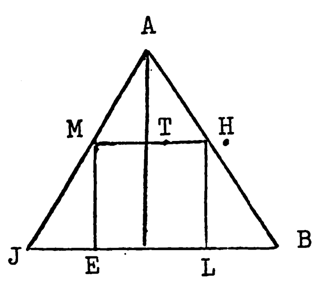
fig. 10



by the square root of $\frac{3}{4}$; it becomes X^2 plus the square root of 75 to equal the square root of $75X^2$. Then take $\frac{1}{2}$ the square root of $75X^2$; it is the square root of $18\frac{1}{2}$ and $\frac{1}{4}$. Multiply it by itself; it becomes $18\frac{1}{2}$ and $\frac{1}{4}$. Then subtract from it the square root of 75; the remainder is $18\frac{1}{2}$ and $\frac{1}{4}$ minus the square root of 75. Then take its root; what comes out plus the square root of $18\frac{1}{2}$ and $\frac{1}{4}$ is the base of the rectangle which is line $\overline{\text{HḤ}}$[34]. That is what we wished to demonstrate.

[XIV] If you are given an equilateral triangle $\overline{\text{ABJ}}$ and in it a square, say $\overline{\text{ḤLEM}}$, and the areas of the square plus the triangle is 10. What is the length of the side of the quadrilateral $\overline{\text{ELḤM}}$? Let each of the quadrilateral's sides be X. The area of the square is X^2 and the area of the triangle is $10 - X^2$. There remains the areas of the triangles $\overline{\text{BLḤ}}$, $\overline{\text{MEJ}}$, and $\overline{\text{AḤM}}$ as 10 minus $2X^2$. We let each side of the square be 10. Therefore, line $\overline{\text{ME}}$ is X and line $\overline{\text{EJ}}$ is the square root of $\frac{1}{3}X^2$. Therefore, the areas of triangles $\overline{\text{MEJ}}$ and $\overline{\text{ḤLB}}$ is the square root of $\frac{1}{3}X^4$. And line $\overline{\text{AṬ}}$ is the square root of $\frac{3}{4}X^2$. Therefore, the area of triangle $\overline{\text{AḤM}}$ is the square root of $\frac{1}{8}X^4$ plus $\frac{1}{2}$ of $\frac{1}{8}X^4$. Therefore, the area of triangle $\overline{\text{AḤM}}$ is the square root of $\frac{1}{8}X^4$ plus $\frac{1}{2}$ of $\frac{1}{8}X^4$. Therefore, the area of

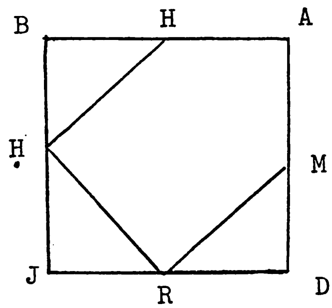
fig. 11



triangles $\overline{\text{MEJ}}$ and $\overline{\text{ḤBL}}$ and $\overline{\text{ḤMA}}$ is the square root of $\frac{1}{3}X^4$ plus the square root of $\frac{1}{8}X^4$ and $\frac{1}{2}$ of $\frac{1}{8}X^4$ to equal $2X^2$. Then, complete the square root of $\frac{1}{3}X^4$ until it becomes the square root of X^4, [75a] which is X^2. You complete this by multiplying it by the square root of 3. Therefore, multiply the square root of $\frac{1}{3}X^4$ by the square root of 3; it becomes the square root of X^4 which is X^2. Multiply the square root of $\frac{1}{8}X^4$ and $\frac{1}{2}$ of $\frac{1}{8}X^4$ by the square root of 3; it becomes the square root of $\frac{1}{2}X^4$ and $\frac{1}{2}$ of $\frac{1}{8}X^4$ which is $\frac{3}{4}X^2$. Then it becomes X^2 and $\frac{3}{4}X^2$. Multiply 10 minus $2X^2$ by the square root of 3; then it becomes the square root of 300 minus the square root of $12X^4$. Then complete the square root of 300 with the square root of $12X^4$. Add it to X^2 and $\frac{3}{4}X^2$; then it becomes X^2 and $\frac{3}{4}X^2$ and the square root of $12X^4$ to equal the square root of 300. Then add everything to X^2 [i.e. multiply it by the square root of $\frac{3072}{20,449}$ minus $\frac{28}{143}$] then it becomes X^2 equal to the square root of 45 plus $\frac{1395}{20,449}$ minus the square root of 11 plus $\frac{10,261}{20,449}$. That is the area of the square $\overline{\text{ḤMLE}}$ which is in the triangle $\overline{\text{ABJ}}$. Then subtract it from 10. The difference is the area of the triangle $\overline{\text{ABJ}}$. If you wish, say, the area of the square it is 6 plus $\frac{102}{143}$ minus the square root of 11 plus $\frac{10,261}{20,449}$ and the area of the triangle remains plus $\frac{41}{143}$ [75b] plus the square root of 11 plus $\frac{10,261}{20,449}$ [35].

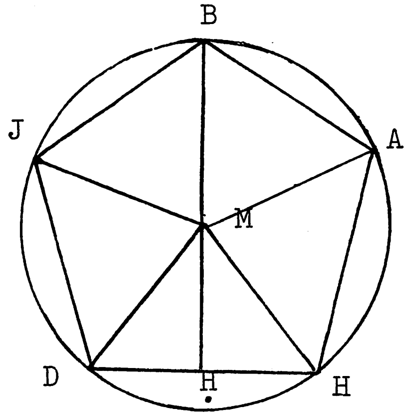
[XV] If you are given a square whose side is 10, that is square $\overline{ABJD}$, we construct on it a pentagon $\overline{AH\underset{\cdot}{H}RM}$. Then, to know each of its sides, let one of them be X. That is, $\overline{AH}$ is X. There remains line $\overline{HB}$ as 10 minus X and line $\overline{J\underset{\cdot}{H}}$ is the square root of $\frac{1}{2}X^2$. Line $\overline{\underset{\cdot}{H}B}$ remains as 10 minus the square root of $\frac{1}{2}X^2$, and line $\overline{HB}$ is 10 minus X. Then we multiply each of them by itself and add them; it is 200 plus $1\frac{1}{2}X^2$ minus 20 parts[36] minus the square root of 200 X^2 to equal X^2. Then we use algebraic confrontation the way I have described for you. It turns out that $\overline{AH}$ which is one of the sides of the pentagon is 200 plus the square root of 320,000 with its square root taken subtracted from 20 and the square root is 200. That is what we wished to demonstrate[37].

fig. 12



[XVI] If you are given a regular pentagon whose area is 50 <u>dhirāʿs</u>[38], what is each side of it? We construct the pentagon $\overline{ABJDH}$ and the center of the circumscribed circle is $\overline{M}$. We draw lines $\overline{AM}$, $\overline{MB}$, $\overline{MJ}$, and $\overline{MH}$. It is, therefore, obvious that it has been divided into 5 equal triangles which are $\overline{AMB}$, $\overline{BMJ}$, $\overline{JMD}$, $\overline{DMH}$, and $\overline{HMA}$. Therefore, triangle $\overline{HMD}$ is 10. We have shown in the previous parts

