An reaction-diffusion approximation of a semilinear wave equation

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Abstract

A semilinear wave equation possesses propagation with a finite speed, while a parabolic equation has propagation with infinite speed. This paper proposes a reaction-diffusion system whose solutions approximate those of a semilinear wave equation under some assumptions of a reaction term. The proof is based on the energy method.

1 Introduction

Wave equations are often used in optics and vibration theory. A typical example is

$$u_{tt} = c^2 u_{xx}$$

in $\mathbb{R}$, where the subscript x or t indicates that the derivatives are taken with respect to the variable x or t. For example, $u_{tt} = \partial^2 u/\partial t^2$, $u_{xx} = \partial^2 u/\partial x^2$. This equation has the solution

$$u(x,t) = g_1(x - ct) + g_2(x + ct)$$

where g_1 and g_2 are arbitrary C^2 functions. Finite propagation is one of the most important properties of the wave equation. On the other hand, the

The first author is partially supported by JSPS KAKENHI Grant Numbers 26287024, 15K04963, 16K13778, 16KT0022.

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parabolic equation has propagation with infinite speed. Because the heat kernel is positive everywhere, the solution to the heat equation instantly becomes positive, even if the initial distribution has compact support. In this paper, we consider the following problem:

Can a semilinear wave equation be approximated by a reaction-diffusion system?

We found the answer to be in the affirmative. This means that finite propagation can be approximated by infinite propagation. The converse, or the approximation of infinite propagation by finite propagation, was studied by Holmes [8]. This study was motivated by the mathematical modeling of animal locomotion. Diffusion models are often used to describe animal distributions, even though an individual animal moves with a finite speed. To discuss this issue with animal movements, Holmes [8] compared the telegraph equation having finite propagation with the diffusion equation having infinite propagation (see also [5]). She showed that the invasion wave speeds of both equations are very close numerically. We emphasize that, for some parameters, the reaction-telegraph equation in [8] becomes a wave equation.

Returning to our problem, we need to construct an appropriate reaction-diffusion system that approximates a semilinear wave equation. This approximation is often called the *reaction-diffusion approximation*. Combining the reactions of components and the diffusivity enables us to approximate the wave equation. This also helps us to understand the solution space of reaction-diffusion systems. Singular limit problems have been studied for several decades. Hilhorst et al. [6] discussed the following singular limit problem:

$$\begin{cases} u_t &= u_{xx} - \dfrac{1}{\varepsilon}uv, \\ v_t &= -\dfrac{1}{\varepsilon}uv, \end{cases}$$

which approximates the one-phase Stefan problem as $\varepsilon \to 0$. This implies that the Stefan problem can be approximated by a reaction-diffusion system. Since their work, many researcher have investigated this kind of singular limit problem, which is called the *fast reaction limit*. For example, a nonlinear diffusion equation such as a cross-diffusion system can also be approximated by a reaction-diffusion system. See [1, 10, 9, 13] and the references in [7]. Recently, Ninomiya et al. [15] successfully constructed a reaction-diffusion system that approximates a non-local evolutionary equation with an even kernel. For singular limit problems of hyperbolic systems, see [11, 14] and the references therein.

In this paper, we consider a reaction-diffusion approximation of the following initial value problem of a semilinear wave equation in $\mathbb{R}^N$:

$$
\begin{cases} w_{tt} = d\Delta w + f(w) & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x) & \text{on } \mathbb{R}^N, \end{cases} \tag{1.1}
$$

where d and T are positive constants and $\Delta u = \sum_{k=1}^N \partial^2 u/\partial x_k^2$. Let $w = w(x,t)$ be a solution to (1.1). We assume that the function $f(u)$ satisfies the following:

$$
\begin{aligned}
&\text{(H1)} && f \in C^1(\mathbb{R}) \ \ \text{(it may allow } \|f\|_{C^1(\mathbb{R})} = \infty), \\
&\text{(H2)} && f(u)u \le f_1|u|^2, \\
&\text{(H3)} && -f_2\left(|u|^2 + |u|^{p+1}\right) \le F(u) \le f_3|u|^2, \\
&\text{(H4)} && |f'(u)| \le f_4 + f_5|u|^q \quad \text{ if } 1 \le N \le 4, \\
&\text{(H5)} && \left|\frac{F(u)}{f(u)}\right| \to \infty \quad \text{ as } u \to \infty,
\end{aligned}
$$

where $p > 1$, $f_j \ge 0$ are constant for $j = 1, \dots, 5$,

$$
\begin{cases} q \ge 0 & \text{if } N = 1, 2, \\ 0 \le q \le \dfrac{N}{2(N-2)} & \text{if } N = 3, 4 \end{cases}
$$

and

$$
F(u) = \int_0^u f(s)\, ds.
$$

A typical example of f is $f(u) = au - |u|^{p-1}u$ where $a \in \mathbb{R}$ and p satisfies

$$
\begin{cases} 1 \le p & \text{if } N = 1, 2, \\ 1 \le p \le \dfrac{5}{2} & \text{if } N = 3, \\ 1 \le p \le 2 & \text{if } N = 4. \end{cases}
$$

***Remark* 1.1.** From (H2) it follows that $f(0) = 0$. It is known that, for $(w_0, w_1) \in H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, there exists a unique solution of (1.1) satisfying

$w \in C^1(\mathbb{R}; H^1(\mathbb{R}^N)) \cap C(\mathbb{R}; H^2(\mathbb{R}^N))$ under the hypotheses (H1), the second inequality of (H3), (H5) and (H4) with

$$\begin{cases} q \geq 0 & \text{if } N = 1, 2, \\ 0 \leq q < \dfrac{4}{N-2} & \text{if } N \geq 3. \end{cases}$$

For example, see [16, Theorems 1–3 in Chap. 3], [2, 4] and references therein. The hypothesis (H5) is only used to guarantee the existence of solutions of the semilinear wave equation (1.1). If the global existence of solutions of semilinear wave equation is guaranteed by (H1)–(H4) together with the alternative condition instead of (H5), (H5) can be replaced by that. We also remark that the semilinear wave equation (1.1) has energy conservation law

$$\frac{d}{dt} \int_{\mathbb{R}^N} \left(\frac{1}{2}|w_t|^2 + \frac{d}{2}|\nabla w|^2 - F(w) \right) dx = 0. \tag{1.2}$$

We propose the following initial value problem of a reaction-diffusion system in order to approximate (1.1):

$$\begin{cases} u_{1,t} = d_1 \varepsilon \Delta u_1 + \dfrac{u_2 - u_1}{\varepsilon}, & \\ & \text{in } \mathbb{R}^N \times (0, T), \\ u_{2,t} = d_2 \varepsilon \Delta u_2 + \dfrac{u_2 - u_1}{\varepsilon} + \varepsilon f(u_1) & \\ u_1(x, 0) = w_0, \quad u_2(x, 0) = w_0 + \varepsilon w_1, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ is small and w_0, w_1 are the initial data of (1.1). Throughout this paper, we also assume that $1 \leq N \leq 4$ and $d_2 > d_1 \geq 0$ and we set $d := d_2 - d_1 > 0$.

