TRAVELING CURVED WAVES
IN TWO DIMENSIONAL EXCITABLE MEDIA

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Abstract. This paper treats a free boundary problem in two-dimensional excitable media arising from a singular limiting problem of a FitzHugh–Nagumo type reaction-diffusion system. The existence and uniqueness up to translations of two-dimensional traveling curved waves solutions is shown. To study the stability of the waves, the local and global existence and uniqueness of solutions to the free boundary problem nearby the waves under certain assumptions is established. The notion of the arrival time is introduced to estimate the propagation speed of solutions to the free boundary problem, which allows us to establish the asymptotic stability of traveling curved waves by using the comparison principle. It is also pointed out that the gradient blowup can take place if the initial data is far from the traveling curved waves, which means the interface may not be always represented by a graph.

1. Introduction

Wave propagation phenomenon occurs in various area such as physics, biology, chemical kinetics and so on. In particular, excitable media which are often modeled by nonlinear partial differential equations can sustain traveling waves, traveling pulses, periodic wave trains, rotating spirals and so on. Examples include propagated waves of electrical or chemical activity in cardiac tissue, in the retina or in the brain cortex, which provide self-organization phenomena in living system [19, 25]. A wide variety of spatiotemporal patterns has been discussed in, for example, [38, 24, 20, 37, 22, 23] and the references therein. Mathematically, self-organized patterns in two-dimensional excitable media such as the spatially distributed models of FitzHugh–Nagumo type are still not understood completely and lead to substantial new mathematical challenges. In order to investigate that, complicated dynamics arising in excitable media should be simplified to be able to analyze them. This motivates Chen et al. [7] to propose a singular limit problem of a FitzHugh–Nagumo type reaction-diffusion system, which is described by the following free boundary problem:

$$\begin{cases} \mathcal{V} = W(v) - \kappa, & (x,y) \in \partial\Omega(t), t > 0, \\ v_t = g(\chi_{\Omega(t)}, v), & (x,y) \in \mathbb{R}^2, t > 0, \end{cases} \tag{1.1}$$

where $\mathcal{V}$ is referred as an outer normal velocity on $\partial\Omega(t)$ which points from $\Omega(t)$ to $\Omega(t)^{\mathrm{c}}$; κ is a curvature of $\Omega(t)$; $\chi_{\Omega(t)}$ is a characteristic function of $\Omega(t)$ which is defined as

$$\chi_{\Omega(t)}(x,y) := \begin{cases} 1, & \text{if } (x,y) \in \Omega(t), \\ 0, & \text{otherwise.} \end{cases}$$

The function $W(v) := a - bv$ for some positive constants a and b. The function g is defined by

$$g(u,v) := g_1 u - \frac{g_2 v}{g_3 v + g_4}$$

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for some positive constants g_i $(i = 1, \cdots, 4)$. In fact, system (1.1) comes from a limiting problem of a FitzHugh–Nagumo type reaction-diffusion system

$$\begin{cases} u_t &= \Delta u + \dfrac{1}{\varepsilon^2}(f_\varepsilon(u) - \varepsilon\beta v), \\ v_t &= g(u,v), \end{cases}$$

where $\varepsilon > 0$ and

$$f_\varepsilon(u) := u(1-u)(u - 1/2 + \varepsilon\alpha), \quad g(u,v) := g_1 u - \frac{g_2 v}{g_3 v + g_4},$$

where u is the activator (membrane potential) and v is the inhibitor (recovery variable). After a formal analysis, $u^\varepsilon \to 1$ or 0 as $\varepsilon \downarrow 0$ and $\Omega(t)$ stands for the region where $u^\varepsilon \to 1$. Since this characteristic function and v correspond to the limiting functions of the activator and the inhibitor, the region $\Omega(t)$ and its complementary region are called an *excited region* and a *resting region*, respectively.

One of the simplest dynamics is the so-called *traveling waves*, which moves with a constant speed c without changing its shape. It is said to be a *planar traveling wave* if it can be represented by a function of the single variable $\mathbf{n}\cdot\mathbf{x} - ct$ with some unit vector $\mathbf{n}$ pointing in the direction of wave propagation. It is essentially a one-dimensional wave. In two-dimensional media, non-planar traveling waves are expected to occur. The existence and uniqueness of traveling spots to the problem (1.1) for the wave speed $c \in (0, a)$ has been discussed in [7], where $\Omega(t)$ is considered as a bounded moving domain. More precisely, Theorem 1.1 of [7] showed that, under the condition $2g_2 \le g_1 g_3$, for any $c \in (0,a)$, there exist a unique constant b, a bounded domain Ω (assuming symmetry along the y-axis) and $v(x,y)$ such that the moving domain

$$\Omega(t) = \{(x, y + ct) \mid (x,y) \in \Omega\}$$

and $v(x, y - ct)$ solve (1.1). Moreover,

$$v(x,y) = 0, \quad |x| \ge L_c; \quad \lim_{R\to\infty} \sup_{x^2+y^2 \ge R^2} v(x,y) = 0,$$

where L_c denotes the width of Ω given by

$$L_c := \max_{(x,y)\in\Omega} |x| = -\frac{\pi}{2c} + \frac{2a}{c\sqrt{a^2-c^2}} \arctan \frac{a+c}{\sqrt{a^2-c^2}}.$$

This traveling wave is called a *traveling spot*. Among other things, the shape of Ω approaches to a disk with radius $1/a$ as $c \searrow 0$ and is non-convex when c is close to a.

The relationship among complicated spatio-temporal patterns in (1.1) are still not completely understood. First let us consider the relationship between the traveling spots and the planar traveling waves. For this, we observe that $L_c \to \infty$ as $c \nearrow a$, which means the width of the traveling spot tends to ∞. More precisely, Chen et al. [8] proved that the traveling spot converges to a planar traveling wave as $c \nearrow a$. It is natural to ask: what happens if c is bigger than a? For simplicity, we look for $C^{2,1}$ functions ϕ_- and ϕ_+ defined on $\{(x,t) : x \in \mathbb{R},\ t > 0\}$ such that the excited region

$$\Omega(t) := \{(x,y) \in \mathbb{R}^2 \mid \phi_-(x,t) < y < \phi_+(x,t)\}.$$

Also, we are interested in the case that the normal velocity on $\{y = \phi_+(x,t)\}$ (resp. $\{y = \phi_-(x,t)\}$) is positive (resp. negative) and call it *the front* (resp. *the back*).

Noting the direction of the outer normal vector, we have

$$\phi_{\pm,t} = \frac{\phi_{\pm,xx}}{1+\phi_{\pm,x}^2} \pm W(v(x,\phi_\pm(x,t),t))\sqrt{1+\phi_{\pm,x}^2}, \quad x \in \mathbb{R},\ t > 0. \tag{1.2}$$

The equation for v can be rewritten as follows:

$$\begin{cases} v_t = g(1, v), & (x, y) \in \Omega(t), \\ v_t = g(0, v), & (x, y) \in \Omega_+(t) \cup \Omega_-(t), \end{cases} \tag{1.3}$$

where

$$\Omega_+(t) := \{(x, y) \in \mathbb{R}^2 \mid y \geq \phi_+(x, t)\}, \quad \Omega_-(t) := \{(x, y) \in \mathbb{R}^2 \mid y \leq \phi_-(x, t)\}.$$

Our goal is to address two questions: (i) Does there exist non-planar traveling waves for (1.2)-(1.3) with an unbounded moving domain when c is bigger than a? (ii) If such wave exists, is it globally asymptotically stable? To answer (i), we consider, without loss of generality, the traveling waves moving in y-direction with a speed c and taking the following forms:

$$v(x, y, t) = \widehat{v}(x, y - ct), \quad \phi_\pm(x, t) = \widehat{\phi}_\pm(x) + ct, \quad \widehat{\Omega} = \{(x, y) \in \mathbb{R}^2 \mid \widehat{\phi}_-(x) < y < \widehat{\phi}_+(x)\},$$
$$\widehat{\Omega}_+ := \{(x, y) \mid y > \widehat{\phi}_+(x)\}, \quad \widehat{\Omega}_- := \{(x, y) \mid y < \widehat{\phi}_-(x)\},$$

which satisfy

$$c\widehat{v}_y(x, y) + g(\chi_{\widehat{\Omega}}(x, y), \widehat{v}(x, y)) = 0, \quad (x, y) \in \widehat{\Omega}_+ \cup \widehat{\Omega} \cup \widehat{\Omega}_-, \tag{1.4}$$

$$c = \frac{\widehat{\phi}_{+,xx}}{1 + \widehat{\phi}_{+,x}^2} + W(\widehat{v}(x, \widehat{\phi}_+(x)))\sqrt{1 + \widehat{\phi}_{+,x}^2}, \quad x \in \mathbb{R}, \tag{1.5}$$

$$c = \frac{\widehat{\phi}_{-,xx}}{1 + \widehat{\phi}_{-,x}^2} - W(\widehat{v}(x, \widehat{\phi}_-(x)))\sqrt{1 + \widehat{\phi}_{-,x}^2}, \quad x \in \mathbb{R}, \tag{1.6}$$

from (1.2) and (1.3).

Traveling curved waves (or V-shaped traveling waves) are one type of non-planar waves which have been studied theoretically in [2] and observed in simulations [33]. See also [1, 31]. Moreover, Peres-Muzunuri et al [33] performed the experiments in a liquid BZ reaction. Using a silver wire with appropriate shape immersed into the liquid, they succeeded to observe that the silver wire constantly emitted V-shaped waves with a period. The mathematical studies can be found in, for example, [9, 16, 17, 29, 30, 35, 36] and the references therein. We emphasize that they have been studied the front waves only and that we consider the traveling curved wave with the front and the back.

The other typical dynamics of two dimensional waves are wave segments, rotating spots and rotating spirals, which have not been studied yet in the system (1.1). These patterns have been studied in [39, 14, 15, 6] for the so-called *wave front interaction model*, which is proposed by Pelcé and Sun [32] and Zykov and Showalter [39]. We remark that the wave front interaction model is simpler than (1.1).