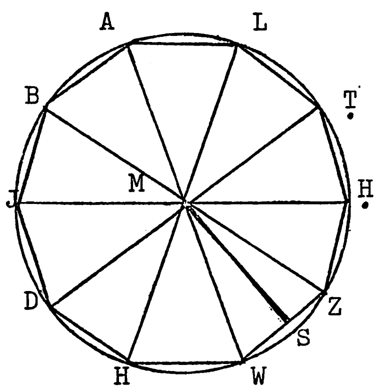
fig. 13



of our book that if there is a regular pentagon whose side is 10 that the square of the diameter of the circumscribed circle is 200 plus the square root of 8,000. Then, it is clear from what we have described that if you want to know the diameter of a circle that is circumscribed about a regular pentagon it is obvious that you multiply one of its sides by itself, then you double it and keep it. Then multiply again, one of its sides by itself, then the result by itself. Then take $\frac{4}{5}$ [76a] of it whose square root you find, then add the results to what you kept, then take the square root of the sum; what is left is the diameter of the circle. In this way, we let line $\overline{HD}$ be X which is a side of the pentagon. Therefore, the square of the diameter of the circumscribed circle, according to what we cleared, is $2X^2$ plus the square root of $\frac{4}{5}X^4$. The square of the radius, which is line $\overline{HM}$, is $\frac{1}{2}X^2$ plus the square root of $\frac{1}{2}$ of $\frac{1}{10}$ X^4. Then, we construct the altitude of triangle $\overline{HMD}$. It is clear that it bisects line $\overline{HD}$; that is line $\overline{M\dot{H}}$. Therefore, line $\overline{H\dot{H}}$ is $\frac{1}{2}X$. Then, multiply it by itself; it becomes $\frac{1}{4}$ X^2. Subtract it from the square of $\overline{HM}$ which is $\frac{1}{2}$ plus the square root of $\frac{1}{2}$ of $\frac{1}{10}$ X^4; there remains the square of line $\overline{M\dot{H}}$ as $\frac{1}{4}X^2$ plus the square root of $\frac{1}{2}$ of $\frac{1}{10}$ X^4. Then, multiply it by line $\overline{H\dot{H}}$[39], which is $\frac{1}{2}X$; it is $\frac{1}{2}$ of $\frac{1}{8}X^4$ plus the square root of $\frac{1}{320}X^8$ to equal 10 by 10 which is 100. Then, complete $\frac{1}{2}$ of $\frac{1}{8}X^4$ until it becomes X^4 [i.e. multiply it by 16]. Therefore, multiply everything you have by 16; it becomes X^8 equal to 1,600. Then add everything you have to X^4 [i.e. multiply it by 5 minus the square root of 200]; then it becomes X^4 equal to 8,000 plus [40]the square root of 51,200,000. Then take its root; that is the side of the pentagon[41].

[XVII] If you are given a regular decagon whose area is 100, what is each of its sides? An example of that is to let the decagon be $\overline{\text{ABJDHWZ}\underset{\cdot}{\text{H}}\underset{\cdot}{\text{T}}\text{L}}$ and the center of the circumscribed circle be $\overline{M}$. It is clear that it is divided by 10 equal triangles. Therefore, triangle $\overline{ZMW}$ is 10. We have said previously that if there is a regular decagon whose side is 10, the diameter of the circumscribed circle [76b] would be 10 plus the square root of 500. It is clear from what we described that if you want

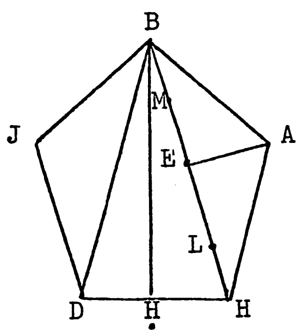
fig. 14



to find the diameter of a circumscribed circle on a given regular decagon you could multiply one of its sides by itself, then by 5, then take the root of what is left and add it to one of the sides of the decagon. What is left is the diameter of the circle. Therefore, according to what we said, we let $\overline{ZW}$ be X which is one of the sides of the decagon. Therefore, the diameter of the circle, according to what we have said, must be X plus the square root of $5X^2$. The radius, which is $\overline{ZM}$, is $\frac{1}{2}X$ plus the square root of $1\frac{1}{4}X^2$. Also, it is clear from what Euclid said that if a line is divided into 2 parts and an equivalent to the larger side has been added to the line, then the sum divided by the larger part is the same as the first line divided by the smaller part[42]. And he said, on another occasion, that if the side of a hexagon has been joined to the side of a decagon in the same circle such that they constitute a straight line then a proportion has been established[43]. Line $\overline{ZM}$, which is one side of the hexagon, is the larger side, and line $\overline{ZW}$ is one side of the decagon. Then if these two lines become one line, we get the extreme and mean ratio. If you set $\overline{ZW}$ to be X and subtract it from $\overline{ZM}$, and then if you add to the difference $\frac{1}{2}$ line $\overline{ZW}$, and multiply the result by itself, it would equal 5 times the square of $\frac{1}{2}\overline{ZW}$, according to what Euclid has said[44].

Then we multiply $\frac{1}{2}\overline{ZW}$ by itself, then by 5, and take its square root, and add it to $\frac{1}{2}$ line $\overline{ZW}$ which is $\frac{1}{2}$ X; therefore we get line $\overline{ZM}$. Then it has become clear that line $\overline{ZW}$ is $\frac{1}{2}X$ plus the square root of $1\frac{1}{4}X^2$[45]. Then we construct the altitude of triangle $\overline{ZMW}$ and that is $\overline{MS}$. It is clear that it bisects the line. Then we multiply line $\overline{ZM}$ by itself, which is $\frac{1}{2}$ X plus the square root of $1\frac{1}{4}X^2$; it becomes [77a] $1\frac{1}{2}X^2$ plus the square root of $1\frac{1}{4}X^4$. Then, multiply $\overline{ZS}$, which is $\frac{1}{2}X$, by itself; it becomes $\frac{1}{4}X^2$. Subtract it from it; there remains $1\frac{1}{4}X^2$ plus the square root of $1\frac{1}{4}X^4$. Then take its root and that is line $\overline{MS}$. Then multiply it by line $\overline{ZS}$[46], which is $\frac{1}{2}$ X, it becomes $\frac{1}{4}X^4$ plus $\frac{1}{2}$ of $\frac{1}{8}$ X^4 plus the square root of $\frac{5^2}{64}$ X^8 to equal 100. Then complete $\frac{1}{4}X^4$ plus $\frac{1}{2}$ of $\frac{1}{8}$ X^4 until it becomes X^4 [i.e. multiply it by 3 plus $\frac{1}{5}$]. Therefore, multiply everything by 3 and $\frac{1}{5}$. Then, it becomes X^4 plus the square root of $\frac{4}{5}X^8$ equal to 320. Add everything to X^4 [i.e. multiply it by 5 minus the square root of 20]; it becomes X^4 equal to 1,600 minus the square root of 2,048,000. Take its root; that is line $\overline{ZW}$ which is one side of the decagon. That is what we wished to demonstrate[47].