In the case of $f(u) \equiv 0$, we explain why we chose the reaction term of (1.3) as $(u_2 - u_1)/\varepsilon$ below. Compared the wave equation with the diffusion equation, we see that the wave equation contains the second-order derivative of the solution with respect to t. Hence, we need to produce the derivative of the solution with respect to t from the reaction term of the system. If $u_2(x, t) \approx u_1(x, t + \varepsilon)$, then

$$u_{1,t}(x, t) \approx \frac{u_1(x, t + \varepsilon) - u_1(x, t)}{\varepsilon} \approx \frac{u_2(x, t) - u_1(x, t)}{\varepsilon}.$$

Borrowing the ideas of Hilhorst et al. [6] and Iida and Ninomiya [10], we consider a reaction-diffusion system that consists of the same reaction term

$(u_2 - u_1)/\varepsilon$ and different diffusion coefficients. That is, we propose

$$u_{1,t} = d_1^\varepsilon \Delta u_1 + \frac{u_2 - u_1}{\varepsilon}, \qquad u_{2,t} = d_2^\varepsilon \Delta u_2 + \frac{u_2 - u_1}{\varepsilon} \tag{1.4}$$

with diffusion coefficients d_j^ε that are specified later. By subtracting the equation of u_1 from that of u_2, we obtain

$$\varepsilon \left(u_{2,t} - u_{1,t} \right) = \left(d_2^\varepsilon - d_1^\varepsilon \right) \Delta u_1 + d_2^\varepsilon \left(\Delta u_2 - \Delta u_1 \right).$$

By $u_{2,t} - u_{1,t} \approx u_{1,t}(t+\varepsilon) - u_{1,t}(t) \approx \varepsilon u_{1,tt}$ and $\Delta u_2 - \Delta u_1 \approx \Delta u_1(t+\varepsilon) - \Delta u_1(t) \approx \varepsilon \Delta u_{1,t}$, we see that

$$u_{1,tt} \approx \frac{d_2^\varepsilon - d_1^\varepsilon}{\varepsilon} \Delta u_1 + d_2^\varepsilon \Delta u_{1,t}.$$

By choosing the diffusion coefficients $d_j^\varepsilon = d_j \varepsilon$ for $j = 1, 2$ where $d_2 > d_1 > 0$, we obtain

$$u_{1,tt} \approx d \Delta u_1 + d_2 \varepsilon \Delta u_{1,t} \approx d \Delta u_1$$

where $d = d_2 - d_1$. Thus, we see that u_1 of (1.4) may converge to the solution of a wave equation.

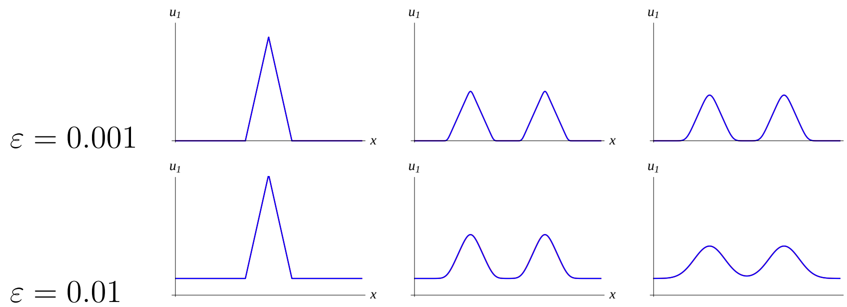
***Remark* 1.2.** Let $d_1, d_2 > 0$ and $1 \leq N \leq 4$. For arbitrary $\varepsilon > 0$, $w_0 \in H^4(\mathbb{R}^N)$ and $w_1 \in H^3(\mathbb{R}^N)$, the local existence of a solution $(u_1^\varepsilon, u_2^\varepsilon)$ of (1.3) follows from the standard semigroup theory ([12]) and the Sobolev embedding.

Theorem 1.3. *Let $1 \leq N \leq 4$. Assume that* (H1)–(H5) *are satisfied, $d_2 > d_1 \geq 0$ and $w_0 \in H^4(\mathbb{R}^N)$, $w_1 \in H^3(\mathbb{R}^N)$. Let w be the solution to* (1.1). *Then, for any positive constant T, there exists a unique solution $(u_1^\varepsilon, u_2^\varepsilon)$ of* (1.3) *in $\mathbb{R}^N \times (0, T)$. Moreover, the following holds:*

$$\sup_{t \in [0,T]} \left\{ \|w - u_1^\varepsilon\|_{H^1(\mathbb{R}^N)} + \left\| w_t - u_{1,t}^\varepsilon \right\|_{L^2(\mathbb{R}^N)} \right\} = O\left(\varepsilon^{1/2} \right)$$

as $\varepsilon \to 0$.

As shown in Figure 1, the numerical solution of (1.3) becomes close to that of (1.1) as ε tends to zero, and the approximation becomes worse as t increases because of the dissipativity.



(a) $t = 0$ (b) $t = 0.2$ (c) $t = 1.2$

Figure 1: Snapshots of the first component of the solution to (1.3) in the cases of $f \equiv 0$ and $\varepsilon = 0.001,\ 0.01$ in the interval $(0, 10)$ under the homogeneous Neumann boundary condition.

2 Boundedness of solutions of a reaction-diffusion system

Now, we show that the principal component u_1 of the solution for (1.3) approximates the solution w of (1.1). We use

$$v_1 := u_1, \qquad v_2 := \frac{u_2 - u_1}{\varepsilon}.$$

Let the pair (v_1, v_2) satisfy the following initial value problem:

$$\begin{cases} v_{1,t} = d_1\varepsilon\Delta v_1 + v_2, & \\ v_{2,t} = d_2\varepsilon\Delta v_2 + d\Delta v_1 + f(v_1), & \text{in } \mathbb{R}^N \times (0,T), \\ v_1(\cdot,0) = w_0, \quad v_2(\cdot,0) = w_1 & \text{in } \mathbb{R}^N. \end{cases} \tag{2.1}$$

Here, we note that $d := d_2 - d_1 > 0$. Based on (2.1), we see that, formally, $v_{1,t}$ tends to v_2 and $v_{2,t}$ tends to $d\Delta v_1 + f(v_1)$ as ε goes to zero. Therefore, we can conjecture that $u_1 = v_1$ satisfies the wave equation $v_{1,tt} = d\Delta v_1 + f(v_1)$ because $v_{1,tt} \to v_{2,t} \to d\Delta v_1 + f(v_1)$ as $\varepsilon \to 0$.

We also differentiate (2.1) with respect to t:

$$v_{1,t} = d_1\varepsilon\Delta v_1 + v_2, \tag{2.2}$$

$$v_{2,t} = d_2\varepsilon\Delta v_2 + d\Delta v_1 + f(v_1), \tag{2.3}$$

$$v_{1,tt} = d_1\varepsilon\Delta v_{1,t} + v_{2,t} \tag{2.4}$$

$$v_{2,tt} = d_2\varepsilon\Delta v_{2,t} + d\Delta v_{1,t} + f'(v_1)v_{1,t}. \tag{2.5}$$

By a priori estimation, we show the boundedness and the convergence of a solution for the initial value problem (2.1).