Without loss of generality, we can assume that the traveling wave is moving in y direction after an appropriate rotation. Before constructing traveling curved wave solutions to the problem (1.2)–(1.3), we give the definition of a traveling curved wave solution as follows:

Definition 1.1. If $(c, \widehat{\phi}_\pm, \widehat{v})$ satisfies (1.4)–(1.6) with

$$\widehat{\phi}_\pm \in C^2(\mathbb{R}), \quad \widehat{v} \geq 0 \text{ in } \mathbb{R}^2, \quad \widehat{v} \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap C^1\Big(\mathbb{R}^2 \setminus \{y = \widehat{\phi}_\pm(x)\}\Big),$$

then $(c, \phi_\pm(x, t), v(x, y, t)) = (c, \widehat{\phi}_\pm(x) + ct, \widehat{v}(x, y - ct))$ is said to be a *traveling wave* of (1.2)–(1.3) (as well as (1.1)) and c is called the wave speed. If the traveling wave is not a planar traveling wave, it is called a *traveling curved wave*.

Suppose that the solution $(c, \widehat{\phi}_\pm, \widehat{v})$ of (1.4)–(1.6) exists with $c > 0$. We can see that $\widehat{v} \equiv 0$ on the region $\widehat{\Omega}_+$. Indeed, from (1.4), $\widehat{v}$ satisfies

$$c\widehat{v}_y = -g(0, \widehat{v}), \quad (x, y) \in \widehat{\Omega}_+.$$

For each $(x_0, y_0) \in \widehat{\Omega}_+$, by integrating the above equation with respect to y, we have

$$-\int_{\widehat{v}(x_0,y_0)}^{\widehat{v}(x_0,y)} \frac{ds}{g(0,s)} = \frac{1}{c}(y - y_0).$$

Thus $\widehat{v}(x_0, y)$ goes to infinity as y tends to ∞ if $\widehat{v}(x_0, y_0) > 0$. Since $\widehat{v}$ is assumed to be bounded, we must have $\widehat{v}(x_0, y_0) = 0$ for all $(x_0, y_0) \in \widehat{\Omega}_+$. Hence $\widehat{v} \equiv 0$ in $\widehat{\Omega}_+$. The continuity of the solution gives that $\widehat{v} = 0$ on the front. Hence the front $\widehat{\phi}_+$ satisfies (1.5) with $W(\widehat{v}(x, \widehat{\phi}_+(x))) = a$. Namely, the front equation (1.5) becomes

$$c = \frac{\widehat{\phi}_{+,xx}}{1 + \widehat{\phi}_{+,x}^2} + a\sqrt{1 + \widehat{\phi}_{+,x}^2}, \quad x \in \mathbb{R}. \tag{1.7}$$

This equation (1.7) is often called a *curvature flow with constant force* and has been studied in, for example, [11, 26, 27, 28]. Especially the existence and the uniqueness of the V-shaped solution of (1.7) was shown in [27, 28]. We recall some results as follows:

Proposition 1.2 (Proposition 1.1 of [27]). *For each $c > a > 0$, there exists a unique solution $\widehat{\phi}^*(x; c)$ of (1.7) with asymptotic lines $y = m_*|x| + \eta$, where*

$$m_* := \frac{\sqrt{c^2 - a^2}}{a} > 0, \quad \eta := -\frac{\sqrt{c^2 - a^2}}{ca} \arctan \frac{\sqrt{c^2 - a^2}}{a} - \frac{1}{c} \log \frac{2(c + a)}{c} < 0.$$

The graph of $y = \widehat{\phi}^*(x; c)$ is characterized by $\theta = \arctan \widehat{\phi}^*_x(x; c)$ as

$$\begin{aligned} x(\theta; c) &= \frac{\theta}{c} + \frac{2}{m_* c} \operatorname{arctanh}\left(\sqrt{\frac{c + k}{c - k}} \tan \frac{\theta}{2}\right), \\ y(\theta; c) &= \frac{1}{c} \log\left(\frac{c - a}{c\cos\theta - a}\right) \end{aligned} \tag{1.8}$$

for $\theta \in (-\arctan m_*, \arctan m_*)$. Furthermore, $\widehat{\phi}^*_{xx} > 0$ for all $x \in \mathbb{R}$.

We remark that the difference between the traveling front in Proposition 1.2 and Proposition 1.1 of [27] is a translation in y. The asymptotic lines in [27] are $y = m_*|x|$.

Proposition 1.3 (Theorem 1.2 of [27]). *For each $c \geq a > 0$, the solution $y = \widehat{\phi}_+(x)$ of (1.7) must be the one of the following two types:*

(i) *a straight line,*
(ii) *a traveling front $y = \widehat{\phi}^*(x + x_0; c) + y_0$ for some x_0 and y_0.*

Furthermore, if $0 \leq c < a$, then there is no smooth traveling front except for the case that the interface forms a stationary circle with radius $1/a$ and $c = 0$.

We make the following assumption in the whole paper:

(H) $g_1 g_3 > 2g_2$.

The assumption **(H)** is needed in the singular limiting process [7]. In our paper, **(H)** can be weaken as $g_1 g_3 > g_2$. However, we still impose **(H)** from the modeling viewpoint.
The main results are given as follows.

Theorem 1.4 (Existence of traveling curved waves)**.** *For each $c > a > 0$, there exists a solution $(c, \widehat{\phi}^*_\pm, \widehat{v}^*)$ of of (1.4)–(1.6) such that*

$$
\begin{aligned}
&\widehat{v}^* \in C(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus \{y = \widehat{\phi}^*_\pm(x)\}) \text{ and } \widehat{\phi}^*_\pm \in C^2(\mathbb{R}),\\
&\widehat{\phi}^*_+(x) = \widehat{\phi}^*(x;c), \quad \widehat{\phi}^*_-(x) = \widehat{\phi}^*(x;c) - c\, G_1^{-1}(2a/b),\\
&\widehat{v}^*(x,y) := \begin{cases} 0 & \text{if } y \ge \widehat{\phi}^*_+(x),\\ G_1\left(\dfrac{\widehat{\phi}^*_+(x) - y}{c}\right) & \text{if } \widehat{\phi}^*_-(x) \le y \le \widehat{\phi}^*_+(x),\\ G_0\left(\dfrac{\widehat{\phi}^*_-(x) - y}{c} + G_0^{-1}\left(\dfrac{2a}{b}\right)\right) & \text{if } y \le \widehat{\phi}^*_-(x),\end{cases}
\end{aligned}
\tag{1.9}
$$

where $\widehat{\phi}^*(x;c)$ is defined in Proposition 1.2 and

$$
G_0^{-1}(v) := \int_1^v \frac{ds}{g(0,s)}, \quad G_1^{-1}(v) := \int_0^v \frac{ds}{g(1,s)}.
$$

We set $G_0^{-1}(0) = \infty$. See Section 2 for more details.

Theorem 1.5 (Uniqueness of traveling curved waves)**.** *For each $c > a > 0$, the traveling curved wave is unique (up to a translation). Namely, if $(c, \phi_\pm, v)$ is a traveling curved wave, then*

$$
\begin{aligned}
&\phi_+(x,t) = \widehat{\phi}^*(x + x_0; c) + ct - y_0, \quad \phi_-(x,t) = \widehat{\phi}^*(x + x_0; c) + ct - c\, G_1^{-1}(2a/b) - y_0,\\
&v(x,y,t) = \widehat{v}^*(x + x_0, y + y_0 - ct) \quad \text{for some } x_0, y_0 \in \mathbb{R}.
\end{aligned}
$$

We next study the asymptotic stability of $(c, \widehat{\phi}_\pm, \widehat{v})$ for any given $c > a$. For this, the global existence and uniqueness of solutions of (1.2) and (1.3) nearby any given traveling curved wave are needed to be established first. We assume that the initial data $(\phi_{\pm,0}, v_0)$ satisfies

$$
\begin{cases}
v_0 \in C^1(\Omega_-(0)) \cap C^1(\overline{\Omega(0)}) \cap C^1(\Omega_+(0)) \cap C(\mathbb{R}^2), \quad v_0 \ge 0 \text{ in } \mathbb{R}^2,\\
\|v_0(x,y) - \widehat{v}^{\phi_{+,0},\phi_{-,0}}(x,y)\|_{C^1(\mathbb{R}^2 \setminus \{y = \phi_{\pm,0}(x)\})} \le \varepsilon,\\
\|\phi_{\pm,0}(x) - \widehat{\phi}^*_\pm(x;c)\|_{C^2(\mathbb{R})} \le \varepsilon
\end{cases}
\tag{1.10}
$$

for some small $\varepsilon > 0$, where $\overline{\Omega(0)}$ is the closure of the set $\Omega(0)$, $v(x,y,0) = v_0(x,y)$, $\phi_\pm(x,0) = \phi_{\pm,0}(x)$ and

$$
\widehat{v}^{\phi_1,\phi_2}(x,y) := \begin{cases} 0 & \text{if } y \ge \phi_1(x),\\ G_1\left(\dfrac{\phi_1(x) - y}{c}\right) & \text{if } \phi_2(x) \le y < \phi_1(x),\\ G_0\left(\dfrac{\phi_2(x) - y}{c} + G_0^{-1} \circ G_1\left(\dfrac{\phi_1(x) - \phi_2(x)}{c}\right)\right) & \text{if } y < \phi_2(x),\end{cases}
\tag{1.11}
$$

if $\phi_1(x) > \phi_2(x)$ for any $x \in \mathbb{R}$. We note that $\widehat{v}^*(x,y) = \widehat{v}^{\widehat{\phi}^*, \widehat{\phi}^* - c\, G_1^{-1}(2a/b)}(x,y)$ for the traveling curved wave. For simplicity, we also denote $\widehat{v}^{\phi_1,-\infty}$ by $\widehat{v}^{\phi_1}$, namely,

$$
\widehat{v}^{\phi_1}(x,y) := G_1\left(\left(\frac{\phi_1(x) - y}{c}\right)_+\right) = \begin{cases} 0 & \text{if } y \ge \phi_1(x),\\ G_1\left(\dfrac{\phi_1(x) - y}{c}\right) & \text{if } y < \phi_1(x),\end{cases}
\tag{1.12}
$$

where we use the notation $(x)_+ := \max\{x, 0\}$. Moreover, we consider a compact perturbation in the sense that there is a compact set $K := [-k_1, k_1] \times [-k_2, k_2] \subset \mathbb{R}^2$ such that

$$\begin{cases} \phi_{\pm,0}(x) - \widehat{\phi}^*_{\pm}(x) \equiv 0, & x \notin [-k_1, k_1], \\ v_0(x,y) - \widehat{v}^*(x,y) \equiv 0, & (x,y) \notin K. \end{cases} \tag{1.13}$$

We also impose the following assumptions:

(A1) $\|\phi'_{\pm,0}\|_{L^\infty(\mathbb{R})} + \|\phi''_{\pm,0}\|_{L^\infty(\mathbb{R})} \leq M_0$ for some positive constant M_0.