fig. 15



[XVIII] If the regular pentagon $\overline{ABJDH}$ is given and the area of triangle $\overline{BDH}$ is 10, what is line $\overline{DH}$ which is the base of the triangle and one side of the pentagon? To find it, you let the length be X. Euclid has made it clear that if line $\overline{DH}$ is divided in an extreme and mean ratio then the larger part is equal to $\overline{DH}$. He also described that if a line is divided in an extreme and mean ratio the smaller part is $\frac{1}{2}$ the larger part and its product by itself. Therefore, its square is 5 times the square of $\frac{1}{2}$ of the larger part. Therefore, we divide $\overline{BH}$ in an extreme and mean ratio on $\overline{M}$. We let the larger part be line $\overline{MH}$. Therefore $\overline{MH}$ equals line [77b] $\overline{HD}$. We divide line $\overline{MH}$ by 2 on $\overline{L}$. Therefore, the product of $\overline{BL}$ by itself equals 5 times the product of $\overline{ML}$ by itself. The product of $\overline{LH}$ by itself is $\frac{1}{4}X^2$ because we let line $\overline{HD}$ be X. Therefore, line $\overline{MH}$ is X and $\overline{LH}$ is $\frac{1}{2}X$; therefore $\overline{BL}$ by itself is $1\frac{1}{4}X^2$. Therefore, $\overline{BL}$ is the square root of $1\frac{1}{4}X^2$ and line $\overline{LH}$ is $\frac{1}{2}$ X. Then, $\overline{BH}$ is $\frac{1}{2}$ X plus the square root of $1\frac{1}{4}X^2$. Then we construct the perpendicular to triangle $\overline{BHD}$, which is $\overline{B\underset{\cdot}{H}}$; therefore, line $H\underset{\cdot}{H}$ is $\frac{1}{2}X$. Then multiply it by itself; it becomes $\frac{1}{4}X^2$. Subtract it from $\overline{BH}$ by itself, which is $1\frac{1}{2}X^2$ plus the square root of $1\frac{1}{4}X^4$; the remainder is $1\frac{1}{4}X^2$ and the square root of $1\frac{1}{4}X^4$ whose square root is line $\overline{B\underset{\cdot}{H}}$. Then we multiply it by $\overline{H\underset{\cdot}{H}}$, which is $\frac{1}{2}$ X ; it becomes $\frac{1}{4}X^4$ plus $\frac{1}{2}$ of $\frac{1}{8}X^4$ plus the square root of $\frac{5}{64}X^8$ equal to 100. Then complete your $\frac{1}{4}X^4$ and $\frac{1}{2}$ of $\frac{1}{8}X^4$ until it becomes X^4 [i.e. multiply it by 3 and $\frac{1}{5}$]. Therefore, multiply everything you have by $3\frac{1}{5}$; it becomes X^4 plus the square root of $\frac{4}{5}X^8$ equal to 320. Then add everything you have to X^4,[i.e. multiply it by 5 minus the square root of 20]; it becomes X^4 equal to 1,600 minus the square root of 2,048,000 whose root is line $\overline{HD}$[48]. Then it is clear that triangle $\overline{BHD}$ of this pentagon which we have described is similar in figure to triangle $\overline{ZMW}$ of the decagon which we have described before, Allah willing.

[XIX] And if you are given the area of the triangle $\overline{ABH}$ as 10, what is line $\overline{BH}$? To find its length, let it be X. Euclid said that if a line has been divided in extreme and mean ratio, then if the larger part has been increased by $\frac{1}{2}$ of the entire line segment, [78a] and its product by itself, then the square that we get is 5 times the square of $\frac{1}{2}$ of the line[49]. We let $\overline{BH}$ be X. Therefore, if we multiply half of it by itself, then by 5, it becomes $1\frac{1}{4}X^2$. It is clear from what we described that line $\overline{AH}$ is the square root of $1\frac{1}{4}X^2$ minus $\frac{1}{2}X$. Then we construct the perpendicular on line $\overline{BH}$, which is $\overline{AE}$. Then we multiply $\overline{AH}$ by itself; it is $1\frac{1}{2}X^2$ minus the square root of $1\frac{1}{4}X^4$. Then we multiply line $\overline{HE}$ by itself, which is $\frac{1}{2}X$; it becomes $\frac{1}{4}X^2$. Then, subtract it from it; there remains $1\frac{1}{4}X^2$ minus the square root of $1\frac{1}{4}X^4$. Then it becomes X^4 whose root is $\overline{AE}$ and which is the altitude of triangle $\overline{BAH}$. Then we multiply it by $\overline{HE}$, which is $\frac{1}{2}X$; then it becomes $\frac{1}{4}X^4$ and $\frac{1}{2}$ of $\frac{1}{8}X^4$ minus the square root of $\frac{5}{64}X^8$ whose root is the area of triangle $\overline{AHB}$ which is 10. Therefore, multiply it by itself; it is 100 equal to $1\frac{1}{4}X^4$ plus

$\frac{1}{2}$ of $\frac{1}{8}$ X^4 minus the square root of $\frac{5}{64}X^8$. Then complete your $\frac{1}{4}$ X^4 and $\frac{1}{2}$ of $\frac{1}{8}$ X^4 until it becomes X^4 [i.e. multiply it by 3 and $\frac{1}{5}$]. Then multiply everything you have by 3 and $\frac{1}{5}$; it is X^4 minus the square root of $\frac{4}{5}$ X^8 equal to 320. Then, complete it until it becomes X^4 [i.e. multiply it by 5 minus the square root of 20]; it becomes X^4 equal to 1,600 plus the square root of 2,048,000 whose root is line $\overline{\text{BH}}$. That is what we wished to demonstrate[50].

[XX] If you are given a regular decagon $\overline{\text{ABJDHWḤṬKR}}$ [51] and the area of the triangle $\overline{\text{KṬḤ}}$ is 10, what is the chord $\overline{\text{KḤ}}$ which is one-fifth of the circumscribed circle? To know its length, let $\overline{\text{KḤ}}$ be X. We have already shown that if the chord which is $\frac{1}{5}$ of the circle was X the square of the diameter would have been $2X^2$ plus the square root of $\frac{4}{5}$ X^4. Therefore, the square of its radius would be $\frac{1}{2}X^2$ plus the square root of $\frac{1}{2}$ of $\frac{1}{10}$ X^4. Then, we subtract it from the square root of the chord, $\frac{1}{5}$ of the circle, which is X^2; the difference is the square of $\overline{\text{KṬ}}$ which is $\frac{1}{2}$ X^2 minus the square root of $\frac{1}{2}$ of $\frac{1}{10}$ X^4. We did this because Euclid [78b] has shown that the $\frac{1}{5}$ chord shows strength vis a vis the $\frac{1}{6}$ chord and the $\frac{1}{10}$ chord if they are in one circle[52]. Then, subtract from it the square of $\overline{\text{KM}}$ which is square of X^2; the remainder is the square of $\overline{\text{ṬM}}$ which is the altitude of the triangle $\overline{\text{KṬḤ}}$, $\frac{1}{4}$ X^2 minus the square root of $\frac{1}{2}$ of $\frac{1}{10}$ X^4. Then we multiply it by the square of $\overline{\text{KM}}$, which is the square of X^2; it is $\frac{1}{2}$ of $\frac{1}{8}$ X^4 minus the square root of $\frac{1}{320}$ X^4 whose square root equals the area of triangle $\overline{M\underset{\cdot}{T}\underset{\cdot}{H}}$ which is 10. Therefore, multiply 10 by 10, it is 100 equal to $\frac{1}{2}$ of $\frac{1}{8}X^4$ minus the square root of $\frac{1}{320}$ X^8. Then, complete your X^4 until it becomes X^4 [i.e. multiply it by 16]. Therefore, multiply everything by 16; it becomes X^4 minus the square root of $\frac{4}{5}$ X^8 equal to 1,600. Then complete it until it becomes X^4, [i.e. multiply it by 5 plus the square root of 20]. Therefore, multiply everything you have by 5 plus the square root of 20; it becomes X^4 equal to 8,000 plus the square root of 51,200,000 whose root is $\overline{K\underset{\cdot}{H}}$ which is the chord $\frac{1}{5}$ of the circumscribed circle. That is what we wished to demonstrate[53].

FOOTNOTES

1. State University of New York, Albany, New York.
2. M. Levey, <u>The Algebra of Abū Kāmil</u> (Madison, 1966) pp. 3-6; <u>Encyclopedia of Islam</u> I, 132-133 (1960).
3. <u>Ibid</u>., pp. 217-220 for elementary algebraic problems used by Leonardo from abū Kāmil's <u>Algebra</u>.
4. <u>Cf</u>. B. Boncompagni, ed. of <u>Scritti di Leonardo Pisano</u> (Roma, 1857-1862) <u>Practica geometriae</u> vol. II, pp. 207-216. <u>Cf</u>. also H. Suter, <u>Bib. Math</u>. **10**, 38-42 (1909-10) where Leonardo's solutions are compared with those of abū Kāmil.
5. M. Levey, in press with <u>Atti della Accademia Nazionale dei Lincei</u> (Roma).
6. From the Munich Codex Heb. 225 by G. Sacerdote in <u>Feschrift z. 80 Geburtstage M. Steunschneiders</u> (Leipzig, 1896) pp. 169-194. The Hebrew translation was by the famous scholar Mordecai Finzi of Mantuo who lived in the fifteenth century. <u>Cf.</u> Levey, <u>Algebra</u>, pp. 11-13.
7. See L.C. Karpinski, "On the Pentagon and Decagon," <u>Bibliotheca Mathematica</u> **12**, 41 (1911-12); German translation is by H. Suter, "Die Abhandlung des Abū Kāmil Shogā[c] b. Aslam über des Fünfeck und Zehneck," <u>Bibliotheca Mathematica</u> **10**, 15-42 (1909-1910). The Latin text is in MS Paris 7377A.
8. Fols. 68a-68b.