In order to prove the boundedness of a solution for (2.1), we prove several lemmas:

Lemma 2.1. *For a solution* (v_1, v_2) *of* (2.1), *we have the following*:

(2.6)
$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}^N} v_{1,t}^2\,dx + \frac{d_1 d_2}{2}\varepsilon^2\int_{\mathbb{R}^N}(\Delta v_1)^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N}|\nabla v_1|^2\,dx - \int_{\mathbb{R}^N}F(v_1)\,dx\right\}$$
$$= -(d_1+d_2)\varepsilon\int_{\mathbb{R}^N}|\nabla v_{1,t}|^2\,dx,$$

(2.7)
$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_{1,t}|^2\,dx + \frac{d_1 d_2}{2}\varepsilon^2\int_{\mathbb{R}^N}|\nabla(\Delta v_1)|^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N}(\Delta v_1)^2\,dx\right\}$$
$$\le \int_{\mathbb{R}^N} f'(v_1)\nabla v_1\cdot\nabla v_{1,t}\,dx,$$

(2.8)
$$\frac{d}{dt}\left\{\varepsilon\int_{\mathbb{R}^N}(\Delta v_{1,t})^2\,dx + d_1 d_2\varepsilon^3\int_{\mathbb{R}^N}(\Delta^2 v_1)^2\,dx + d\varepsilon\int_{\mathbb{R}^N}|\nabla(\Delta v_1)|^2\,dx\right\}$$
$$\le \frac{1}{2(d_1+d_2)}\int_{\mathbb{R}^N}|f'(v_1)|^2\,|\nabla v_1|^2\,dx.$$

Proof. First, we show (2.6). By substituting (2.3) for (2.4), we obtain

$$v_{1,tt} = d_1\varepsilon\Delta v_{1,t} + d_2\varepsilon\Delta v_2 + d\Delta v_1 + f(v_1).$$

From (2.2), it holds that

$$v_2 = v_{1,t} - d_1\varepsilon\Delta v_1,$$
$$\Delta v_2 = \Delta v_{1,t} - d_1\varepsilon\Delta(\Delta v_1).$$

Gathering these equalities implies that

$$v_{1,tt} = d_1\varepsilon\Delta v_{1,t} + d_2\varepsilon\left\{\Delta v_{1,t} - d_1\varepsilon\Delta(\Delta v_1)\right\} + d\Delta v_1 + f(v_1).$$

Hence, we obtain

$$v_{1,tt} = -d_1 d_2\varepsilon^2\Delta^2 v_1 + (d_1+d_2)\varepsilon\Delta v_{1,t} + d\Delta v_1 + f(v_1). \tag{2.9}$$

Note that (2.9) is represented only by v_1. By multiplying (2.9) with $v_{1,t}$ and integrating it over $\mathbb{R}^N$, we have

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}^N} v_{1,t}^2\,dx + \frac{d_1d_2}{2}\varepsilon^2\int_{\mathbb{R}^N}(\Delta v_1)^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N}|\nabla v_1|^2\,dx - \int_{\mathbb{R}^N}F(v_1)\,dx\right\}$$
$$= -(d_1+d_2)\varepsilon\int_{\mathbb{R}^N}|\nabla v_{1,t}|^2\,dx.$$

Next, let us show the inequality (2.7). Multiplying (2.9) by $\Delta v_{1,t}$ and integrating it over $\mathbb{R}^N$ yields that

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_{1,t}|^2\,dx + \frac{d_1d_2}{2}\varepsilon^2\int_{\mathbb{R}^N}|\nabla(\Delta v_{1,t})|^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N}(\Delta v_1)^2\,dx\right\}\\ &= -(d_1+d_2)\varepsilon\int_{\mathbb{R}^N}(\Delta v_{1,t})^2\,dx + \int_{\mathbb{R}^N}f'(v_1)\nabla v_1\cdot\nabla v_{1,t}\,dx.\end{aligned}$$

Since $\int_{\mathbb{R}^N}(\Delta v_{1,t})^2\,dx$ is non-negative, we obtain (2.7).

Finally, we prove (2.8). By applying the Laplacian Δ to (2.9), we have

$$\Delta v_{1,tt} = -d_1d_2\varepsilon^2\Delta^3 v_1 + (d_1+d_2)\varepsilon\Delta^2 v_{1,t} + d\Delta^2 v_1 + \nabla\cdot\left(f'(v_1)\nabla v_1\right).$$

Multiplying this equation by $\Delta v_{1,t}$ and integrating it over $\mathbb{R}^N$ yield

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}^N}(\Delta v_{1,t})^2\,dx + \frac{d_1d_2}{2}\varepsilon^2\int_{\mathbb{R}^N}(\Delta^2 v_1)^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N}|\nabla(\Delta v_1)|^2\,dx\right\}\\ &= -(d_1+d_2)\varepsilon\int_{\mathbb{R}^N}|\nabla(\Delta v_{1,t})|^2\,dx - \int_{\mathbb{R}^N}f'(v_1)\nabla v_1\cdot\nabla(\Delta v_{1,t})\,dx\\ &\le \frac{1}{4(d_1+d_2)\varepsilon}\int_{\mathbb{R}^N}|f'(v_1)|^2\,|\nabla v_1|^2\,dx.\end{aligned}$$

We multiply this by 2ε to obtain (2.8). $\square$

We prepare the following lemma to derive a priori bounds.

Lemma 2.2. *Assume that A, B, and g are non-negative constants. Suppose that X is a non-negative C^2 function, and Y is a non-negative C^1 function that satisfies*

$$X'' + AY' \le g^2X + B \tag{2.10}$$

for $t > 0$ where the prime $'$ denotes the differentiation with respect to the time t. If $g > 0$, then

$$X(t) \le X(0)\cosh gt + \frac{1}{g}\{X'(0) + AY(0)\}\sinh gt + \frac{B}{g^2}(\cosh gt - 1)\ ; \tag{2.11}$$

If $g = 0$, then

$$X(t) \le X(0) + (X'(0) + AY(0))t + \frac{B}{2}t^2. \tag{2.12}$$

Proof. First we consider the case of $g = 0$. By (2.10), we have

$$X'(t) + AY(t) \le X'(0) + AY(0) + Bt.$$

Because $X'(t) \le X'(0) + AY(0) + Bt$ from $Y(t) \ge 0$, by integrating it once more, we obtain (2.12).

Next, we consider the case where $g > 0$. By multiplying (2.10) with e^{-gt}, it follows that

$$\left(X'e^{-gt} + gXe^{-gt}\right)' + A(Ye^{-gt})' + AgYe^{-gt} \le Be^{-gt}.$$

Because $AgYe^{-gt} \ge 0$ holds, we obtain

$$\left(X'e^{-gt} + gXe^{-gt}\right)' + A(Ye^{-gt})' \le Be^{-gt}.$$

Integrating this over $[0,t]$ yields

$$\left(X'e^{-gt} + gXe^{-gt}\right) - (X'(0) + gX(0)) + A\left(Ye^{-gt} - Y(0)\right) \le -\frac{B}{g}\left(e^{-gt} - 1\right).$$

Noting that $Ye^{-gt} \ge 0$, we have

$$X'e^{-gt} + gXe^{-gt} \le X'(0) + gX(0) + AY(0) - \frac{B}{g}\left(e^{-gt} - 1\right).$$

By multiplying this with e^{2gt}, integrating it over $[0,t]$ and then multiplying it with e^{-gt}, we can calculate the following:

$$\begin{aligned}
X(t) - X(0)e^{-gt} &\le \frac{1}{2g}\{X'(0) + gX(0) + AY(0)\}(e^{gt} - e^{-gt}) \\
&\quad -\frac{B}{g}\left(\frac{1}{g} - \frac{1}{g}e^{-gt} - \frac{1}{2g}e^{gt} + \frac{1}{2g}e^{-gt}\right) \\
&= X(0)\left\{\frac{1}{2}e^{gt} - \frac{1}{2}e^{-gt}\right\} + \frac{1}{g}\{X'(0) + AY(0)\}\sinh(gt) \\
&\quad +\frac{B}{g^2}(\cosh(gt) - 1).
\end{aligned}$$