(A2) $\dfrac{\phi''_{\pm,0}(x)}{1+\phi'_{\pm,0}(x)^2} \pm W(v_0(x, \phi_{\pm,0}(x)))\sqrt{1+\phi'_{\pm,0}(x)^2} \geq \zeta_0$ in $\mathbb{R}$ for some positive constant ζ_0.

We remark that since $v_0 \geq 0$ in $\mathbb{R}^2$, it follows from (1.3) that $v \geq 0$ as long as it exists. Before we state the local and global existence and uniqueness result, the definition of classical solutions to (1.2) and (1.3) is given as follows.

Definition 1.6. A classical solution of (1.2) and (1.3) for $t \in [0, T]$ is a pair $(\phi_\pm, v)$ such that

(i) $\phi_+(x,t) > \phi_-(x,t)$ for $(x,t) \in \mathbb{R} \times [0,T]$,
(ii) $\phi_\pm \in C(\mathbb{R} \times [0,T]) \cap C^{2,1}(\mathbb{R} \times (0,T])$,
(iii) $v \in C(\mathbb{R}^2 \times [0,T]) \cap C^1(\mathbb{R}^2 \times (0,T] \setminus \{y = \phi_\pm(x,t)\})$
(iv) $(\phi_\pm, v)$ satisfies (1.2)-(1.3) for $t \in (0,T]$.

Theorem 1.7. *Let* $(\phi_{\pm,0}, v_0)$ *satisfy* (1.10)-(1.13) *and* **(A1)-(A2)**. *Then for any* $c > a$, *the problem* (1.2)-(1.3) *has a unique classical solution* $(\phi_\pm, v)$ *for all* $t \in [0, \infty)$. *Moreover, the following estimates hold:*

$$\begin{aligned} &\|\phi_{\pm,x}\|_{L^\infty(Q_\infty)} + \|\phi_{\pm,t}\|_{L^\infty(Q_\infty)} \leq M^\pm, \\ &\phi_{+,t}(\cdot,t) \geq \zeta_+ \quad \textit{for all } x \in \mathbb{R} \textit{ and } t \in [0,\infty) \textit{ with some positive constant } \zeta_+, \\ &\phi_{-,t}(\cdot,t) \geq e^{-\zeta_- t}\zeta_0 \quad \textit{for all } x \in \mathbb{R} \textit{ and } t \in [0,\infty) \textit{ with some positive constant } \zeta_-, \end{aligned}$$

where $Q_\infty := \mathbb{R} \times (0,\infty)$ *and* $M^\pm$ *(resp.* $\zeta_\pm$*) is a positive constant depending only on* M_0 *(resp.* M_0, ζ_0 *and* ε*).*

The system (1.1) with diffusion

$$\begin{cases} \mathcal{V} = W(v) - \kappa, & (x,y) \in \partial\Omega(t), t > 0, \\ v_t = \Delta v + g(u_\pm(v), v), & (x,y) \in \mathbb{R}^2, t > 0 \end{cases} \tag{1.14}$$

has been discussed by many researchers, where $u_\pm(v)$ are roots of $F(u_\pm(v)) - v = 0$ for some cubic function F with three zeros. The local existence and uniqueness of solutions of (1.14) was shown by [3, 4]; the global existence of weak solutions has been established in [13]. The existence of diffusion plays an important role in showing the local existence of solutions. Due to the lack of diffusivity, we only show the existence of solutions to (1.1) near traveling curve waves. Moreover, the above theorem guarantees the global existence of solutions. See [5, 18] for one dimensional case of (1.14).

Our final result shows that the traveling curved wave of (1.1) is asymptotically stable if the given initial perturbation satisfies (1.10)-(1.13) and **(A1)-(A2)**.

Theorem 1.8. *Suppose that* $(\phi_{\pm,0}, v_0)$ *satisfying* (1.10)-(1.13) *and* **(A1)-(A2)**. *Then the solution* $(\phi_\pm, v)$ *of* (1.2)-(1.3) *satisfies*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\left|\phi_+(x,t)-(\widehat{\phi}^*(x;c)+ct)\right|=0,$$
$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\left|\phi_-(x,t)-\left(\widehat{\phi}^*(x;c)+ct-c\,G_1^{-1}\Big(\frac{2a}{b}\Big)\right)\right|=0,$$
$$\lim_{t\to\infty}\sup_{(x,y)\in\mathbb{R}^2}\left|v(x,y,t)-\widehat{v}^*(x,y-ct)\right|=0.$$

The rest of the paper is organized as follows. In Section 2, we show Theorem 1.4 by the help of Proposition 1.2 and prove Theorem 1.5 by constructing suitable super-sub solutions and applying the comparison principle. In Section 3, we divide it into two subsections. First, the global existence and uniqueness of the free boundary problem is discussed. For this, a priori estimates for $\phi_\pm$ and $\phi_{\pm,x}$ are investigated. Some difficulties occur due to the presence of $v(x,\phi_\pm,t)$ in the nonlinear term. Here the notion of the arrival time is introduced to overcome the difficulties. In the successive subsection, we study the asymptotic stability of traveling curved waves. The notion of the arrival time helps us estimate the propagation speed of solutions to the free boundary problem, which allows us to establish the asymptotic stability by using the comparison principle. In section 4, we give an example to illustrate that the gradient blowup can take place if the initial data is far from the traveling curved waves by using a geometric approach proposed in [12].

2. Existence and uniqueness of traveling curved waves

In this section, we shall prove Theorem 1.4 and Theorem 1.5. Before we start, we introduce some notations which will be used in the whole paper. Define

$$G_1^{-1}(v):=\int_0^v\frac{ds}{g(1,s)}.$$

Due to **(H)**, $G_1^{-1}(v)$ is well-defined for $v\ge 0$ and strictly increasing in $[0,\infty)$. Hence its inverse function, denoted by G_1, is well-defined and is also strictly increasing in $[0,\infty)$. Also, define

$$G_0^{-1}(v):=\begin{cases}\displaystyle\int_1^v\frac{ds}{g(0,s)} & \text{if } v>0,\\ \infty & \text{if } v=0.\end{cases}$$

By direct calculations, we have

$$G_0^{-1}(v)=-\frac{g_3}{g_2}(v-1)-\frac{g_4}{g_2}\log v,\quad v>0. \tag{2.1}$$

It is easy to see that $G_0^{-1}(\cdot)$ is strictly decreasing to $-\infty$ and $G_0^{-1}(1)=0$. By some simple computations, its inverse function satisfies

$$G_0(v)=\exp\left[-\frac{g_2}{g_4}v-\frac{g_3}{g_4}G_0(v)+\frac{g_3}{g_4}\right],\quad v>0. \tag{2.2}$$

Basic results about G_0 and G_1 are given as follows.

Lemma 2.1. *The function G_0 and G_1 satisfy*

$$0 \le G_0(G_0^{-1}(s)+t) \le s, \quad s,t \ge 0,$$

$$\frac{d}{ds}G_0(G_0^{-1}(s)+t) = \begin{cases} \dfrac{g(0, G_0(G_0^{-1}(s)+t))}{g(0,s)}, & s>0,\ t \ge 0, \\ e^{-g_2 t/g_4}, & s=0,\ t\ge 0, \end{cases}$$

$$\frac{d}{dt}G_1(G_1^{-1}(s)+t) = g(1, G_1(G_1^{-1}(s)+t)), \quad s,t \ge 0.$$

Proof. Since $G_0(\cdot)$ is decreasing to zero,

$$0 \le G_0(G_0^{-1}(s)+t) \le G_0(G_0^{-1}(s)) = s, \quad s,t \ge 0. \tag{2.3}$$

The chain rule immediately induces the derivative when $s>0$ and $t \ge 0$.

For $s=0$, by using (2.2) and (2.1), we have

$$\begin{aligned} &\lim_{s\searrow 0} \frac{G_0(G_0^{-1}(s)+t)}{s} \\ &= \lim_{s\searrow 0} \frac{1}{s}\exp\left\{-\frac{g_2}{g_4}(G_0^{-1}(s)+t) - \frac{g_3}{g_4}G_0(G_0^{-1}(s)+t) + \frac{g_3}{g_4}\right\} \\ &= \lim_{s\searrow 0} \exp\left\{\frac{g_3}{g_4}(s-1) - \frac{g_2}{g_4}t - \frac{g_3}{g_4}G_0(G_0^{-1}(s)+t) + \frac{g_3}{g_4}\right\} \\ &= \exp\left\{-\frac{g_2}{g_4}t\right\}. \end{aligned}$$

The last equality follows from (2.3) with taking $s \to 0$. Hence the proof is completed. $\square$

Next, we recall some properties of the solution $\widehat{\phi}^*(x;c)$ of (1.7) which will be used later frequently.

Lemma 2.2 (Lemma 2.1 in [28]). *The following properties hold for the solution $\widehat{\phi}^*(x;c)$ of (1.7).*

(1) *The following asymptotic estimates*

$$\begin{aligned} \widehat{\phi}^*_x(x;c) &= \pm\frac{\sqrt{c^2-a^2}}{a} + O(e^{-c\sqrt{c^2-a^2}|x|/a}) \qquad \text{as } x \to \pm\infty, \\ \widehat{\phi}^*(x;c) &= \frac{\sqrt{c^2-a^2}}{a}|x| + \eta + O(|x|e^{-c\sqrt{c^2-a^2}|x|/a}) \qquad \text{as } x \to \pm\infty \end{aligned}$$

hold.