9. Fols. 70b-71a.

10. Fols. 75b-76a.

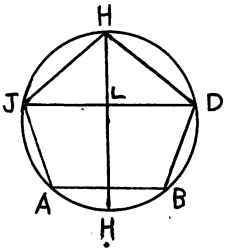
11. Fol. 72a.

12. Fol. 76b.

13. <u>Cf</u>. Suter, <u>op. cit.</u>, p. 37.

14. <u>Al-jabr wa'l-muqābala.</u>

15. $\overline{HD} = X$



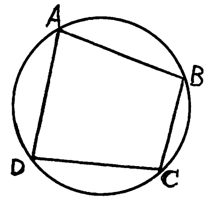
$\overline{HD}^2 = \overline{HL} \cdot \overline{HḤ}$

$$\overline{HL} = \frac{\overline{HD}^2}{\overline{HḤ}} = \frac{1}{10} X^2$$

$$\overline{JL} = \sqrt{X^2 - \frac{1}{100} X^4} \quad \text{But } \overline{JL} = \frac{1}{2} \overline{JD}$$

$$\therefore \ \overline{JD} = \sqrt{4X^2 - \frac{2}{5} \cdot \frac{1}{10} X^4}$$

$\overline{AB} \cdot \overline{JD} + \overline{BD} \cdot \overline{AJ} = \overline{JD}^2$ (Ptolemy's theorem) "If ABCD is a cyclic quadrilateral, then AB $\cdot$ CD + BC $\cdot$ DA = AC $\cdot$ BD" C.J. Tranter, <u>Advanced Pure Mathematics</u> (The English University Press Ltd. 1963), p. 346.



But $\overline{JD}^2 = 4X^2 - \frac{2}{5} \cdot \frac{1}{10} X^4$ and $\overline{BD} \times \overline{AJ} = \overline{HD}^2 = X^2$

replacing $\overline{JD}^2$ with its value and subtracting X^2 for $\overline{BD} \times \overline{AJ}$ we get $\overline{AB} \cdot \overline{JD} = 3X^2 - \frac{2}{5} \cdot \frac{1}{10} X^4$

Divide by $\overline{AB} = X \therefore \quad \overline{JD} = 3X - \frac{2}{5} \quad \frac{1}{10} X^3$

$$\overline{JD}^2 = 9X^2 + \frac{X^6}{625} - \frac{6}{25} X^4 = 4X^2 - \frac{2}{5} \cdot \frac{1}{10} X^4$$

$$5X^2 + \frac{1}{625} X^6 = \frac{1}{5} X^4$$

$$5 + \frac{1}{625} X^4 = \frac{1}{5} X^2$$

$$3125 + X^4 = 125 X^2$$

$$X^4 - 125X^2 + 3125 = 0$$

$$X^2 = \frac{125 \pm \sqrt{15625 - 12500}}{2} = 62\frac{1}{2} - \sqrt{\frac{3125}{4}}$$

$$X^2 = 62\frac{1}{2} - \sqrt{781\frac{1}{4}} \quad \therefore X = \overline{HD} = \sqrt{62\frac{1}{2} - \sqrt{781\frac{1}{4}}}$$

16. $\overline{DH}^2 = 62\frac{1}{2} - \sqrt{781\frac{1}{4}}$

$\overline{AJ} \times \overline{DH} + \overline{JD} \times \overline{AH} = \overline{AD} \times \overline{JH}$; $\overline{AD} = \overline{JH}$, $\overline{AJ} = \overline{DH}$

$\therefore \overline{DH}^2 + \overline{JD} \times \overline{AH} = \overline{JH}^2$

Let $\overline{JD} = X$ Then $62\frac{1}{2} - \sqrt{781\frac{1}{4}} + 10X = \overline{JH}^2$

Add $\overline{AJ}^2 = 62\frac{1}{2} - \sqrt{781\frac{1}{4}}$ to both sides

$\therefore 125 + 10X - \sqrt{3125} = \overline{AJ}^2 + \overline{JH}^2 = \overline{AH}^2 = 100$

$\therefore 10X = \sqrt{3125} - 25$

$\therefore X = \overline{JD} = \sqrt{31\frac{1}{4}} - 2\frac{1}{2}$

17. $\overline{TH}^2 = 62\frac{1}{2} - \sqrt{781\frac{1}{4}}$

$$\overline{TL}^2 = \left(\frac{\overline{TH}}{2}\right)^2 = \frac{62\frac{1}{2} - \sqrt{781\frac{1}{4}}}{4} = 15\frac{5}{8} - \sqrt{\frac{781\frac{1}{4}}{16}}$$

$$= 15\frac{5}{8} - \sqrt{\frac{3125}{4 \text{ x } 16}} = 15\frac{5}{8} - \sqrt{\frac{3000}{4 \text{ x } 16} + \frac{100}{4 \text{ x } 16} + \frac{20}{4 \text{ x } 16} + \frac{5}{4 \text{ x } 16}}$$

$$= 15\frac{5}{8} - \sqrt{48\frac{1}{2} + \frac{1}{4} + \frac{5}{8} \text{ of } \frac{1}{8}}$$

18. $\dfrac{X^2}{\overline{KH}^2} = \dfrac{\overline{TH}^2}{\overline{HL}^2}$ where $\overline{HL}^2 = \overline{HH}^2 - \overline{HL}^2$

$$\overline{HL}^2 = 25 - 15\frac{5}{8} + \sqrt{48\frac{1}{2} + \frac{1}{4} + \frac{5}{8} \text{ of } \frac{1}{8}}$$

$$\therefore \overline{HL}^2 = 9\frac{1}{4} + \frac{1}{8} + \sqrt{48\frac{1}{2} + \frac{1}{4} + \frac{5}{8} \text{ of } \frac{1}{8}}$$

$$\therefore \frac{X^2}{25} = \frac{62\frac{1}{2} - \sqrt{781\frac{1}{4}}}{9\frac{1}{4} + \frac{1}{8} + \sqrt{48\frac{1}{2} + \frac{1}{4} + \frac{5}{8} \text{ of } \frac{1}{8}}}$$

19. $\therefore 9\frac{3}{8}X^2 + \sqrt{48\frac{3}{4}X^4 + \frac{5}{8} \cdot \frac{1}{8}X^4} = 1562\frac{1}{2} - \sqrt{488{,}281\frac{1}{4}}$

He says, multiply both sides by $\frac{1}{5} + \frac{1}{25} - \sqrt{\frac{20}{625}}$.