Therefore, we obtain (2.11). □

Lemma 2.3. *Assume that $w_0 \in H^3(\mathbb{R}^N)$ and $w_1 \in H^1(\mathbb{R}^N)$. Let $d_2 > d_1 \geq 0$, $\varepsilon_0 > 0$ and $T > 0$. Let (v_1, v_2) be a solution of (2.1) for $\varepsilon \in (0, \varepsilon_0]$. Then, the following integrations are bounded for any $t \in [0, T]$ and $\varepsilon \in (0, \varepsilon_0]$:*

$$\int_{\mathbb{R}^N} v_1^2\,dx, \quad \int_{\mathbb{R}^N} v_{1,t}^2\,dx, \quad \int_{\mathbb{R}^N} |\nabla v_1|^2\,dx, \quad \int_{\mathbb{R}^N} |\nabla v_{1,t}|^2\,dx, \quad \int_{\mathbb{R}^N} (\Delta v_1)^2\,dx.$$

Proof. By (2.6) of Lemma 2.1, it follows that

$$\begin{aligned}\frac{d}{dt}\Bigg\{\frac{1}{2}\int_{\mathbb{R}^N} v_{1,t}^2\,dx + \frac{d_1 d_2}{2}\varepsilon^2 \int_{\mathbb{R}^N} (\Delta v_1)^2\,dx \\ + \frac{d}{2}\int_{\mathbb{R}^N} |\nabla v_1|^2\,dx - \int_{\mathbb{R}^N} F(v_1)\,dx\Bigg\} \leq 0.\end{aligned}$$

Hence, by integrating this with respect to t over $[0, t]$, we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^N} v_{1,t}^2\,dx + \frac{d_1 d_2}{2}\varepsilon^2 \int_{\mathbb{R}^N} (\Delta v_1)^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N} |\nabla v_1|^2\,dx - \int_{\mathbb{R}^N} F(v_1)\,dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} w_1^2\,dx + \frac{d_1 d_2}{2}\varepsilon^2 \int_{\mathbb{R}^N} (\Delta w_0)^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N} |\nabla w_0|^2\,dx - \int_{\mathbb{R}^N} F(w_0)\,dx.\end{aligned}$$

Because $-f_3|v_1|^2 \leq -F(v_1)$ holds from (H3),

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^N} v_{1,t}^2\,dx + \frac{d_1 d_2}{2}\varepsilon^2 \int_{\mathbb{R}^N} (\Delta v_1)^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N} |\nabla v_1|^2\,dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} w_1^2\,dx + \frac{d_1 d_2}{2}\varepsilon^2 \int_{\mathbb{R}^N} (\Delta w_0)^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N} |\nabla w_0|^2\,dx \\ &\quad - \int_{\mathbb{R}^N} F(w_0)\,dx + f_3\int_{\mathbb{R}^N} v_1^2\,dx.\end{aligned}$$

By doubling this and using the boundedness of w_0 and w_1, we obtain

$$\int_{\mathbb{R}^N} v_{1,t}^2\,dx + d_1 d_2 \varepsilon^2 \int_{\mathbb{R}^N} (\Delta v_1)^2\,dx + d\int_{\mathbb{R}^N} |\nabla v_1|^2\,dx \leq 2f_3\int_{\mathbb{R}^N} v_1^2\,dx + C_1, \tag{2.13}$$

where

$$\begin{aligned}C_1 := \int_{\mathbb{R}^N} w_1^2\,dx + d_1 d_2\varepsilon^2 \int_{\mathbb{R}^N} (\Delta w_0)^2\,dx \\ + d\int_{\mathbb{R}^N} |\nabla w_0|^2\,dx - 2\int_{\mathbb{R}^N} F(w_0)\,dx.\end{aligned} \tag{2.14}$$

By $w_0 \in H^2(\mathbb{R}^N)$, we use the Sobolev embedding $H^2(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for all $2 \le r < \infty$ when $1 \le N \le 4$. Together with $-F(w_0) \le f_2(|w_0|^2 + |w_0|^{p+1})$ in (H3), we see that

$$\begin{aligned}
-\int_{\mathbb{R}^N} F(w_0)\, dx &\le f_2 \int_{\mathbb{R}^N} \left(|w_0|^2 + |w_0|^{p+1}\right)\, dx \\
&\le C\left(\|w_0\|_{H^2(\mathbb{R}^N)}^2 + \|w_0\|_{H^2(\mathbb{R}^N)}^{p+1}\right).
\end{aligned}$$

Thus, the upper bound of C_1 is finite for any $t \in [0, T]$ and $\varepsilon \in (0, \varepsilon_0]$.

Next, by multiplying (2.9) with v_1 and integrating it over $\mathbb{R}^N$, we can calculate the following:

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left\{\frac{\partial}{\partial t}(v_1 v_{1,t}) - v_{1,t}^2\right\} dx \\
&= -d_1 d_2 \varepsilon^2 \int_{\mathbb{R}^N} (\Delta v_1)^2\, dx - \frac{d}{dt}\left\{\frac{(d_1+d_2)\varepsilon}{2}\int_{\mathbb{R}^N} |\nabla v_1|^2\, dx\right\} \\
&\quad -d\int_{\mathbb{R}^N} |\nabla v_1|^2\, dx + \int_{\mathbb{R}^N} f(v_1) v_1\, dx.
\end{aligned}$$

Hence, we obtain the following identity:

$$\begin{aligned}
&\frac{d^2}{dt^2}\left(\frac{1}{2}\int_{\mathbb{R}^N} v_1^2\, dx\right) + \frac{d}{dt}\left\{\frac{(d_1+d_2)\varepsilon}{2}\int_{\mathbb{R}^N} |\nabla v_1|^2\, dx\right\} \\
&= \int_{\mathbb{R}^N} v_{1,t}^2\, dx - d_1 d_2 \varepsilon^2 \int_{\mathbb{R}^N} (\Delta v_1)^2\, dx - d\int_{\mathbb{R}^N} |\nabla v_1|^2\, dx + \int_{\mathbb{R}^N} f(v_1) v_1\, dx.
\end{aligned}$$

By (2.13), it follows that

$$\int_{\mathbb{R}^N} v_{1,t}^2\, dx \le 2f_3 \int_{\mathbb{R}^N} v_1^2\, dx - d\int_{\mathbb{R}^N} |\nabla v_1|^2\, dx + C_1.$$

Therefore, we see that

$$\begin{aligned}
&\frac{d^2}{dt^2}\left(\frac{1}{2}\int_{\mathbb{R}^N} v_1^2\, dx\right) + \frac{d}{dt}\left\{\frac{(d_1+d_2)\varepsilon}{2}\int_{\mathbb{R}^N} |\nabla v_1|^2\, dx\right\} \\
&\le 2f_3 \int_{\mathbb{R}^N} v_1^2\, dx - 2d\int_{\mathbb{R}^N} |\nabla v_1|^2\, dx + \int_{\mathbb{R}^N} f(v_1) v_1\, dx + C_1.
\end{aligned}$$