(2) *A function $x\widehat{\phi}^*_x(x;c) - \widehat{\phi}^*(x;c)$ is strictly monotone increasing in $|x|$ with*

$$\begin{aligned} 0 \le x\widehat{\phi}^*_x(x;c) - \widehat{\phi}^*(x;c) &< |\eta|, \qquad x \in \mathbb{R}, \\ \left\{x\widehat{\phi}^*_x(x;c) - \widehat{\phi}^*(x;c)\right\}\Big|_{x=0} &= 0, \\ \lim_{|x|\to\infty}\left(x\widehat{\phi}^*_x(x;c) - \widehat{\phi}^*(x;c)\right) &= |\eta|. \end{aligned}$$

(3) *A function $c - a\sqrt{1+\widehat{\phi}^*_x(x;c)^2}$ is strictly monotone decreasing in $|x|$ with*

$$\begin{aligned} \left\{c - a\sqrt{1+\widehat{\phi}^*_x(x;c)^2}\right\}\Big|_{x=0} &= c-a, \\ \lim_{|x|\to\infty}\left(c - a\sqrt{1+\widehat{\phi}^*_x(x;c)^2}\right) &= 0, \end{aligned}$$

(4) *For* $0<\alpha<1$, $\widehat{\phi}^*(x;c)-\alpha^{-1}\widehat{\phi}^*(\alpha x;c)$ *is strictly monotone increasing in* $|x|$ *with*

$$0\le \widehat{\phi}^*(x;c)-\frac{1}{\alpha}\widehat{\phi}^*(\alpha x;c)<|\eta|\left(\frac{1}{\alpha}-1\right),\qquad 0<\alpha<1,\ x\in\mathbb{R}.$$

For $\alpha>1$, $\alpha^{-1}\widehat{\phi}^*(\alpha x;c)-\widehat{\phi}^*(x;c)$ *is strictly monotone increasing in* $|x|$ *with*

$$0\le \frac{1}{\alpha}\widehat{\phi}^*(\alpha x;c)-\widehat{\phi}^*(x;c)<|\eta|\left(1-\frac{1}{\alpha}\right),\qquad \alpha>1,\ x\in\mathbb{R}.$$

From Lemma 2.2, we have the following result.

Lemma 2.3. *The following properties hold for the solution* $\widehat{\phi}^*(x;c)$ *of* (1.7)*.*

(1) $\widehat{\phi}^*_x(x;c)\uparrow m_*:=\sqrt{c^2-a^2}/a$ *as* $x\to\infty$ *and* $\widehat{\phi}^*_x(x;c)\downarrow -m_*$ $x\to-\infty$*.*
(2) $\widehat{\phi}^*_x(x;c)$ *converges to* $\pm m_*$ *exponentially as* x *tends to* $\pm\infty$ *respectively. More precisely,*

$$\lim_{x\to\pm\infty}\frac{\widehat{\phi}^*_x(x;c)\mp m_*}{e^{-cm_*|x|}}=\mp\lambda_0\quad \textit{for some positive constant } \lambda_0$$

Proof. The statement (1) follows from Lemma 2.2(1) and (3) immediately. It suffices to deal with (2). By (1.8), we can see easily that

$$\lim_{x\to\infty}\frac{1-\sqrt{\frac{c+k}{c-k}}\tan(\frac{\theta}{2})}{e^{-cm_*x}}=\lambda_1 \tag{2.4}$$

for some positive constant λ_1. By differentiating (1.8) in x, we have

$$1=\frac{\theta_x}{c}+\frac{\sec^2(\frac{\theta}{2})}{2cm_*}\sqrt{\frac{c+k}{c-k}}\left[\frac{\theta_x}{1+\sqrt{\frac{c+k}{c-k}}\tan(\frac{\theta}{2})}+\frac{\theta_x}{1-\sqrt{\frac{c+k}{c-k}}\tan(\frac{\theta}{2})}\right].$$

Taking $x\to\infty$ i.e., $\theta_x\to 0$, we have

$$\lim_{x\to\infty}\frac{\theta_x}{1-\sqrt{\frac{c+k}{c-k}}\tan(\frac{\theta}{2})}=\lambda_2 \tag{2.5}$$

for some positive constant λ_2. Note that $\widehat{\phi}^*_x=\tan\theta$. By l'Hôpital's rule, (2.4) and (2.5),

$$\lim_{x\to\infty}\frac{\widehat{\phi}^*_x(x;c)-m_*}{e^{-cm_*x}}=\lim_{x\to\infty}\frac{\theta_x\sec^2\theta}{(-cm_*)e^{-cm_*x}}=-\lambda_0\quad\text{for some positive constant }\lambda_0.$$

By the symmetry, we can obtain the result as $x\to-\infty$. Hence the proof of Lemma 2.3 is completed. □

Now we are ready to show Theorem 1.4.

Proof of Theorem 1.4. For any given $c>a>0$, by Proposition 1.2 and the argument in front of it, the front is given by $\widehat{\phi}_+(x)=\widehat{\phi}^*(x;c)$ and $\widehat{v}(x,y)=0$ for all $(x,y)\in\widehat{\Omega}_+$. The value of $\widehat{v}$ on $\widehat{\Omega}$ is determined by (1.4). In fact, (1.4) gives us that

$$-\frac{c\widehat{v}_y}{g(1,\widehat{v})}=1,\quad x\in\mathbb{R}\text{ and }\widehat{\phi}_-(x)\le y\le\widehat{\phi}_+(x).$$

Integrating the above equation respect to y over $(y,\widehat{\phi}_+(x))$, we obtain

$$\widehat{v}(x,y)=G_1\left(\frac{\widehat{\phi}_+(x)-y}{c}\right),\quad x\in\mathbb{R}\text{ and }\widehat{\phi}_-(x)\le y\le\widehat{\phi}_+(x). \tag{2.6}$$

Note that $\widehat{v}\in C^2(\widehat{\Omega})$.

Next, we construct the back. Define

$$\widehat{\phi}_-(x) := \widehat{\phi}_+(x) - c\,G_1^{-1}\left(\frac{2a}{b}\right) = \widehat{\phi}^*(x;c) - c\,G_1^{-1}\left(\frac{2a}{b}\right).$$

From (2.6), we see that $\widehat{v}(x, \widehat{\phi}_-(x)) = 2a/b$ for $x \in \mathbb{R}$, which implies $W(\widehat{v}(x, \widehat{\phi}_-(x))) = -a$ for $x \in \mathbb{R}$. It follows that $\widehat{\phi}_+(x;c) - c\,G_1^{-1}(2a/b)$ satisfies (1.6). Thus, the existence of a traveling back has been established.

Finally, we need to define the value of $\widehat{v}$ for $\widehat{\Omega}_-$. Note that (1.4) gives us that

$$\frac{\widehat{v}_y}{g(0,\widehat{v})} = -\frac{1}{c}, \quad x \in \mathbb{R} \text{ and } y \le \widehat{\phi}_-(x).$$

Integrating the above equation respect to y over $(y, \widehat{\phi}_-(x))$, we have

$$G_0^{-1}(\widehat{v}(x,y)) = \frac{\widehat{\phi}_-(x) - y}{c} + G_0^{-1}\left(\frac{2a}{b}\right).$$

Thus, we obtain

$$\widehat{v}(x,y) = G_0\left(\frac{\widehat{\phi}_-(x) - y}{c} + G_0^{-1}\left(\frac{2a}{b}\right)\right)$$

in $\widehat{\Omega}_-$. Also, notice that $\widehat{v} \in C(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus \{y = \widehat{\phi}_\pm(x)\})$ and $\widehat{\phi}_\pm \in C^2(\mathbb{R})$. This completes the proof. □

As in the proof, the front of the traveling curved wave is uniquely determined up to the shift. The proof of the uniqueness is based on the comparison principle for the back. For this, we recall the following Phragmèn-Lindelöf principle (see e.g., [34]).

Proposition 2.4. *Let $\mathcal{L}$ be a second order differential operator:*

$$\mathcal{L} := \frac{\partial}{\partial t} - \alpha(x,t)\frac{\partial^2}{\partial x^2} - \beta(x,t)\frac{\partial}{\partial x},$$

where $\alpha, \beta \in L^\infty(Q_T)$ and $Q_T := (-\infty, \infty) \times (0, T]$. Suppose that $(\mathcal{L} + h)[u] \le 0$ in Q_T for some $h = h(x,t)$ which is bounded from below in Q_T and

$$\liminf_{R\to\infty} e^{-\gamma R^2}\left[\max_{|x|=R, t\in[0,T]} u(x,t)\right] \le 0 \text{ for some } \gamma > 0.$$

If $u(x,0) \le 0$ in $\mathbb{R}$, then $u(x,t) \le 0$ in Q_T.

We shall apply Proposition 2.4 to establish a certain comparison principle for the following general equation:

$$\mathcal{N}[\phi] := \phi_t - \frac{\phi_{xx}}{1+\phi_x^2} - F(x,t,\phi,\phi_x) = 0 \quad \text{in } Q_T, \tag{2.7}$$

where $F \in C^1\Big(\{(x,t,u,p) |\, x, u \in \mathbb{R},\, t \ge 0,\, k_1 \le p \le k_2\}\Big)$ for some $k_1, k_2 \in \mathbb{R}$ and

$$F_p \in L^\infty, \quad F_u \text{ is bounded from below.} \tag{2.8}$$

Then $\overline{\phi}(x,t)$ is called a supersolution of (2.7) in Q_T if $\mathcal{N}[\overline{\phi}] \ge 0$ in Q_T; $\underline{\phi}(x,t)$ is called a subsolution of (2.7) in Q_T if $\mathcal{N}[\underline{\phi}] \le 0$ in Q_T.

Lemma 2.5. *Suppose that $\overline{\phi}(x,t)$ and $\underline{\phi}(x,t)$ are a supersolution and a subsolution of (2.7) in Q_T, respectively, which satisfy the following:*

$$\overline{\phi}_x,\ \underline{\phi}_x,\ \overline{\phi}_{xx} \text{ and } \underline{\phi}_{xx} \in L^\infty(Q_T) \tag{2.9}$$

with $k_1 \le \overline{\phi}_x,\ \underline{\phi}_x \le k_2$. If $\overline{\phi}(x,0) \ge \underline{\phi}(x,0)$ in $\mathbb{R}$, then $\overline{\phi}(x,t) \ge \underline{\phi}(x,t)$ in $Q_T := \mathbb{R} \times (0,T]$.