After multiplying we get:

$$\frac{15}{8}X^2 + \frac{3}{8}X^2 - \frac{3}{4}\sqrt{5}\ X^2 + \frac{5}{8}\sqrt{5}\ X^2 + \frac{1}{8}\sqrt{5}\ X^2 - \frac{5}{4}X^2 = X^2$$

$$\therefore X^2 = \left(1562\frac{1}{2} - \sqrt{488{,}281\frac{1}{4}}\right)\left(\frac{1}{5} + \frac{1}{25} - \sqrt{\frac{20}{625}}\right)$$

after solving we get:

$$X^2 = 500 - \left(\frac{150 + 250}{2}\sqrt{5}\right) = 500 - \sqrt{200{,}000}$$

20. $$\overline{TH} = \sqrt{31\tfrac{1}{4}} - 2\tfrac{1}{2} \quad ;$$

$$\overline{TL} = \frac{1}{2}\,\overline{TH} = \sqrt{7\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{2} \text{ of } \tfrac{1}{8}} - 1\tfrac{1}{4}$$

$$\therefore\ \overline{TL}^2 = 9\tfrac{1}{4} + \tfrac{1}{8} - \sqrt{48\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{5}{8} \text{ of } \tfrac{1}{8}}$$

$$\overline{TH}^2 - \overline{TL}^2 = \overline{HL}^2 = 15\tfrac{5}{8} + \sqrt{48\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{5}{8} \text{ of } \tfrac{1}{8}}$$

$$\overline{TH}^2 = 37\tfrac{1}{2} - \sqrt{781\tfrac{1}{4}}$$

Let $\overline{AB} = X \quad \therefore \quad \overline{AB}^2 = X^2$

$$\therefore\ \frac{X^2}{\overline{TH}^2} = \frac{\overline{KH}^2}{\overline{HL}^2} \qquad X^2 \cdot \overline{HL}^2 = \overline{KH}^2 \cdot \overline{TH}^2$$

$$X^2 \cdot \left[15\tfrac{5}{8} + \sqrt{48\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{5}{8} \text{ of } \tfrac{1}{8}}\right] = 25\left[37\tfrac{1}{2} - \sqrt{781\tfrac{1}{4}}\right]$$

$$15\tfrac{5}{8}X^2 + \sqrt{48\tfrac{3}{4}X^4 + \tfrac{5}{8} \cdot \tfrac{1}{8}X^4} = 937\tfrac{1}{2} - \sqrt{488{,}281\tfrac{1}{4}}$$

$$X^2 = 75 + \sqrt{625} - \sqrt{3125} - \sqrt{1125}$$

$$X^2 = 100 - \sqrt{8{,}000}$$

$$\therefore\ X = \sqrt{100 - \sqrt{8{,}000}}$$

Note : $\sqrt{3125} + \sqrt{1125} = \sqrt{125 \cdot 25} + \sqrt{45 \cdot 25}$

$= \sqrt{25}\left(\sqrt{125} + \sqrt{45}\right) = \sqrt{25}\left(5\sqrt{5} + 3\sqrt{5}\right) \quad = \sqrt{8,000}$

21. Let $\overline{HJ} = X$

$\therefore \overline{HJ} \times \overline{AB} + \overline{AB}^2 = \overline{HJ}^2$

$10X + 100 = X^2 \Rightarrow X^2 - 10X - 100 = 0$

$X = 5 + \sqrt{125} = \overline{HJ}$

$\frac{1}{2}\,\overline{HJ} = 2\frac{1}{2} + \sqrt{31\frac{1}{4}} = \overline{HL}$

$\overline{HL}^2 = 37\frac{1}{2} + \sqrt{781\frac{1}{4}}$

$\overline{HD}^2 - \overline{HL}^2 = \overline{DL}^2 = 62\frac{1}{2} - \sqrt{781\frac{1}{4}}$

Now let $\overline{KD} = X$

$$\therefore \frac{X^2}{\overline{DH}^2} = \frac{\overline{DH}^2}{\overline{DL}^2} \qquad X^2 \cdot \overline{DL}^2 = \overline{DH}^4$$

$$\therefore X^2 \cdot \left[62\frac{1}{2} - \sqrt{781\frac{1}{4}}\right] = 10,000$$

$$62\frac{1}{2}X^2 - \sqrt{781\frac{1}{4}X^4} = 10,000$$

Now, he says multiply both sides of the equation

by $\frac{1}{5} \cdot \frac{1}{10} + \sqrt{1\frac{1}{4} \text{ of } \frac{1}{15,625}}$

$$\left[62\frac{1}{2}X^2 - \sqrt{781\frac{1}{4}\ X^4}\right] \cdot \left[\frac{1}{50} + \sqrt{\frac{5}{4} \cdot \frac{1}{15,625}}\right]$$

$$= 10,000\left[\frac{1}{50} + \sqrt{\frac{5}{4 \cdot 15,625}}\right]$$

$$\frac{5}{4}X^2 + \frac{1}{4}\sqrt{5}\ X^2 - \frac{1}{4}\sqrt{5}\ X^2 - \frac{1}{4}\ X^2$$

$$= 200 + \sqrt{8,000}$$

$$\therefore X^2 = 200 + \sqrt{8,000}$$

22. $\overline{AM}^2 = 50 + \sqrt{500}$ since $2\ \overline{AM}^2 = 200 + \sqrt{8,000}$

$$\overline{AM}^2 - \overline{KA}^2 = \overline{KM}^2 = 50 + \sqrt{500} - 25 + \sqrt{500}$$

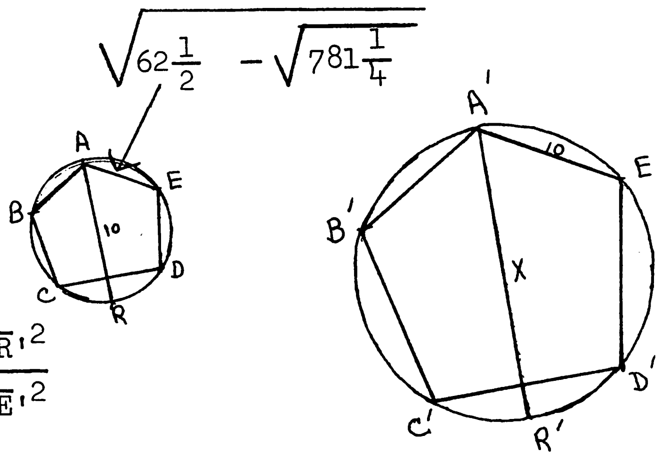
$$\therefore 2\ \overline{KM}^2 = 100 + \sqrt{8,000}$$

23. $200 + \sqrt{8,000} - 100 = 100 + \sqrt{8,000}$

$\sqrt{100 + \sqrt{8,000}}$ = the diameter of the inscribed circle in this case.

24. "Similar polygons inscribed in circles are to one another as the squares on the diameters." Book XII, Proposition I, Sir Thomas Heath, Euclid The Elements Vol. III (Cambridge, 1947), p. 369.

25.



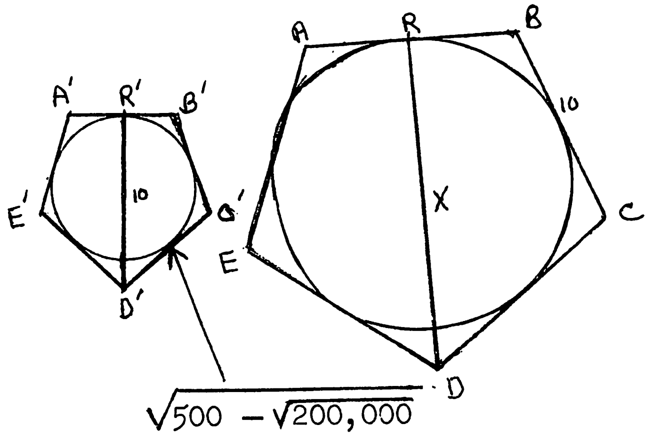
$$\frac{\overline{AR}^2}{\overline{AE}^2} = \frac{\overline{A'R'}^2}{\overline{A'E'}^2}$$

$$\frac{100}{62\frac{1}{2} - \sqrt{781\frac{1}{4}}} = \frac{X^2}{100}$$

$$10{,}000 = 62\frac{1}{2}X^2 - \sqrt{781\frac{1}{4}X^4}$$

We can solve this in the same way as we solved it in (7). $\therefore X^2 = 200 + \sqrt{8{,}000}$