By (H2), we also see that

$$\begin{aligned}&\frac{d^2}{dt^2}\left(\int_{\mathbb{R}^N} v_1^2\,dx\right)+\frac{d}{dt}\left\{(d_1+d_2)\varepsilon\int_{\mathbb{R}^N}|\nabla v_1|^2\,dx\right\}\\&\leq (4f_3+2f_1)\int_{\mathbb{R}^N} v_1^2\,dx-4d\int_{\mathbb{R}^N}|\nabla v_1|^2\,dx+2C_1.\end{aligned}$$

Here, we set $g_1=\sqrt{4f_3+2f_1}$. Then, the following holds:

(2.15)

$$\frac{d^2}{dt^2}\left(\int_{\mathbb{R}^N} v_1^2\,dx\right)+\frac{d}{dt}\left\{(d_1+d_2)\varepsilon\int_{\mathbb{R}^N}|\nabla v_1|^2\,dx\right\}\leq g_1^2\int_{\mathbb{R}^N} v_1^2\,dx+2C_1$$

To simplify the description, let

$$X(t):=\int_{\mathbb{R}^N} v_1^2\,dx \quad \text{and} \quad Y(t):=\int_{\mathbb{R}^N}|\nabla v_1|^2\,dx.$$

For each $t\geq 0$, $X(t)$ and $Y(t)$ are non-negative and (2.15) becomes

$$X''(t)+(d_1+d_2)\varepsilon Y'(t)\leq g_1^2X(t)+2C_1. \tag{2.16}$$

Hence, Lemma 2.2 is applicable to (2.16). If $g_1=0$, then using (2.11) or (2.12) with $A=(d_1+d_2)\varepsilon$, $B=2C_1$, and $g=0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} v_1^2\,dx \;&\leq\; \int_{\mathbb{R}^N} w_0^2\,dx+2T\int_{\mathbb{R}^N}|w_0||w_1|\,dx\\&\quad+(d_1+d_2)\varepsilon T\int_{\mathbb{R}^N}|\nabla w_0|^2\,dx+C_1T^2 \quad \text{for } t\in[0,T],\end{aligned}$$

which implies the boundedness of $\int_{\mathbb{R}^N} v_1^2\,dx$.

Next, we consider the case where $g_1>0$. Lemma 2.2 implies that

$$\begin{aligned}X(t)\;&\leq\; X(0)\cosh(g_1t)+\frac{1}{g_1}\left\{X'(0)+(d_1+d_2)\varepsilon Y(0)\right\}\sinh(g_1t)\\&\quad+\frac{2C_1}{g_1^2}\left(\cosh(g_1t)-1\right).\end{aligned}$$

By (2.14), we have the boundedness of the initial data of X, Y and X' as follows:

$$\begin{aligned}X(0)&=\int_{\mathbb{R}^N} w_0^2\,dx,\\Y(0)&=\int_{\mathbb{R}^N}|\nabla w_0|^2\,dx,\\X'(0)&=2\int_{\mathbb{R}^N} w_0w_1\,dx\leq\int_{\mathbb{R}^N} w_0^2\,dx+\int_{\mathbb{R}^N} w_1^2\,dx.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} v_1^2\,dx &\le X(0)\cosh(g_1 t) + \frac{1}{g_1}\left\{X'(0) + (d_1+d_2)\varepsilon Y(0)\right\}\sinh(g_1 t)\\ &\quad + \frac{2C_1}{g_1^2}\left(\cosh(g_1 t) - 1\right).\end{aligned}$$

Because $\cosh(g_1 t)$ and $\sinh(g_1 t)$ are increasing for all $t \in [0,T]$, we see that

$$\int_{\mathbb{R}^N} v_1^2\,dx \le C_2\cosh(g_1 T) + C_3\sinh(g_1 T) =: K_{1,T} \quad \text{for } t \in [0,T] \tag{2.17}$$

where

$$\begin{aligned} C_2 &= \int_{\mathbb{R}^N} w_0^2\,dx + \frac{2C_1}{g_1^2},\\ C_3 &= \frac{1}{g_1}\left\{\int_{\mathbb{R}^N} w_0^2\,dx + \int_{\mathbb{R}^N} w_1^2\,dx + (d_1+d_2)\varepsilon_0\int_{\mathbb{R}^N}|\nabla w_0|^2\,dx\right\}.\end{aligned}$$

By (2.13), it holds for all $t \in [0,T]$ that

$$\int_{\mathbb{R}^N} v_{1,t}^2\,dx + d_1 d_2\varepsilon^2\int_{\mathbb{R}^N}(\Delta v_1)^2\,dx + d\int_{\mathbb{R}^N}|\nabla v_1|^2\,dx \le 2f_3 K_{1,T} + C_1. \tag{2.18}$$

Therefore,

$$\int_{\mathbb{R}^N} v_1^2\,dx, \qquad \int_{\mathbb{R}^N} v_{1,t}^2\,dx \quad \text{and} \quad \int_{\mathbb{R}^N}|\nabla v_1|^2\,dx$$

are bounded for any $t \in [0,T]$.

Next, we show the boundedness of $\int_{\mathbb{R}^N}|\nabla v_{1,t}|^2\,dx$ and $\int_{\mathbb{R}^N}(\Delta v_1)^2\,dx$ for all $t \in [0,T]$. We note that $v_1 \in H^1(\mathbb{R}^N)$ for all $t \in [0,T]$ by the previous proof. We estimate the the right-hand side of (2.7) as follows:

$$\int_{\mathbb{R}^N} f'(v_1)\nabla v_1\cdot\nabla v_{1,t}\,dx \le \int_{\mathbb{R}^N}|\nabla v_{1,t}|^2\,dx + \int_{\mathbb{R}^N}|f'(v_1)|^2|\nabla v_1|^2\,dx. \tag{2.19}$$

In the case of $N = 1$, by $v_1 \in H^1(\mathbb{R}) \hookrightarrow BC(\mathbb{R})$ and (H1), $|f'(v_1)|^2$ is bounded for all $t \in [0,T]$. Hence, there exists a constant $C_f > 0$ depending on f such that

$$\int_{\mathbb{R}^N} f'(v_1)\nabla v_1\cdot\nabla v_{1,t}\,dx \le \int_{\mathbb{R}^N}|\nabla v_{1,t}|^2\,dx + C_f\int_{\mathbb{R}^N}|\nabla v_1|^2\,dx.$$

To estimate $\int_{\mathbb{R}^N} |f'(v_1)|^2|\nabla v_1|^2\,dx$, we consider the assumption (H4). By considering the cases $|u| \ge 1$ and $|u| < 1$, (H4) holds with $q = 1$ when $0 \le q < 1$. Thus we can assume that $q \ge 1$. Hence, using (H4), we estimate the following:

$$\begin{aligned}\int_{\mathbb{R}^N} |f'(v_1)|^2|\nabla v_1|^2\,dx &\le \int_{\mathbb{R}^N} (f_4 + f_5|v_1|^q)^2\,|\nabla v_1|^2\,dx \\ &\le 2\int_{\mathbb{R}^N} \left({f_4}^2 + f_5^2|v_1|^{2q}\right)|\nabla v_1|^2\,dx \\ &\le 2{f_4}^2\|\nabla v_1\|^2_{L^2(\mathbb{R}^N)} + 2f_5^2\|v_1\|^2_{L^{4q}(\mathbb{R}^N)}\|\nabla v_1\|^2_{L^4(\mathbb{R}^N)}.\end{aligned}$$