Proof. Set $\Psi(x,t) := \underline{\phi}(x,t) - \overline{\phi}(x,t)$. Using $\mathcal{N}[\overline{\phi}] \geq 0$ and $\mathcal{N}[\underline{\phi}] \leq 0$, we have

$$\Psi_t \leq \frac{\Psi_{xx}}{1+\underline{\phi}_x^2} + \beta(x,t)\Psi_x - F_u\Psi \quad \text{in } Q_T,$$

where

$$\beta := \frac{\overline{\phi}_{xx}(\overline{\phi}_x + \underline{\phi}_x)}{(1+\overline{\phi}_x^2)(1+\underline{\phi}_x^2)} + F_p(x,t,\underline{\phi},\theta\underline{\phi}_x + (1-\theta)\overline{\phi}_x) \quad \text{for some } 0<\theta<1,$$
$$F_u := F_u(x,t,\rho\underline{\phi} + (1-\rho)\overline{\phi},\overline{\phi}_x) \quad \text{for some } 0<\rho<1.$$

The boundedness of β follows from (2.8) and (2.9). Finally, by (2.9), we see that

$$\liminf_{R\to\infty} e^{-\gamma R^2}\left[\max_{|x|=R,t\in[0,T]} \Psi(x,t)\right] = 0 \text{ for any } \gamma > 0.$$

Hence Lemma 2.5 follows from Proposition 2.4. □

To establish Theorem 1.5, we prepare two lemmas as follows:

Lemma 2.6. *Assume that $c > a > 0$. Suppose that $(c, \widehat{v}, \widehat{\phi}_\pm)$ is a traveling curved wave with the width $w(x) := \widehat{\phi}_+(x) - \widehat{\phi}_-(x)$. Then*

$$0 < \liminf_{x\to\pm\infty} w(x) \leq \limsup_{x\to\pm\infty} w(x) < \infty.$$

Proof. By Proposition 1.3, we may assume, without loss of generality, that $\widehat{\phi}_+(x) = \widehat{\phi}^*(x;c)$. We first derive that $\ell_\pm := \liminf_{x\to\pm\infty} w(x) > 0$. From (1.5) and (1.6), we have

$$\begin{aligned} w''(x) &= c(\widehat{\phi}'_+ + \widehat{\phi}'_-)w'(x) - a\left[(1+(\widehat{\phi}'_+)^2)^{3/2} + (1+(\widehat{\phi}'_-)^2)^{3/2}\right] \\ &\quad + bG_1\left(\frac{w(x)}{c}\right)\left[1+(\widehat{\phi}'_-)^2\right]^{3/2}. \end{aligned} \tag{2.10}$$

Here we have used (1.9), i.e.,

$$\widehat{v}(x,\widehat{\phi}_-(x)) = G_1\left(\frac{\widehat{\phi}_+(x) - \widehat{\phi}_-(x)}{c}\right) = G_1\left(\frac{w(x)}{c}\right).$$

We prove that $\ell_+ > 0$. For contradiction, we assume that $\ell_+ = 0$. Under this assumption, first, we show that $w'(x) < 0$ for all large x. Otherwise, there exists $x_n \to \infty$ such that $\lim_{n\to\infty} w(x_n) = 0$, $w'(x_n) = 0$ and $w''(x_n) \geq 0$ for all n. Since $\ell_+ = 0$,

$$G_1\left(\frac{w(x_n)}{c}\right) \to G_1(0) = 0 \quad \text{as } n \to \infty. \tag{2.11}$$

Also, note that $w'(x_n) = 0$ if and only if $\widehat{\phi}'_+(x_n) = \widehat{\phi}'_-(x_n)$. Thus, from (2.10), we have

$$0 \leq \left[-2a + bG_1\left(\frac{w(x_n)}{c}\right)\right][1+(\widehat{\phi}'_-)^2(x_n)]^{3/2} < 0 \quad \text{for all large } n,$$

which leads us to a contradiction. Therefore, $w'(x) < 0$, i.e., $\widehat{\phi}'_+(x) < \widehat{\phi}'_-(x)$ for all large x and $\lim_{x\to+\infty} w(x) = 0$. As seen in Lemma 2.3, we have that $\widehat{\phi}'_+(\infty) = (\widehat{\phi}^*)'(\infty) = m_*$ increasingly. Thus, $\widehat{\phi}'_\pm(x) \geq m_*/2 > 0$ for all large x. From (2.10) we see that

$$w''(x) < -2a\left[1+\left(\frac{m_*}{2}\right)^2\right]^{3/2} + bG_1\left(\frac{w(x)}{c}\right)[1+m_*^2]^{3/2}$$

for all large x. Together with (2.11), there exists $\kappa > 0$ such that $w''(x) \leq -\kappa$ for all large x. However, together with that $w'(x) < 0$ for all large x, it follows that $w(\xi) = 0$ for some large ξ, which contradicts with $w(x) > 0$ for all x. Thus, we must have $\ell_+ > 0$. We can show $\ell_- > 0$ by the same argument.

To prove that $L_\pm := \limsup_{x\to\pm\infty} w(x) > 0$. We only prove that $L_+ < \infty$ since the same argument can apply to prove that $L_- < \infty$. For contradiction we assume that $L_+ = \infty$. Then using the same argument as the above, we can show that $w'(x) > 0$ for all large x and $\lim_{x\to\pm\infty} w(x) = \infty$. Next, we rewrite (1.6) as

$$(\tan^{-1} \widehat{\phi}'_-(x))' = c + \left(a - bG_1\Big(\frac{w(x)}{c}\Big)\right)\sqrt{1 + (\widehat{\phi}'_-)^2}.$$

Taking $\xi_n \to \infty$ and integrating the above equation over $[\xi_n, \xi_n + 1]$, we have

$$\begin{aligned}&\tan^{-1} \widehat{\phi}'_-(\xi_n + 1) - \tan^{-1} \widehat{\phi}'_-(\xi_n)\\ &= \int_{\xi_n}^{\xi_n+1} \left[c + \left(a - bG_1\left(\frac{w(x)}{c}\right)\right)\sqrt{1 + (\widehat{\phi}'_-(x))^2}\right] dx.\end{aligned} \tag{2.12}$$

Letting $n \to \infty$ and noting that $G_1(w(x)/c) \to \infty$ as $x \to \infty$, we see that the left-hand side of (2.12) is uniformly bounded but the right-hand side of (2.12) tends to $-\infty$. We then reach a contradiction and so $L_+ < \infty$. This completes the proof. □

The next lemma is to construct a super/sub solution. To do so, we consider the following ordinary differential equation:

$$\rho'(t) = 2a - bG_1\left(\frac{\rho(t)}{c}\right), \ t > 0, \quad \rho(0) = \rho_0. \tag{2.13}$$

Due to the monotonicity of G_1, we easily see that

$$\begin{cases} \rho(t) \uparrow c\, G_1^{-1}(2a/b) & \text{as } t \to \infty \text{ if } \rho_0 \in (0, c\, G_1^{-1}(2a/b)), \\ \rho(t) \downarrow c\, G_1^{-1}(2a/b) & \text{as } t \to \infty \text{ if } \rho_0 \in (c\, G_1^{-1}(2a/b), \infty). \end{cases} \tag{2.14}$$

Lemma 2.7. *Define*

$$\overline{\phi}(x,t) := \widehat{\phi}^*(x;c) + ct - \overline{\rho}(t), \quad \underline{\phi}(x,t) := \widehat{\phi}^*(x;c) + ct - \underline{\rho}(t),$$

where $\overline{\rho}(t)$ *and* $\underline{\rho}(t)$ *satisfy* (2.13); $\overline{\rho}(0) = \overline{\rho}_0 \in (0, c\, G_1^{-1}(2a/b))$ *and* $\underline{\rho}(0) = \underline{\rho}_0 \in (c\, G_1^{-1}(2a/b), \infty)$, *respectively. Then* $\overline{\phi}(x,t)$ *and* $\underline{\phi}(x,t)$ *are a supersolution and a subsolution of* (2.7) *with*

$$F(x,t,\phi,\phi_x) := -\left[a - bG_1\Big(\frac{\widehat{\phi}^*(x;c) + ct - \phi(x,t)}{c}\Big)\right]\sqrt{1 + \phi_x^2},$$

respectively. Furthermore,

$$\lim_{t\to\infty} [\underline{\phi}(x,t) - ct] = \lim_{t\to\infty} [\overline{\phi}(x,t) - ct] = \widehat{\phi}^*(x;c) - c\, G_1^{-1}(2a/b). \tag{2.15}$$

Proof. Recall that $\mathcal{N}$ is defined in (2.7). We only show that $\mathcal{N}[\overline{\phi}(x,t)] \geq 0$ for all $x \in \mathbb{R}$ and $t > 0$ since the proof of $\mathcal{N}[\overline{\phi}(x,t)] \leq 0$ is similar.

Note that

$$\mathcal{N}[\overline{\phi}(x,t)] = -\overline{\rho}'(t) + c - \frac{\widehat{\phi}^*_{xx}}{1 + (\widehat{\phi}^*_x)^2} + \left[\left(a - bG_1\Big(\frac{\overline{\rho}(t)}{c}\Big)\right)\sqrt{1 + (\widehat{\phi}^*_x)^2}\right].$$

Note that $\widehat{\phi}^*$ satisfies (1.7), we have

$$\begin{aligned}\mathcal{N}[\overline{\phi}(x,t)] &= -\overline{\rho}'(t) + \left[2a - bG_1\left(\frac{\overline{\rho}(t)}{c}\right)\right]\sqrt{1+(\widehat{\phi}^*_x)^2} \\ &\geq -\overline{\rho}'(t) + \left[2a - bG_1\left(\frac{\overline{\rho}(t)}{c}\right)\right] = 0,\ x\in\mathbb{R} \text{ and } t\geq 0\end{aligned}$$

where we used (2.14).

Finally, it is easy to check that (2.8) holds; also (2.15) follows from (2.14). Hence we have completed the proof. □

We are ready to verify Theorem 1.5.