26. $\dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \dfrac{\overline{RD}^2}{\overline{R'D'}^2}$



$$\therefore \frac{100}{500 - \sqrt{200{,}000}} = \frac{X^2}{100}$$

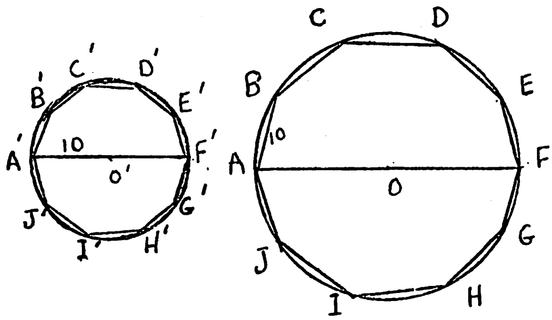
$$500\,X^2 - \sqrt{200{,}000\,X^4} = 10{,}000$$

multiply both sides by $\frac{1}{10} \cdot \frac{1}{10} + \sqrt{\frac{4}{5} \cdot \frac{1}{10{,}000}}$ you get

$$\left[500X^2 - \sqrt{200{,}000}\right]\left[\frac{1}{10} \cdot \frac{1}{10} + \sqrt{\frac{4}{5} \cdot \frac{1}{10{,}000}}\right] = 10{,}000\left[\frac{1}{100} + \sqrt{\frac{4}{5 \cdot 10{,}000}}\right]$$

$5X^2 + \sqrt{20}X^2 - \sqrt{20}\,X^2 - 4X^2 = 100 + \sqrt{8,000}$

$\therefore\ X^2 = 100 + \sqrt{8,000}$



27. Let $\overline{AF} = X$

$\therefore\ \dfrac{\overline{AF}}{\overline{AB}} = \dfrac{\overline{A'F'}}{\overline{A'B'}}$

$\dfrac{X}{10} = \dfrac{10}{\sqrt{31\frac{1}{4}} - 2\frac{1}{2}}$

$100 = \sqrt{31\frac{1}{4}\ X^2} - 2\frac{1}{2}X \quad \therefore\ 100 + 2\frac{1}{2}X = \sqrt{31\frac{1}{4}\ X^2}$

$10,000 + 6\frac{1}{4}X^2 + 500X = 31\frac{1}{4}X^2 \quad \therefore\ 25X^2 - 500X - 10,000 = 0$

$X^2 - 20X - 400 = 0 \quad \therefore\ X = 10 + \sqrt{500}$

28. The process is the same as in the previous problems.

29. Prove: $\overline{AJ}^2 + \overline{KE}^2 = \overline{AD}^2$

Proof: draw from L, $\overline{LM}$ such that $\overline{LM} \perp \overline{AJ}$. It meets the circle at Z. $\therefore\ \overline{LZ} = \overline{KE}$.

Join Z to D, and D to J.

$\overline{AM} = \overline{MD}$, $\overline{ML} = \overline{MZ}$, $\angle\ \overline{ZMD} = \angle\ \overline{AML}$

$\therefore\ \overline{DZ} = \overline{AL} \Rightarrow \triangle\ \overline{DMZ} = \triangle\ \overline{ALM} \Rightarrow\ \overline{ZD} \parallel \overline{LJ}$

Since $\overline{ZD} \parallel$ and $= \overline{LJ} \Rightarrow \overline{ZL} \parallel$ and $= \overline{DJ}$.

And since $\angle\overline{AJD} = 90°$ Then $\overline{JD}^2 + \overline{AJ}^2 = \overline{AD}^2$

$\overline{DJ} = \overline{LZ} = \overline{KE} \quad \therefore\ \overline{AJ}^2 + \overline{KE}^2 = \overline{AD}^2$

30. To find $\overline{JD}$ we know $\overline{BD} = \sqrt{31\frac{1}{4}} - 2\frac{1}{2}$

$$\overline{AD} = \sqrt{62\frac{1}{2} + \sqrt{781\frac{1}{4}}}$$

$\overline{BJ} = 5$ because it is equal to the radius

$$\overline{AJ} = \sqrt{75}$$

$$\overline{AD} \times \overline{BJ} = \sqrt{1562\frac{1}{2} + \sqrt{488,281\frac{1}{4}}}$$

$$\overline{BD} \times \overline{AJ} = \sqrt{2343\frac{3}{4}} - \sqrt{468\frac{3}{4}}$$

$$\overline{AD} \times \overline{BJ} - \overline{BD} \times \overline{AJ} = \sqrt{1562\frac{1}{2} + \sqrt{488,281\frac{1}{4}} + 468\frac{3}{4}}$$

$$-\sqrt{2343\frac{3}{4}} = \overline{AB} \times \overline{JD}$$

$$\therefore \overline{JD} = \frac{\overline{AD} \times \overline{BJ} - \overline{BD} \times \overline{AJ}}{\overline{AB}} = \sqrt{15\frac{5}{8} + \sqrt{48\frac{3}{4}} + \frac{5}{8} \cdot \frac{1}{8}}$$

$$+\sqrt{4\frac{1}{2} + \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{8}} - \sqrt{23\frac{3}{8} + \frac{1}{2} + \frac{1}{8}}$$

31. Find $\overline{AD}$ where the area of $\triangle \overline{ABJ} + \overline{AD} = 10$

Let $\overline{AD} = X$ Then $\overline{JD} = \sqrt{\frac{1}{3}X^2}$ because

$\overline{JD}^2 = \overline{AJ}^2 - \overline{AD}^2$ Since $\overline{JD} = \frac{1}{2}\overline{AJ}^2$

$$\therefore \overline{JD}^2 = 4\overline{JD}^2 - \overline{AD}^2 \Rightarrow 3\overline{JD}^2 - \overline{AD}^2 = \overline{JD}^2 = \frac{1}{3}X^2$$

$$\therefore \overline{JD}^2 = \sqrt{\frac{1}{3}X^2}$$

$\therefore$ the area of the triangle is $\dfrac{2\sqrt{\frac{1}{3}}\,X \cdot X}{2} = \sqrt{\frac{1}{3}}\;X^2$

$$\therefore \sqrt{\frac{1}{3}X^2} + X = 10 \Rightarrow X^2 + \sqrt{3}\,X = \sqrt{300}$$

$$\Rightarrow X^2 + \sqrt{3}\,X - \sqrt{300} = 0$$

$$X = \frac{-\sqrt{3} \pm \sqrt{3+4\sqrt{300}}}{2} = \frac{-\sqrt{3}}{2} \pm \sqrt{\frac{3}{4} + \sqrt{300}}$$

$$\therefore \quad X = \overline{AD} = \sqrt{\frac{3}{4} + \sqrt{300}} - \sqrt{\frac{3}{4}}$$

32. A right angled, long quadrilater al.

33. Dirham is the absolute number.

34. Given: Each side of the equilateral triangle is 10.

The area of the rectangle is 10.

Find the base of the rectangle.

$$\underset{\cdot}{\overline{BH}} = \sqrt{\frac{1}{3}X^2} = \underset{\cdot}{\overline{TJ}} = \underset{\cdot\cdot}{\overline{TH}} = 10 - 2\sqrt{\frac{1}{3}X^2} = 10 - \sqrt{1\frac{1}{3}X^2}$$

$$\therefore \underset{\cdot}{\overline{HH}} \cdot \underset{\cdot\cdot}{\overline{TH}} = 10$$

$$10X - \sqrt{1\frac{1}{3}X^4} = 10$$

$$\therefore \quad 10X = 10 + \sqrt{1\frac{1}{3}X^4}$$

$$\sqrt{\frac{3}{4}}\,(10X) = \sqrt{\frac{3}{4}}\left(10 + \sqrt{1\frac{1}{3}X^4}\right)$$

$$\sqrt{75X^2} = X^2 + \sqrt{75} \Rightarrow X^2 - \sqrt{75X} + \sqrt{75} = 0$$

$$X = \frac{\sqrt{75} \pm \sqrt{75 - 4\sqrt{75}}}{2}$$

$$\therefore \quad X = \sqrt{18\frac{3}{4}} + \sqrt{18\frac{3}{4} - \sqrt{75}} = \underset{\cdot}{\overline{HH}}$$

35. Let $\overline{EM} = X$; The area of the square is X^2

So, the area of the triangle is $10 - X^2$; the areas of $\overline{BLH} + \overline{MEJ} + \overline{AHM} = 10 - 2X^2 = S$

We know that $\overline{EJ} = \sqrt{\frac{1}{3}X^2}$ $\therefore$ Area of $\triangle$ $\overline{MEJ}$ + Area of $\triangle$ $\overline{HLB} = \sqrt{\frac{1}{3}X^4}$.