Consider the case when $N = 1, 2$. Since $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $2 \le r < \infty$, $\|v_1\|_{L^{4q}(\mathbb{R}^N)} \le C_S\|v_1\|_{H^1(\mathbb{R}^N)}$ holds. If $N = 3, 4$, then $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $2 \le r \le 2N/(N-2)$. Noting that $4q \le 2N/(N-2)$, we also obtain $\|v_1\|_{L^{4q}(\mathbb{R}^N)} \le C_S\|v_1\|_{H^1(\mathbb{R}^N)}$. Therefore, $\|v_1\|_{L^{4q}(\mathbb{R}^N)}$ is bounded for all $t \in [0, T]$. Similarly, using the Sobolev embedding of $H^1(\mathbb{R}^N)$, we estimate $\|\nabla v_1\|^2_{L^4(\mathbb{R}^N)}$ as follows:

$$\|\nabla v_1\|^2_{L^4(\mathbb{R}^N)} \le C_S^2\|\nabla v_1\|^2_{H^1(\mathbb{R}^N)} = C_S^2\left(\|\nabla v_1\|^2_{L^2(\mathbb{R}^N)} + \|D^2 v_1\|^2_{L^2(\mathbb{R}^N)}\right),$$

where $\|D^2v_1\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |D^2v_1|^2\,dx$ and $|D^2v_1|^2 = \sum_{i,j=1}^N |\frac{\partial^2 v_1}{\partial x_i \partial x_j}|^2$. By the divergence theorem,

$$\begin{aligned}\|D^2v_1\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |D^2v_1|^2\,dx = \sum_{i,j=1}^N \int_{\mathbb{R}^N} \left(\frac{\partial^2 v_1}{\partial x_i \partial x_j}\right)\left(\frac{\partial^2 v_1}{\partial x_i \partial x_j}\right)dx \\ &= \sum_{i,j=1}^N \int_{\mathbb{R}^N} \left(\frac{\partial^2 v_1}{\partial x_i^2}\right)\left(\frac{\partial^2 v_1}{\partial x_j^2}\right)dx = \|\Delta v_1\|^2_{L^2(\mathbb{R}^N)}\end{aligned}$$

for any $v_1 \in H^2(\mathbb{R}^N)$. Hence, we see that

$$\|\nabla v_1\|^2_{L^4(\mathbb{R}^N)} \le C_S^2\left(\|\nabla v_1\|^2_{L^2(\mathbb{R}^N)} + \|\Delta v_1\|^2_{L^2(\mathbb{R}^N)}\right).$$

In both cases $N = 1$ and $2 \le N \le 4$, it follows from (2.17) and (2.18) that

$$\int_{\mathbb{R}^N} |f'(v_1)|^2|\nabla v_1|^2\,dx \le C_4\int_{\mathbb{R}^N} (\Delta v_1)^2\,dx + C_5 \tag{2.20}$$

where C_4 and C_5 are positive constants independent of $t \in [0, T]$ and $\varepsilon \in (0, \varepsilon_0]$. Hence, by (2.19) and (2.20), we obtain

$$\int_{\mathbb{R}^N} f'(v_1)\nabla v_1 \cdot \nabla v_{1,t}\, dx \leq \int_{\mathbb{R}^N} |\nabla v_{1,t}|^2\, dx + C_4 \int_{\mathbb{R}^N} (\Delta v_1)^2\, dx + C_5.$$

By this estimate and (2.7), we see that

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_{1,t}|^2\, dx + \frac{d_1 d_2}{2}\varepsilon^2 \int_{\mathbb{R}^N} |\nabla(\Delta v_1)|^2\, dx + \frac{d}{2}\int_{\mathbb{R}^N} (\Delta v_1)^2\, dx\right\} \\
&\leq \int_{\mathbb{R}^N} f'(v_1)\nabla v_1 \cdot \nabla v_{1,t}\, dx \\
&\leq \int_{\mathbb{R}^N} |\nabla v_{1,t}|^2\, dx + C_4 \int_{\mathbb{R}^N} (\Delta v_1)^2\, dx + C_5 \\
&\leq C_6 \left\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_{1,t}|^2\, dx + \frac{d_1 d_2}{2}\varepsilon^2 \int_{\mathbb{R}^N} |\nabla(\Delta v_1)|^2\, dx + \frac{d}{2}\int_{\mathbb{R}^N} (\Delta v_1)^2\, dx\right\} + C_5,
\end{aligned}$$

where $C_6 = 2\max\{1, C_4/d\}$. Hence, setting

$$Z(t) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_{1,t}|^2\, dx + \frac{d_1 d_2}{2}\varepsilon^2 \int_{\mathbb{R}^N} |\nabla(\Delta v_1)|^2\, dx + \frac{d}{2}\int_{\mathbb{R}^N} (\Delta v_1)^2\, dx,$$

we see that the previous inequality becomes $Z'(t) \leq C_6 Z(t) + C_5$, that is, $Z'(t) - C_6 Z(t) \leq C_5$. For any $t \in [0, T]$ and $\varepsilon \in (0, \varepsilon_0]$, multiplying this with $e^{-C_6 t}$ and integrating it over $[0, t]$, we have that

$$Z(t) \leq Z(0)e^{C_6 t} + \frac{C_5}{C_6}\left(e^{C_6 t} - 1\right) \leq C_7 Z(0) + C_7, \tag{2.21}$$

where C_7 is a positive constant depending on C_5, C_6 and T. Since $(u_1^\varepsilon, u_2^\varepsilon)$ is a classical solution for any $\varepsilon > 0$, $v_1 = u_1^\varepsilon$ and $v_2 = (u_2^\varepsilon - u_1^\varepsilon)/\varepsilon$, we see that for each $t \in [0, T]$, $v_{1,t}$ is continuously differentiable with respect to x and v_2 is continuously twice differentiable with respect to x. Hence, the equation $v_{1,t} = d_1\varepsilon\Delta v_1 + \nabla v_2$ is continuously differentiable with respect to x. Since

$\nabla v_{1,t} = d_1\varepsilon\nabla(\Delta v_1) + \nabla v_2$, $w_0 \in H^4(\mathbb{R}^N)$ and $w_1 \in H^3(\mathbb{R}^N)$, we see that

$$\begin{aligned}
Z(0) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_{1,t}|^2\,dx\bigg|_{t=0} + \frac{d_1d_2}{2}\varepsilon^2\int_{\mathbb{R}^N} |\nabla(\Delta v_1)|^2\,dx\bigg|_{t=0} \\
&\quad + \frac{d}{2}\int_{\mathbb{R}^N} (\Delta v_1)^2\,dx\bigg|_{t=0} \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \left|d_1\varepsilon\nabla(\Delta w_0) + \nabla w_1\right|^2\,dx \\
&\quad + \frac{d_1d_2}{2}\varepsilon^2\int_{\mathbb{R}^N} |\nabla(\Delta w_0)|^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N}(\Delta w_0)^2\,dx.
\end{aligned}$$

Thus, we obtain that $Z(0)$ is estimated by $\|w_0\|_{H^3(\mathbb{R}^N)}$ and $\|w_1\|_{H^1(\mathbb{R}^N)}$. By (2.21) and the definition of $Z(t)$, for all $t \in [0,T]$ and $\varepsilon \in (0,\varepsilon_0]$, we have that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_{1,t}|^2\,dx + \frac{d_1d_2}{2}\varepsilon^2\int_{\mathbb{R}^N} |\nabla(\Delta v_1)|^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N}(\Delta v_1)^2\,dx \le K_{2,T},$$

where $K_{2,T}$ is a positive constant depending on d_1, d_2, ε_0, C_7, $\|w_0\|_{H^3(\mathbb{R}^N)}$, $\|w_1\|_{H^1(\mathbb{R}^N)}$ and T. Thus, we complete the proof. □

For the higher derivatives of v_1, we present Lemma 2.4.