Proof of Theorem 1.5. Let $(c, \widehat{v}, \widehat{\phi}_\pm)$ be any solution of (1.4)-(1.6). By Proposition 1.3, we have

$$\widehat{\phi}_+(x) = \widehat{\phi}^*(x+x_0) - y_0 \quad \text{for some } x_0, y_0 \in \mathbb{R}.$$

Since traveling curved waves are translation invariant, without loss of generality, we may assume $x_0 = 0 = y_0$, i.e., $\widehat{\phi}_+(x) = \widehat{\phi}^*(x;c)$. To complete the proof, it suffices to prove that $\widehat{\phi}_-(x) = \widehat{\phi}^*(x;c) - c\,G_1^{-1}(2a/b)$. Thanks to Lemma 2.6, we can choose $\overline{\rho}_0 \in (0, c\,G_1^{-1}(2a/b))$ and $\underline{\rho}_0 > c\,G_1^{-1}(2a/b)$ such that

$$\overline{\rho}_0 < \widehat{\phi}_+(x) - \widehat{\phi}_-(x) < \underline{\rho}_0, \quad x\in\mathbb{R}. \tag{2.16}$$

Next, we consider a supersolution $\overline{\phi}(x,t)$ and a subsolution $\underline{\phi}(x,t)$ defined in Lemma 2.7 with $\overline{\rho}(0) = \overline{\rho}_0$ and $\underline{\rho}(0) = \underline{\rho}_0$. By (2.16), we have

$$\underline{\phi}(x,0) < \widehat{\phi}_-(x) < \overline{\phi}(x,0), \quad x\in\mathbb{R}.$$

By (1.6) and (2.6), we see that $\widehat{\phi}_-(x) + ct$ is a solution of (2.7) with initial data $\widehat{\phi}_-(x)$. Also, it is easy to check that (2.9) holds since $\widehat{\phi}^*{}_x, \widehat{\phi}^*{}_{xx} \in L^\infty(\mathbb{R})$. Hence Lemma 2.5 is available to conclude

$$\underline{\phi}(x,t) \leq \widehat{\phi}_-(x) + ct \leq \overline{\phi}(x,t), \quad x\in\mathbb{R} \text{ and } t\geq 0.$$

Taking $t\to\infty$ and using (2.15), we obtain $\widehat{\phi}_-(x) = \widehat{\phi}^*(x;c) - c\,G_1^{-1}(2a/b)$. Hence the proof of Theorem 1.5 is completed. □

3. Asymptotic stability of traveling curved waves

We divide this section into two subsections. In the former subsection, we discuss the existence and uniqueness of solutions of (1.2)-(1.3), i.e., Theorem 1.7. In the latter subsection, we will show the asymptotic stability, i.e., Theorem 1.8.

The key ingredient is the introduction of the arrival time, denoted as a function of (x,y). It allows us to analyze the behaviour of $v(x,y,t)$ and provide some important estimates. More precisely, for each $(x,y)\in\mathbb{R}^2$ with $y\geq\phi_\pm(x,0)$, the arrival times of the front ϕ_+ and the back ϕ_- at the position (x,y) are defined as $T_+ := T_+(x,y)$ and $T_- := T_-(x,y)$, respectively, satisfying

$$y = \phi_\pm(x, T_\pm(x,y)), \quad x\in\mathbb{R} \text{ and } y\geq\phi_\pm(x,0).$$

For convenience, we also define $T_\pm(x,y) = 0$ if $y < \phi_\pm(x,0)$; while $T_\pm(x,y) = \infty$ if $y \geq \sup_{t\geq 0}\phi_\pm(x,t)$ respectively. Then $T_\pm(x,y)$ is well-defined for all $(x,y)\in\mathbb{R}^2$ if $\phi_\pm(\cdot,t)$ is strictly increasing in t. In particular, if $\phi_\pm(\cdot,t)$ is strictly increasing to infinity as $t\to\infty$, it can be seen that $T_\pm(x,y)$ is finite for all $(x,y)\in\mathbb{R}^2$.

3.1. **The existence and uniqueness of the free boundary problem.** In this subsection, we always assume that the initial data $(v_0, \phi_{\pm,0})$ satisfying (1.10)-(1.13) and **(A1)-(A2)**.

The proof of Theorem 1.7 can be carried out in three steps:

Step 1. Solve (ϕ_+, v) uniquely satisfying the following system:

$$\begin{cases} \phi_{+,t} &= \dfrac{\phi_{+,xx}}{1+\phi_{+,x}^2} + W(v(x,\phi_+(x,t),t))\sqrt{1+\phi_{+,x}^2}, \\ v_t &= g(\chi_{\{y<\phi_+(x,t)\}}, v), \quad t>0, \ (x,y)\in\Omega_+(t). \end{cases} \tag{3.1}$$

Step 2. Solve (ϕ_-, v) uniquely satisfying the following system:

$$\begin{cases} \phi_{-,t} &= \dfrac{\phi_{-,xx}}{1+\phi_{-,x}^2} - W(v(x,\phi_-(x,t),t))\sqrt{1+\phi_{-,x}^2}, \\ v_t &= g(\chi_{\{y<\phi_+(x,t)\}}, v), \quad t>0, \ (x,y)\in\Omega(t). \end{cases} \tag{3.2}$$

Step 3. Solve v over $\Omega_-(t)$ using the value of v on the back. Moreover, we confirm that $\phi_+(x,t) > \phi_-(x,t)$ for $x\in\mathbb{R}$ and $t>0$.

We remark that a classical solution of (3.1) and (3.2) is defined parallel to that of Definition 1.6. However, we do not need Definition 1.6 (i) for classical solutions of (3.1) since ϕ_- is not involved in problem (3.1).

In order to get the well-definedness of T_+, we need the monotonicity of ϕ_+.

Lemma 3.1. *If (ϕ_+, v) is a solution of (3.1) for $t\in[0,T]$, then $\phi_{+,t}>0$ for $t\in[0,T]$.*

Proof. Note that if $\phi_{+,t}\ge 0$ for $x\in\mathbb{R}$ and $t\in[0,\tau]$, we have $\Omega_+(t)\subset\Omega_+(0)$ for all $t\in[0,\tau]$. It follows from the second equation of (3.1) that

$$\frac{cv_t(x,y,t)}{g(0,v(x,y,t))} = 1 \quad t\in[0,\tau] \text{ and } (x,y)\in\Omega_+(\tau).$$

Integrating the above equation with respect to t over $[0,\tau]$ gives

$$v(x,y,\tau) = G_0(G_0^{-1}(v_0(x,y))+\tau), \quad (x,y)\in\Omega_+(\tau).$$

Hence

$$\phi_{+,t} = \frac{\phi_{+,xx}}{1+\phi_{+,x}^2} + W(G_0(G_0^{-1}(v_0(x,\phi_+)+t))\sqrt{1+\phi_{+,x}^2}$$

as long as $\phi_{+,t}\ge 0$.

By differentiating the above equation in t and setting $\omega := \phi_{+,t}$, we have

$$\begin{cases} \omega_t = \dfrac{\omega_{xx}}{1+\phi_{+,x}^2} + \alpha(x,t,\phi_{+,x},\phi_{+,xx})\omega_x - b\Big[v_y\omega + g(0,v(x,\phi_+,t))\Big]\sqrt{1+\phi_{+,x}^2}, \\ \omega(x,0) = \phi_{+,t}(x,0), \quad x\in\mathbb{R}, \end{cases}$$

where

$$\alpha(x,t,\phi_{+,x},\phi_{+,xx}) := -2\frac{\phi_{+,x}\phi_{+,xx}}{(1+\phi_{+,x}^2)^2} + W(v(x,\phi_+(x,t),t))\frac{\phi_{+,x}}{\sqrt{1+\phi_{+,x}^2}}. \tag{3.3}$$

Since $g(0,v(x,\phi_+,t))<0$ and $\omega(x,0)>0$ by **(A2)**, the maximum principle gives $\omega=\phi_{+,t}>0$ as long as ϕ_+ exists. This completes the proof. □

By the help of Lemma 3.1 and the notion of T_+, the form of $v(x,y,t)$ can be derived explicitly via functions G_0 and G_1 defined in Section 2.

Lemma 3.2. *Let $(\phi_\pm, v)$ be a solution of (3.1)-(3.2). Then $v(x,y,t) = v^{\phi_+,0,T_+}(x,y,t)$ where*

$$v^{\phi_+,0,T_+}(x,y,t) := \begin{cases} G_0(G_0^{-1}(v_0(x,y)) + t), & t \le T_+(x,y), \\ G_1(G_1^{-1}(v^{\phi_+,0,T_+}(x,y,T_+)) + t - T_+), & t > T_+(x,y). \end{cases} \tag{3.4}$$

In particular,

$$v(x, \phi_+(x,t), t) = G_0(G_0^{-1}(v_0(x, \phi_+(x,t))) + t), \quad t > 0. \tag{3.5}$$

By definition, we also denote $v(x,y,t) = G_0(G_0^{-1}(v_0(x,y)) + t)$ by $v^{-\infty,\infty}(x,y,t)$. We note that the definitions of $\widehat{v}^{\phi_1,\phi_2}(x,y)$ in (1.11) and $v^{\phi_+,0,T_+}(x,y,t)$ in (3.4) are different.

Proof. By Lemma 3.1, $T_+ \in [0,\infty]$ in $\mathbb{R}^2$. When $t \le T_+(x,y)$, it follows from the second equation of (3.1) that

$$\frac{c v_t(x,y,\tau)}{g(0, v(x,y,\tau))} = 1, \quad \tau \in [0,t].$$

Integrating the above equation with respect to τ over $[0,t]$, we have

$$v(x,y,t) = G_0(G_0^{-1}(v_0(x,y)) + t). \tag{3.6}$$

When $T_+(x,y) < \infty$, we have

$$\frac{c v_t(x,y,\tau)}{g(1, v(x,y,\tau))} = 1, \quad \tau > T_+(x,y).$$

Integrating the above equation with respect to τ over (T_+, t), we have

$$v(x,y,t) = G_1(G_1^{-1}(v(x,y,T_+)) + t - T_+), \quad y > \phi_+(x,0) \text{ and } t > T_+(x,y).$$

Finally, (3.5) follows from (3.6). Hence we complete the proof of Lemma 3.2. □

Lemma 3.2 tells us that v can be solved explicitly by (3.5). To finish Step 1, the standard theory of quasilinear parabolic PDEs [21, Chapter 5] can be applied if we can establish a priori estimates. More precisely, since v can be solved by (3.5), it suffices to focus on the equation for ϕ_+. Let

$$w(x,t) = \phi_+(x,t) - \widehat{\phi}^*(x;c) - ct,$$

where $\widehat{\phi}^*(x;c)$ is defined in Proposition 1.2. By (3.5), w satisfies

$$\begin{aligned} w_t &= \Big(\arctan(w_x + \widehat{\phi}^*_x)\Big)_x - c \\ &\quad + W(G_0(G_0^{-1}(v_0(x, w + \widehat{\phi}^* + ct)) + t))\sqrt{1 + (w_x + \widehat{\phi}^*_x)^2}. \end{aligned} \tag{3.7}$$

Note that $w(\cdot,0) = \phi_{+,0}(x) - \widehat{\phi}^*(x;c) \in L^\infty(\mathbb{R})$. A priori estimates for w and w_x are required to apply [21, Theorem 8.1 of Chapter 5] to the problem (3.7) and to establish the global existence and uniqueness of the solution w to (3.7) and then so does ϕ_+.