$\overline{AT} = \sqrt{\frac{3}{4}X^2}$, $\therefore$ Area of $\triangle$ $\overline{AHM}$ is $\sqrt{\frac{3}{16}X^4}$

$$\therefore \quad S = \sqrt{\frac{1}{3}}X^2 + \sqrt{\frac{3}{4}}X^2 = 10 - 2X^2$$

$$\sqrt{\frac{1}{3}}X^2 + \sqrt{\frac{3}{4}}X^2 = 10 - 2X^2$$

$$X^2 + \frac{3}{4}X^2 = \sqrt{300} - \sqrt{12}X^2$$

$$X^2 + \frac{3}{4}X^2 + \sqrt{12}X^2 = \sqrt{300}$$

He says multiply both sides of this equation by $\sqrt{\frac{3072}{20,449}} - \frac{28}{143}$

$$\therefore \left(X^2 + \frac{3}{4}X^2 + \sqrt{12}X^2\right)\left(\sqrt{\frac{3072}{20,449}} - \frac{28}{143}\right)$$

$$= \sqrt{300}\left(\sqrt{\frac{3072}{20,449}} - \frac{28}{143}\right)$$

$$\left(\frac{7 + 8\sqrt{3}}{4}X^2\right)\left(\sqrt{\frac{3072}{20,449}} - \frac{28}{143}\right) = \sqrt{300}\left(\sqrt{\frac{3072}{20,449}} - \frac{28}{143}\right)$$

$$\frac{1}{4}X^2 \left[7\sqrt{\frac{3072}{20,449}} + 8\sqrt{3}\sqrt{\frac{3072}{20,449}} - \frac{7 \cdot 28}{143} - \frac{8\sqrt{3} \cdot 28}{143}\right] =$$

$$\sqrt{45 + \frac{1395}{20,449}} - \sqrt{11 + \frac{10,261}{20,449}}$$

$$\frac{1}{4}X^2 \left[\frac{7 \cdot 32\sqrt{3}}{143} + \frac{8 \cdot 32 \cdot 3}{143} - \frac{7 \cdot 28}{143} - \frac{8 \cdot 28\sqrt{3}}{143}\right]$$

$$= \sqrt{45 + \frac{1395}{20,449}} - \sqrt{11 + \frac{10,261}{20,449}}$$

$$\therefore \quad \frac{1}{4}X^2 \left[\frac{572}{143}\right] = X^2 = 6 + \frac{102}{143} - \sqrt{11 + \frac{10,261}{20,449}}$$

$\therefore$ The area of the triangle is $10 - X^2 = 3 + \frac{41}{143}$

$$+ \sqrt{11 + \frac{10,261}{20,449}}$$

36. He means 20X.

37. Given: square $\overline{ABJD}$, where $\overline{AB} = 10$

Find the side of the regular pentagon $\overline{AH\dot{H}RM}$.

Let $\overline{AH} = X$ Then $\overline{HB} = 10 - X$

$$\overline{J\dot{H}} = \sqrt{\frac{1}{2}X^2} = \overline{\dot{H}B} = 10 - \sqrt{\frac{1}{2}X^2}$$

$$\overline{HB}^2 + \overline{\dot{H}B}^2 = \overline{H\dot{H}}^2$$

$$(10 - X)^2 + \left(10 - \sqrt{\frac{1}{2}X^2}\right) = X^2$$

$$100 - 20X + X^2 + 100 - \sqrt{200X^2} + \frac{1}{2}X^2 = X^2$$

$$200 + 1\frac{1}{2}X^2 - 20X - \sqrt{200X^2} = X^2$$

after solving the equation

$$\overline{AH} = X = 20 + 200 - \sqrt{200 + \sqrt{320,000}}$$

38. Here, unit of area. A dhirā[c] is about an arm or foot in length ranging from .58 meter upward. In this context, it is 50 times a certain unit area.

39. I.e. $\overline{H\d{H}}^2 = \frac{1}{4} X^2$.

40. He means minus instead of plus.

41. $\therefore \; 2\,\overline{MH}^2 = 2X^2 + \sqrt{\frac{4}{5} X^4}$

$$\therefore \; \overline{MH}^2 = \frac{1}{2} X^2 + \sqrt{\frac{1}{2} \cdot \frac{1}{10} X^4}$$

Draw $\overline{M\d{H}} \perp \overline{HD}$ $\therefore$ $\overline{\d{H}D} = \overline{H\d{H}} \Rightarrow \overline{H\d{H}} = \frac{1}{2} X$

$$\overline{HM}^2 - \overline{H\d{H}}^2 = \overline{M\d{H}}^2$$

$$\therefore \; \frac{1}{2} X^2 + \sqrt{\frac{1}{2} \cdot \frac{1}{10} X^4} - \frac{1}{4} X^2 = \frac{1}{4} X^2 + \sqrt{\frac{1}{20} X^4}$$

$$10^2 = \overline{M\d{H}}^2 \cdot \overline{H\d{H}}^2 = \frac{1}{2} \cdot \frac{1}{8} X^4 + \sqrt{\frac{1}{320} X^8} = 100$$

multiply by 16,

$$X^4 + \sqrt{\frac{256}{320}} X^4 = 1,600$$

$$X^4 + \sqrt{\frac{4}{5}} X^4 = 1,600$$

multiply both sides by $5 - \sqrt{20}$, we get:

$$X^4 = 8{,}000 - \sqrt{51{,}200{,}000}$$

After taking the 4th root $X = \left[8{,}000 - \sqrt{51{,}200{,}000}\right]^{1/4}$

42. This is called "extreme and mean ratio". Book VI def. 3.

43.

$$\frac{\overline{ZM}}{\overline{WZ'}} = \frac{\overline{ZW}}{\overline{WM}} \qquad \text{where } \overline{ZM} = \overline{MZ'}$$

44.

"Let AB be divided in extreme and mean ratio at C, AC being the greater segment; and let AD = $\frac{1}{2}$AB I say that (sq. on CD) = 5 (sq. on AD)"
Heath, p. 442, Book XIII Proposition I.