Lemma 2.4. *Assume that $w_0 \in H^4(\mathbb{R}^N)$ and $w_1 \in H^2(\mathbb{R}^N)$. Let $d_2 > d_1 \ge 0$, $\varepsilon_0 > 0$, and $T > 0$. Then, the following integrations are bounded for any $t \in [0,T]$ and $\varepsilon \in (0,\varepsilon_0]$:*

$$\varepsilon\int_{\mathbb{R}^N}(\Delta v_{1,t})^2\,dx, \qquad \varepsilon^3\int_{\mathbb{R}^N}(\Delta^2 v_1)^2\,dx, \qquad \varepsilon\int_{\mathbb{R}^N}|\nabla(\Delta v_1)|^2\,dx$$

where $d = d_2 - d_1$.

Proof. By integrating (2.8) with respect to t and using (2.20) and Lemma 2.3, we have for any $t \in [0,T]$,

$$\varepsilon\int_{\mathbb{R}^N}(\Delta v_{1,t})^2\,dx + d_1d_2\varepsilon^3\int_{\mathbb{R}^N}(\Delta^2 v_1)^2\,dx + d\varepsilon\int_{\mathbb{R}^N}|\nabla(\Delta v_1)|^2\,dx \le C_8,$$

where C_8 is a positive constant depending on $\varepsilon_0\|w_0\|^2_{H^4(\mathbb{R}^N)} + \varepsilon_0\|w_1\|^2_{H^2(\mathbb{R}^N)}$. □

3 Convergence

In this section, we give a proof of Theorem 1.3.

Proof of Theorem 1.3. We can rewrite (1.1) to

$$
\begin{cases} V_{1,t} = V_2, & \text{in } \mathbb{R}^N \times (0,T), \\ V_{2,t} = d\Delta V_1 + f(V_1), & \\ V_1(x,0) = w_0, \quad V_2(x,0) = w_1 & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}
$$

Since we assume that $w_0 \in H^4(\mathbb{R}^N)$ and $w_1 \in H^2(\mathbb{R}^N)$, at least V_1 belongs to $H^2(\mathbb{R}^N)$ for all $t \in [0,T]$. Recall that

$$
\begin{cases} v_{1,t} = d_1\varepsilon\Delta v_1 + v_2, & \text{in } \mathbb{R}^N \times (0,T], \\ v_{2,t} = d_2\varepsilon\Delta v_2 + d\Delta v_1 + f(v_1), & \\ v_1(\cdot,0) = w_0, & \text{in } \mathbb{R}^N. \\ v_2(\cdot,0) = w_1, & \end{cases}
$$

By comparing two systems, we have

$$
\begin{cases} (v_1 - V_1)_t = d_1\varepsilon\Delta v_1 + (v_2 - V_2), \\ (v_2 - V_2)_t = d_2\varepsilon\Delta v_2 + d\Delta(v_1 - V_1) + f(v_1) - f(V_1), \end{cases}
$$

Then, we get

$$
(v_1 - V_1)_{tt} - d_1\varepsilon\Delta v_{1,t} = d_2\varepsilon\Delta v_2 + d\Delta(v_1 - V_1) + f(v_1) - f(V_1). \tag{3.2}
$$

Multiplying (3.2) by $(v_1 - V_1)_t$ and integrating it over $\mathbb{R}^N$ yield

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^N}(v_{1,t} - V_{1,t})^2\,dx + \frac{d}{2}\int_{\mathbb{R}^N}|\nabla(v_1 - V_1)|^2\,dx\right) \\
&= \int_{\mathbb{R}^N}(d_1\varepsilon\Delta v_{1,t} + d_2\varepsilon\Delta v_2)\,(v_1 - V_1)_t\,dx \\
&\quad + \int_{\mathbb{R}^N}\{f(v_1) - f(V_1)\}\,(v_1 - V_1)_t\,dx.
\end{aligned}
$$

By Lemma 2.4 and $\Delta v_2 = \Delta v_{1,t} - d_1\varepsilon\Delta^2 v_1$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N}(\varepsilon\Delta v_{1,t})^2\,dx &\leq C_9\varepsilon, \\
\int_{\mathbb{R}^N}(\varepsilon\Delta v_2)^2\,dx &= \int_{\mathbb{R}^N}(\varepsilon\Delta v_{1,t} - d_1\varepsilon^2\Delta^2 v_1)^2\,dx \\
&\leq 2\int_{\mathbb{R}^N}(\varepsilon\Delta v_{1,t})^2\,dx + 2\int_{\mathbb{R}^N}(d_1\varepsilon^2\Delta^2 v_1)^2\,dx \leq C_9\varepsilon.
\end{aligned}
$$

By the previous estimates, we see that

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^N}(v_{1,t}-V_{1,t})^2\,dx+\frac{d}{2}\int_{\mathbb{R}^N}|\nabla(v_1-V_1)|^2\,dx\right)\\
&=2C_9\varepsilon\int_{\mathbb{R}^N}(v_{1,t}-V_{1,t})^2\,dx+\int_{\mathbb{R}^N}\{f(v_1)-f(V_1)\}(v_1-V_1)_t\,dx.
\end{aligned}
\tag{3.3}
$$

By multiplying (3.2) by v_1-V_1 and integrating it over $\mathbb{R}^N$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N}(v_1-V_1)^2\,dx\\
&=\int_{\mathbb{R}^N}(v_{1,t}-V_{1,t})^2\,dx+\int_{\mathbb{R}^N}(v_{1,tt}-V_{1,tt})(v_1-V_1)\,dx\\
&=\int_{\mathbb{R}^N}(v_{1,t}-V_{1,t})^2\,dx+\int_{\mathbb{R}^N}(d_1\varepsilon\Delta v_{1,t}+d_2\varepsilon\Delta v_2)(v_1-V_1)\,dx\\
&\qquad-\int_{\mathbb{R}^N}d|\nabla(v_1-V_1)|^2\,dx+\int_{\mathbb{R}^N}\{f(v_1)-f(V_1)\}(v_1-V_1)\,dx\\
&\le\int_{\mathbb{R}^N}(v_{1,t}-V_{1,t})^2\,dx+\int_{\mathbb{R}^N}(v_1-V_1)^2\,dx+C_{10}\varepsilon\\
&\qquad-\int_{\mathbb{R}^N}d|\nabla(v_1-V_1)|^2\,dx+\int_{\mathbb{R}^N}\{f(v_1)-f(V_1)\}(v_1-V_1)\,dx.
\end{aligned}
\tag{3.4}
$$