In order to derive a priori estimates for w and w_x for $x \in \mathbb{R}$ and $t \in [0,T]$, we shall show that there exist two positive constants C_1 and C_2 such that

$$\begin{cases} |\phi_+(x,t) - \widehat{\phi}^*(x;c) - ct| \le C_1, & x \in \mathbb{R} \text{ and } t \in [0,T], \\ |\phi_{+,x}(x,t)| \le C_2, & x \in \mathbb{R} \text{ and } t \in [0,T]. \end{cases} \tag{3.8}$$

The first estimate of (3.8) will be established in Proposition 3.5. The second estimate is a little complicated due to the presence of $v(x, \phi_+(x,t), t)$ arising from the nonlinear term. In general, C_1 and C_2 can depend on T. However, we need the uniform in time estimates to ensure that the back cannot catch up with the front (see Remark 3.11). This is essential to establish the well-posedness to the problem (1.2) and (1.3). We shall separate our estimates for the front into two time intervals. The outline of the argument is as follows. First, if we have known $\phi_{+,t} > 0$ for all time, then from (3.5) and (1.13), we see that

$$v(x, \phi_+(x,t), t) = 0 \quad \text{if } \phi_+(x,t) \geq k_2,$$

where k_2 is given in (1.13). Hence if there exists $T^* \gg 1$ such that $\phi_+(x, T^*) \geq k_2$ for all $x \in \mathbb{R}$, then after that time, ϕ_+ always satisfies

$$\phi_{+,t} \;=\; \frac{\phi_{+,xx}}{1+\phi_{+,x}^2} + a\sqrt{1+\phi_{+,x}^2},\ x \in \mathbb{R},\ t \geq T^*.$$

Putting $t = T^*$ as a new initial time, this equation has been studied in [27], where the global existence and uniqueness of ϕ_+ and its uniform in time gradient estimate have been established for all $t \geq T^*$. In other words, if we can show ϕ_+ exists for all $t \in [0, T^*]$, then ϕ_+ can be extended to any positive time. However, such idea is not applicable to deal with Step 2 and some more complicated process will be needed since $v(x, \phi_-(x,t), t)$ does not vanish. Hence the argument of the uniform in time gradient estimate for the back is different from that of the front.

Our first goal is to establish upper and lower estimates for the front ϕ_+ (Proposition 3.5) by using Lemma 2.2 and the comparison principle. Set

$$L[\phi] := \phi_t - \frac{\phi_{xx}}{1+\phi_x^2} - a\sqrt{1+\phi_x^2}. \tag{3.9}$$

As in proving Lemma 2.5, we have

Lemma 3.3. *Suppose that $\overline{\phi}, \underline{\phi} \in C^{2,1}(Q_T)$ such that*

$$\overline{\phi}_x(x,t),\ \underline{\phi}_x(x,t),\ \overline{\phi}_{xx}(x,t) \text{ and } \underline{\phi}_{xx}(x,t) \text{ are all bounded in } Q_T. \tag{3.10}$$

and

$$L[\overline{\phi}] - L[\underline{\phi}] \geq 0 \quad \text{in } Q_T := \mathbb{R} \times (0, T]$$

If $\overline{\phi}(x,0) \geq \underline{\phi}(x,0)$ in $\mathbb{R}$, then $\overline{\phi}(x,t) \geq \underline{\phi}(x,t)$ in Q_T.

Because we can prove this lemma similarly to Lemma 2.5, we omit the proof.

Lemma 3.4. *Let $\varepsilon > 0$ be defined in (1.10) and L be given in (3.9). Assume that (ϕ_+, v) is a solution of (3.1). Then*

$$\left(-b\varepsilon e^{-\gamma_0 t}\sqrt{1+\phi_+^2}\right)\chi_{[-k_1,k_1]}(x) \leq L[\phi_+] \leq 0, \quad x \in \mathbb{R},\ t \geq 0,$$

where χ is a characteristic function and $\gamma_0 := g_2/(g_3\varepsilon + g_4)$.

Proof. Plugging ϕ_+ into L, we have

$$L[\phi_+] \;=\; \phi_{+,t} - \frac{\phi_{+,xx}}{1+\phi_{+,x}^2} - a\sqrt{1+\phi_{+,x}^2} = -bv(x, \phi_+(x,t), t)\sqrt{1+\phi_{+,x}^2}. \tag{3.11}$$

It follows from (1.10) and (1.13) that

$$\begin{cases} v_t = -\dfrac{g_2 v}{g_3 v + g_4} \leq -\gamma_0 v,\ (x,y) \in \Omega_+(t), \\ 0 \leq v_0(x,y) \leq \varepsilon\chi_{[-k_1,k_1]}(x),\ (x,y) \in \Omega_+(0). \end{cases} \tag{3.12}$$

By Lemma 3.1, we have $\Omega_+(t) \subset \Omega_+(0)$ for $t > 0$ so that

$$0 \leq bv(x, \phi_+(x,t), t) \leq b\varepsilon e^{-\gamma_0 t}\chi_{[-k_1,k_1]}(x), \quad x \in \mathbb{R} \text{ and } t \geq 0.$$

Combining the above inequalities and (3.11), we complete the proof. $\square$

Proposition 3.5. *Let $\varepsilon > 0$ be given in (1.10) small enough and $T > 0$. Assume that (ϕ_+, v) be a solution of (3.1) for $t \in [0, T]$ with*

$$\sup_{x\in\mathbb{R}, t\in[0,T]} |\phi_{+,x}(x,t)| \le K_0 \quad \textit{for some } K_0 > 0 \textit{ independent of } \varepsilon. \tag{3.13}$$

Then there exist constants $\delta_i = \delta_i(\varepsilon) \in (0,1)$ with $i = 1, 2$ such that $\lim_{\varepsilon\to 0} \delta_i(\varepsilon) = 0$ and

$$\frac{\widehat{\phi}^*((1+\delta_1)x;c)}{1+\delta_1} - \frac{|\eta|\delta_1}{1+\delta_1} \le \phi_+(x,t) - ct \le \frac{\widehat{\phi}^*((1-\delta_2)x;c)}{1-\delta_2} + \frac{|\eta|\delta_2}{1-\delta_2} \tag{3.14}$$

for all $x \in \mathbb{R}$ and $t \in [0,T]$, where $\eta < 0$ and $\widehat{\phi}^*(x;c)$ are defined in Proposition 1.2. Moreover,

$$\left|\phi_+(x,t) - \widehat{\phi}^*(x;c) - ct\right| \le \frac{|\eta|\delta_2}{1-\delta_2}, \quad x \in \mathbb{R}, \ t \in [0,T]. \tag{3.15}$$

Proof. Let us define

$$\bar{w}(x,t) := \frac{\widehat{\phi}^*((1-\delta_2)x;c)}{1-\delta_2} + \frac{|\eta|\delta_2}{1-\delta_2} + ct$$

and plug it into the operator L given by (3.9), we have

$$L[\bar{w}] \;=\; c - \frac{(1-\delta_2)\widehat{\phi}^*_{xx}}{1+(\widehat{\phi}^*_x)^2} - a\sqrt{1+(\widehat{\phi}^*_x)^2} = \frac{\delta_2\widehat{\phi}^*_{xx}((1-\delta_2)x;c)}{1+(\widehat{\phi}^*_x((1-\delta_2)x;c))^2} > 0.$$

It follows that $L[\bar{w}] - L[\phi_+] \ge 0$ for all $x \in \mathbb{R}$ and $t \ge 0$ since $L[\phi_+] \le 0$ (Lemma 3.4). Hence, to compare $\bar{w}$ and ϕ_+ for $t \in [0,T]$, it suffices to show $\bar{w}(x,0) \ge \phi_+(x,0)$ for all $x \in \mathbb{R}$. For this, we divide our discussion into two parts: $|x| \le k_1$ and $|x| > k_1$, where $k_1 > 0$ is given in (1.13).