45. He says that $\left(\overline{ZM} - X + \frac{1}{2}X\right)^2 = 5\left(\frac{1}{2}X\right)^2$

$$\therefore \overline{ZM} - \frac{1}{2}X = \sqrt{\frac{5}{4}}\ X$$

$$\therefore \overline{ZM} = \frac{1}{2}X + \sqrt{1\frac{1}{4}\ X^2}$$

obviously, abū Kāmil has made use of a proportion to arrive at his conclusion. For example, if we use the proportion 43, we will have $\frac{\overline{ZM}}{X + 2\overline{ZM}} = \frac{X}{X + \overline{ZM}}$.

$$\therefore \overline{ZM}^2 - X \ \overline{ZM} - X^2 = 0 \Rightarrow \overline{ZM} = \frac{1}{2} X + \sqrt{1\frac{1}{4} X^2}$$

46. From his calculations, it is obvious that he means $\overline{ZS}^2 \times \overline{MS}^2 = 100$

47. $\overline{ZM}^2 - \overline{ZS}^2 = \overline{MS}^2$

$$\left(\frac{1}{2}X + \sqrt{1\frac{1}{4} X^2} \right)^2 - \left(\frac{1}{2}X\right)^2 = 1\frac{1}{4} X^2 + \sqrt{1\frac{1}{4} X^4}$$

$$\overline{MS} = \left[1\frac{1}{4} X^2 + \sqrt{1\frac{1}{4} X^4} \right]^{1/2}$$

$$\overline{ZS}^2 \cdot \overline{MS}^2 = \left(\frac{1}{4} X^2\right) \left(1\frac{1}{4} X^2 + \sqrt{1\frac{1}{4} X^4}\right)$$

$$= \frac{5}{16} X^4 + \sqrt{\frac{5}{64} X^8} = 100$$

multiply both sides by $3 + \frac{1}{5} = \frac{16}{5}$

we get $X^4 + \sqrt{\frac{4}{5} X^8} = 320$

multiply both sides by $5 - \sqrt{20}$ we get

$$X^4 = 1,600 - \sqrt{2,048,000}$$

$$X = \left[1,600 - \sqrt{2,048,000} \right]^{1/4}$$

48. $\overline{BL}^2 = 5 \ \overline{ML}^2$

$$\overline{MH} = X = \overline{HD} \Rightarrow \overline{LH} = \frac{1}{2} \overline{MH} = \frac{1}{2} X \Rightarrow \overline{LH}^2 = \frac{1}{4} X^2$$

$$\overline{BL}^2 = 1\frac{1}{4} X^2 \Rightarrow \overline{BL} = \sqrt{1\frac{1}{4} X^2}$$

$$\therefore\ \overline{BH} = \frac{1}{2}X + \sqrt{1\frac{1}{4}\ X^2}$$

$$\therefore\ \overline{BH}^2 - \overline{H\underset{\bullet}{H}}^2 = \overline{B\underset{\bullet}{H}}^2$$

$$\therefore\ 1\frac{1}{2}\ X^2 + \sqrt{1\frac{1}{4}X^4} - \frac{1}{4}X^2 = 1\frac{1}{4}X^2 + \sqrt{1\frac{1}{4}\ X^4}$$

$$\overline{B\underset{\bullet}{H}}^2 \times \overline{H\underset{\bullet}{H}}^2 = 100$$

$$\left(1\frac{1}{4}\ X^2 + \sqrt{1\frac{1}{4}\ X^4}\right)\ \frac{1}{4}\ X^2 = 100$$

$$\frac{5}{16}X^4 + \sqrt{\frac{5}{64}\ X^8} = 100$$

multiply both sides by $3 + \frac{1}{5} = \frac{16}{5}$

$X^4 + \sqrt{\frac{4}{5}X^8} = 320$ multiply by $5 - \sqrt{20}$

$$X^4 = 1,600 - \sqrt{2,048,000}$$

$$X = \left[1,600 - \sqrt{2,048,000}\right]^{1/4}$$

49. See note 44.

50. Let $\overline{BH} = X$

$$5\left(\frac{1}{2}\ \overline{BH}\right)^2 = 1\frac{1}{4}\ X^2$$

$$\therefore\ \overline{AH} = \sqrt{1\frac{1}{4}\ X^2} - \frac{1}{2}\ X$$

$$\overline{AH}^2 - \overline{EH}^2 = \overline{AE}^2 = 1\frac{1}{2}\ X^2 - \sqrt{1\frac{1}{4}X^4} - \frac{1}{4}\ X^2$$

$$= 1\frac{1}{4}\ X^2 - \sqrt{1\frac{1}{4}\ X^4}$$

$$\therefore \overline{AE}^2 \cdot \overline{HE}^2 = 10 \cdot 10 = 100$$

$$\therefore \frac{5}{16} X^4 - \sqrt{\frac{5}{64} X^8} = 100$$

multiply by $\frac{16}{5}$ we get $X^4 - \sqrt{\frac{4}{5} X^8} = 320$

$$\sqrt{5}\, X^4 - 2X^4 = 320 \cdot \sqrt{5}$$

$$X^4(\sqrt{5} - 2) = 320 \cdot \sqrt{5}$$

$$X^4 = \frac{320 \cdot \sqrt{5}}{\sqrt{5} - 2} = \frac{320 \cdot \sqrt{5}\,(\sqrt{5} + 2)}{(\sqrt{5} - 2)(\sqrt{5} + 2)}$$

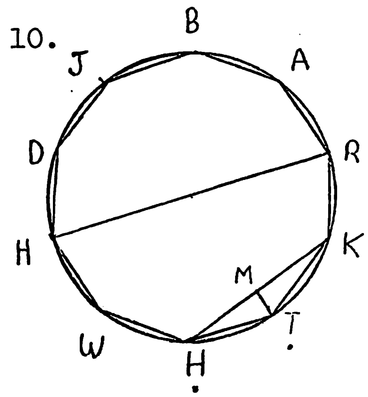
$$X^4 = 320 \cdot 5 + 320 \cdot 2 \cdot \sqrt{5}$$

$$X^4 = 1{,}600 + \sqrt{2{,}048{,}000}$$

$$X = \overline{BH} = \left(1{,}600 + \sqrt{2{,}048{,}000}\right)^{\frac{1}{4}}$$

51. Letter R has been added. It was missing.

52. "If an equilateral pentagon be inscribed in a circle, the square on the side of the pentagon is equal to the squares on the side of the hexagon and on that of the decagon inscribed in the same circle." Heath, p. 457; Euclid, Book XIII, Prop. 10.



53. Area $\overline{KTH} = 10$

Find $\overline{KH}$

Let $\overline{KH} = X$

$$\overline{RH}^2 = 2X^2 + \sqrt{\frac{4}{5} X^4}$$

$$\therefore \quad \frac{1}{2}\,\overline{RH}^2 = \frac{1}{2}X^2 \qquad \sqrt{\frac{1}{20}X^4}$$

$$\therefore \quad \overline{KT}^2 = \overline{KH}^2 - \frac{1}{2}\,\overline{RH}^2$$

$$\overline{KT}^2 = X^2 - \frac{1}{2}X^2 - \sqrt{\frac{1}{20}X^4}$$

$$\overline{KT}^2 = \frac{1}{2}X^2 - \sqrt{\frac{1}{20}X^4}$$

$$\overline{TM}^2 = \overline{TK}^2 - \overline{KM}^2 = \frac{1}{2}X^2 - \sqrt{\frac{1}{20}X^4} - \frac{1}{4}X^2$$

$$= \frac{1}{4}X^2 - \sqrt{\frac{1}{20}X^4}$$

$$\overline{TM}^2 \cdot \overline{KM}^2 = 10 \cdot 10 = 100$$

$$\left(\frac{1}{4}X^2 - \sqrt{\frac{1}{20}X^4}\right) \cdot \frac{1}{4}X^2 = 100$$

$$\frac{1}{16}X^4 - \sqrt{\frac{1}{320}X^8} = 100 \implies X^4 - \sqrt{\frac{4}{5}X^8} = 1{,}600$$

$$\left(5 + \sqrt{20}\right)\left(X^4 - \sqrt{\frac{4}{5}X^8}\right) = 1{,}600\left(5 + \sqrt{20}\right)$$

$$5X^4 - 5\sqrt{\frac{4}{5}X^8} + \sqrt{20}\,X^4 - \sqrt{\frac{20 \cdot 4}{5}X^8}$$

$$= 8{,}000 + \sqrt{51{,}200{,}000}$$

$$X^4 = 8{,}000 + \sqrt{51{,}200{,}000}$$

$$X = \left(8{,}000 + \sqrt{51{,}200{,}000}\right)^{1/4} = \overline{KH}$$

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Supplement 2

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