Since $H^2(\mathbb{R}^N)\hookrightarrow BC(\mathbb{R}^N)$ in the case of $1\le N\le 3$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\{f(v_1)-f(V_1)\}(v_1-V_1)\,dx\le C_{11}\int_{\mathbb{R}^N}(v_1-V_1)^2\,dx,\\
&\int_{\mathbb{R}^N}\{f(v_1)-f(V_1)\}(v_1-V_1)_t\,dx\\
&\quad\le\ C_{11}\int_{\mathbb{R}^N}(v_{1,t}-V_{1,t})^2\,dx+C_{11}\int_{\mathbb{R}^N}(v_1-V_1)^2\,dx.
\end{aligned}
$$

In the case of $N=4$, let us estimate $\int_{\mathbb{R}^4}\{f(v_1)-f(V_1)\}(v_1-V_1)\,dx$ and $\int_{\mathbb{R}^4}\{f(v_1)-f(V_1)\}(v_{1,t}-V_{1,t})\,dx$. Noting that $q=1$ in (H4) when $N=4$, we have $|f'(u)|\le f_4+f_5|u|$. By the mean value theorem, we see that

$$
\begin{aligned}
|f(v_1)-f(V_1)|\ &\le\ \int_0^1|f'(\theta v_1+(1-\theta)V_1)|d\theta\cdot|v_1-V_1|\\
&\le\ \big(f_4+f_5|v_1|+f_5|V_1|\big)|v_1-V_1|.
\end{aligned}
$$

Hence, we see that

$$\int_{\mathbb{R}^4} \{f(v_1) - f(V_1)\} (v_1 - V_1)\, dx \quad \le \quad \int_{\mathbb{R}^4} (f_4 + f_5|v_1| + f_5|V_1|) (v_1 - V_1)^2\, dx,$$

and

$$\begin{aligned} &\int_{\mathbb{R}^4} \{f(v_1) - f(V_1)\} (v_1 - V_1)_t\, dx \\ &\le \int_{\mathbb{R}^4} (|v_1| + |V_1|) |v_1 - V_1||v_{1,t} - V_{1,t}|\, dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^4} (v_{1,t} - V_{1,t})^2\, dx + \frac{1}{2}\int_{\mathbb{R}^4} (f_4 + f_5|v_1| + f_5|V_1|)^2 (v_1 - V_1)^2\, dx. \end{aligned}$$

By the Schwarz inequality and the Sobolev inequality, we have that for each $m = 1, 2$,

$$\begin{aligned} &\int_{\mathbb{R}^4} (f_4 + f_5|v_1| + f_5|V_1|)^m (v_1 - V_1)^2\, dx \\ &\quad \le \quad C_{12}\|v_1 - V_1\|^2_{L^2(\mathbb{R}^4)} + C_{12}(\|v_1\|^m_{L^{2m}(\mathbb{R}^4)} + \|V_1\|^m_{L^{2m}(\mathbb{R}^4)})\|v_1 - V_1\|^2_{L^4(\mathbb{R}^4)} \\ &\quad \le \quad C_{12}\|v_1 - V_1\|^2_{L^2(\mathbb{R}^4)} + C_{12}(\|v_1\|^m_{L^{2m}(\mathbb{R}^4)} + \|V_1\|^m_{L^{2m}(\mathbb{R}^4)})C_S^2\|v_1 - V_1\|^2_{H^1(\mathbb{R}^4)} \end{aligned}$$

where C_{12} is a positive constant. Since $v_1,\ V_1 \in H^2(\mathbb{R}^4)$, $\|v_1\|^m_{L^{2m}(\mathbb{R}^4)}$ and $\|V_1\|^m_{L^{2m}(\mathbb{R}^4)}$ are bounded for each $m = 1, 2$ and $t \in [0, T]$. Therefore, we obtain that

$$\begin{aligned} &\int_{\mathbb{R}^4} \{f(v_1) - f(V_1)\} (v_1 - V_1)\, dx \le C_{13}\|v_1 - V_1\|^2_{H^1(\mathbb{R}^N)} \\ &\int_{\mathbb{R}^N} \{f(v_1) - f(V_1)\} (v_1 - V_1)_t\, dx \le \|v_{1,t} - V_{1,t}\|^2_{L^2(\mathbb{R}^N)} + C_{13}\|v_1 - V_1\|^2_{H^1(\mathbb{R}^N)}. \end{aligned}$$

Thus, from (3.3) and (3.4) with the above inequalities, we have

$$\begin{aligned} X'' &\le C_{14}\varepsilon + C_{14}X + C_{15}Y, \\ Y' &\le C_{14}\varepsilon + C_{14}X + C_{15}Y, \end{aligned}$$

where

$$X = \int_{\mathbb{R}^N} |v_1 - V_1|^2 dx, \quad Y = \int_{\mathbb{R}^N} |v_{1,t} - V_{1,t}|^2 dx + \int_{\mathbb{R}^N} d|\nabla(v_1 - V_1)|^2 dx.$$

By adding the previous inequalities, we have

$$X'' + Y' \le 2C_{14}\varepsilon + 2C_{14}X + 2C_{15}Y.$$

By choosing a sufficiently large C_{14} with $\sqrt{2C_{14}} > 2C_{15}$ and taking $g = \sqrt{2C_{14}}$, we obtain

$$\left\{(X' + gX)\, e^{-gt}\right\}' + \left(Y e^{-gt}\right)' \le 2C_{14}\varepsilon\, e^{-gt} \quad \text{for } t > 0.$$

The argument similar to Lemma 2.2 yields

$$\int_{\mathbb{R}^N} (v_1 - V_1)^2 dx \le C_{14}\varepsilon, \quad \int_{\mathbb{R}^N} |\nabla(v_1 - V_1)|^2 dx \le C_{14}\varepsilon$$
$$\text{and} \quad \int_{\mathbb{R}^N} (v_{1,t} - V_{1,t})^2 dx \le C_{14}\varepsilon,$$

where we use $(v_1 - V_1)|_{t=0} = 0$ and $(v_{1,t} - V_{1,t})|_{t=0} = 0$. $\square$

4 Concluding remarks

We proposed a 2-component reaction-diffusion system (1.3) approximating a semilinear wave equation. The semilinear wave equation has energy conservation law (1.2) where w is a unique solution of (1.1). However, as seems in (2.6) that the first component u of the solution to (1.3) does not have the conservation law such as the above. Although it is difficult to investigate the details of solutions to the wave equation by (1.3), we have proved the convergenece in $C^0([0,T]; H^1(\mathbb{R}^N)) \cap C^1((0,T]; L^2(\mathbb{R}^N))$ for any positive time $T > 0$. Moreover, we emphasize that the reaction-diffusion approximation is useful because of the facility of the numarical scheme. In particular, when $d_1 = 0$, the system (2.1) of (v_1, v_2) approximating the solution of (1.1) becomes

$$\begin{cases} v_{1,t} = v_2, \\ v_{2,t} = d_2\varepsilon\Delta v_2 + d_2\Delta v_1 + f(v_1) \end{cases} \quad \text{in } \mathbb{R}^N.$$

This system is approximately the same as (3.1), and we conjecture that the case $d_1 = 0$ is slightly better than the case $d_1 > 0$ if the initial data w_0 and w_1 are sufficiently smooth. However, to stabilize the original system (1.3), it might be better to use the system in the case $d_1 > 0$.

Acknowledgements

The authors would like to thank the anonymous referee for his helpful suggestions and comments. The first author was partially supported by JSPS KAKENHI Grant Numbers JP16KT0022 and JP20H01816. The second author was partially supported by JSPS KAKENHI Grant Number JP19K14588.

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