In the former part, since $\widehat{\phi}^*(x;c) + \varepsilon \ge \phi_+(x,0)$ by (1.10), it suffices to show that

$$\bar{w}(x,0) \ge \widehat{\phi}^*(x;c) + \varepsilon, \quad |x| \le k_1. \tag{3.16}$$

We now use an idea in [28, Lemma 3.1] to show (3.16). To do so, we set

$$\begin{aligned} &M := |\eta| - (k_1\widehat{\phi}^*_x(k_1;c) - \widehat{\phi}^*(k_1;c)) > 0 \quad \text{(by Lemma 2.2(2))},\\ &\delta_2(\varepsilon) := \frac{\varepsilon}{\varepsilon + M}. \end{aligned}$$

Then for $|x| \le k_1$, we have

$$\begin{aligned} &\bar{w}(x,0) - [\widehat{\phi}^*(x;c) + \varepsilon]\\ &\quad = \frac{\widehat{\phi}^*((1-\delta_2)x;c)}{1-\delta_2} - \widehat{\phi}^*(x;c) + \frac{|\eta|\delta_2}{1-\delta_2} - \varepsilon\\ &\quad \ge \frac{\widehat{\phi}^*((1-\delta_2)k_1;c)}{1-\delta_2} - \widehat{\phi}^*(k_1;c) + \frac{|\eta|\delta_2}{1-\delta_2} - \varepsilon \end{aligned}$$

by Lemma 2.2 (4). Furthermore, using Lemma 2.2 (2), we have

$$
\begin{aligned}
&\bar{w}(x,0) - [\widehat{\phi}^*(x;c)+\varepsilon]\\
&= -\int_{1-\delta_2}^{1} \frac{\partial}{\partial\xi}\left[\frac{1}{\xi}\widehat{\phi}^*(\xi x;c)\right] d\xi + \frac{|\eta|\delta_2}{1-\delta_2} - \varepsilon\\
&= -\int_{1-\delta_2}^{1} \left[\frac{1}{\xi^2}(\xi x\widehat{\phi}_x^*(\xi x;c) - \widehat{\phi}^*(\xi x;c))\right] d\xi + \frac{|\eta|\delta_2}{1-\delta_2} - \varepsilon\\
&\geq -\left[(k_1\widehat{\phi}_x^*(k_1;c) - \widehat{\phi}^*(k_1;c))\right]\int_{1-\delta_2}^{1} \frac{d\xi}{\xi^2} + \frac{|\eta|\delta_2}{1-\delta_2} - \varepsilon\\
&= M\frac{\delta_2}{1-\delta_2} - \varepsilon = 0.
\end{aligned}
$$

Hence (3.16) holds.

For $|x| > k_1$, we have $\phi_+(x,0) = \widehat{\phi}^*(x;c)$. By Lemma 2.2(4),

$$
\bar{w}(x,0) = \frac{\widehat{\phi}^*((1-\delta_2)x;c)}{1-\delta_2} + \frac{|\eta|\delta_2}{1-\delta_2} > \widehat{\phi}^*(x;c) = \phi_+(x,0), \quad |x| > k_1. \tag{3.17}
$$

Combining (3.16) and (3.17), we have $\bar{w}(x,0) \geq \phi_+(x,0)$ for all $x \in \mathbb{R}$. Also, it is easy to check (3.10) holds with $\overline{\phi} = \bar{w}$ and $\underline{\phi} = \phi_+$ in Lemma 3.3. By Lemma 3.3, $\bar{w}(x,t) \geq \phi_+(x,t)$ for all $x \in \mathbb{R}$ and $t \geq 0$, which implies the right-hand inequality of (3.14).

To derive the left-hand inequality of (3.14), we set

$$
\begin{aligned}
&\underline{w}(x,t) := \frac{\widehat{\phi}^*((1+\delta_1)x;c)}{1+\delta_1} - \frac{|\eta|\delta_1}{1+\delta_1} + ct,\\
&\delta_1 := \max\left\{\frac{b\varepsilon\sqrt{1+K_0^2}}{c - a\sqrt{1+[\widehat{\phi}_x^*(2k_1;c)]^2}}, \frac{\varepsilon}{|\eta| - 2k_1\widehat{\phi}_x^*(2k_1;c) + \widehat{\phi}^*(2k_1;c) - \varepsilon}\right\}.
\end{aligned}
$$

Note that $\delta_1 > 0$ because of Lemma 2.2; $\delta_1 < 1$ as long as $\varepsilon > 0$ small enough. By Lemma 3.4 and direct computations, we have

$$
\begin{aligned}
&L[\phi_+] - L[\underline{w}]\\
&\geq \delta_1\left(c - a\sqrt{1+[\widehat{\phi}_x^*((1+\delta_1)x;c)]^2}\right) - \left(b\varepsilon e^{-\gamma_0 t}\sqrt{1+\phi_{+,x}^2(x,t)}\right)\chi_{[-k_1,k_1]}(x)
\end{aligned}
$$

for $x \in \mathbb{R}$ and $t \in [0,T]$.

We separate two parts: $|x| \leq k_1$ and $|x| > k_1$, respectively. For $|x| \geq k_1$, we have $\chi_{[-k_1,k_1]}(x) = 0$. Then

$$
L[\phi_+] - L[\underline{w}] = \delta_1\left(c - a\sqrt{1+[\widehat{\phi}_x^*((1+\delta_1)x;c)]^2}\right) \geq 0,
$$

which follows from Lemma 2.2 (3).

For $|x| \leq k_1$, by (3.13) and the definition of δ_1,

$$
L[\phi_+] - L[\underline{w}] \geq \delta_1\left(c - a\sqrt{1+[\widehat{\phi}_x^*(2k_1;c)]^2}\right) - b\varepsilon\sqrt{1+K_0^2} \geq 0.
$$

Hence $L[\phi_+] - L[\underline{w}] \geq 0$ for all $x \in \mathbb{R}$ and $t \in [0,T]$.

It suffices to show that $\phi_+(x,0) \geq \underline{w}(x,0)$ for all $x \in \mathbb{R}$. Again, we divide into two parts: $|x| > k_1$ and $|x| \leq k_1$. As in deriving (3.17), we have $\phi_+(x,0) \geq \underline{w}(x,0)$ for all $|x| > k_1$.

For $|x| \le k_1$, the same argument as in the proof of (3.16) gives us

$$\begin{aligned}
&\underline{w}(x,0) - (\widehat{\phi}^*(x;c) - \varepsilon) \\
&= \int_1^{1+\delta_1} \left[\frac{1}{\xi^2}(\xi x \widehat{\phi}^*_x(\xi x;c) - \widehat{\phi}^*(\xi x;c))\right] d\xi - \frac{|\eta|\delta_1}{1+\delta_1} + \varepsilon \\
&\le -\left[|\eta| - (1+\delta_1)k_1\widehat{\phi}^*_x((1+\delta_1)k_1;c) + \widehat{\phi}^*((1+\delta_1)k_1;c)\right]\frac{\delta_1}{1+\delta_1} + \varepsilon \\
&\le -\left[|\eta| - 2k_1\widehat{\phi}^*_x(2k_1;c) + \widehat{\phi}^*(2k_1;c)\right]\frac{\delta_1}{1+\delta_1} + \varepsilon \\
&\le 0
\end{aligned}$$

where we used the definition of δ_1 and $\delta_1 < 1$. Consequently, we have $\phi_+(x,0) \ge \underline{w}(x,0)$ for all $x \in \mathbb{R}$. By the comparison (Lemma 3.3), the left-hand inequality of (3.14) follows.

Finally, (3.15) follows from (3.14) and Lemma 2.2 (4). Hence the proof of Proposition 3.5 is completed. ☐

By the help of Proposition 3.5, we can establish the global existence and uniqueness of the front with uniform and gradient estimates which are uniform in time.

Proposition 3.6. *The problem* (3.1) *with initial data* $(v_0, \phi_{\pm,0})$ *has a unique classical solution* (ϕ_+, v) *in* $Q_\infty := \mathbb{R} \times [0,\infty)$. *Moreover, the following hold:*

(i) $\|\phi_{+,x}\|_{L^\infty(Q_\infty)} + \|\phi_{+,t}\|_{L^\infty(Q_\infty)} \le M^+$ *for some* M^+ *depending only on* M_0,

(ii) $\phi_{+,t}(\cdot,t) \ge \zeta_+$ *in* Q_∞,*where* ζ_+ *is a positive constant depending only on* ζ_0, ε *and* M_0.

Proof. We first establish a priori estimates for $\|\phi_{+,x}\|_{Q_T}$ and $\|\phi_{+,t}\|_{Q_T}$ if the solution exists for $t \in [0,T]$. Assume that the solution exists, then by Lemma 3.2, v satisfies (3.6).

We now estimate $\phi_{+,x}$. Since v in (3.1) is given by (3.6), we have

$$\phi_{+,t} = \frac{\phi_{+,xx}}{1+\phi_{+,x}^2} + W(v^{-\infty,\infty}(x,\phi_+(x,t),t))\sqrt{1+\phi_{+,x}^2}. \tag{3.18}$$

By setting $\psi := \phi_{+,x}$, then

$$\begin{cases} \psi_t = \dfrac{\psi_{xx}}{1+\phi_{+,x}^2} + \alpha(x,t,\phi_{+,x},\phi_{+,xx})\psi_x - b\big[v_x + v_y\psi\big]\sqrt{1+\phi_{+,x}^2} & \text{in } Q_T, \\ \psi(x,0) = \phi'_{+,0}(x), \quad x \in \mathbb{R}, \end{cases} \tag{3.19}$$

where $v_x = v_x^{-\infty,\infty}(x,\phi_+(x,t),t)$, $v_y = v_y^{-\infty,\infty}(x,\phi_+(x,t),t)$ and α is given in (3.3).

We now construct a supersolution ψ_+ of (3.19) satisfying

$$\psi_+{}'(t) = 4b\varepsilon\psi_+{}^2(t), \quad \psi_+(0) := \max\{1, \|\phi'_{+,0}\|_{L^\infty(\mathbb{R})}\}.$$

The solution ψ_+ can be expressed as

$$\psi_+(t) = \left[\frac{1}{\psi_+(0)} - 4b\varepsilon t\right]^{-1}, \quad t \in \Big[0, \frac{1}{4b\varepsilon\psi_+(0)}\Big)$$

which is strictly increasing in time and blows up at $t = 1/(4b\varepsilon\psi_+(0))$. Lemma 2.1, (3.6) and (1.10) imply

$$\begin{aligned}
&-b\big(v_x + v_y\psi\big)\sqrt{1+\phi_{+,x}^2} \\
&= -b\sqrt{1+\phi_{+,x}^2}\,\frac{g(0, G_0(G_0^{-1}(v_0(x,\phi_+)) + t))}{g(0, v_0(x,\phi_+))}\Big(v_{0,x}(x,\phi_+) + v_{0,y}(x,\phi_+)\psi\Big) \\
&\le b\varepsilon\sqrt{1+\psi^2}\big(1+\psi\big).
\end{aligned} \tag{3.20}$$

Thus we see easily that ψ_+ is a supersolution of (3.19) for $t \in [0, 1/(4b\varepsilon\psi_+(0)))$, which implies

$$\|\psi\|_{L^\infty(Q_{T^*})} \leq \psi_+(T^*) = 2\psi_+(0), \tag{3.21}$$

where

$$T^* = T^*(\varepsilon) := \frac{1}{8b\varepsilon\psi_+(0)}. \tag{3.22}$$

Similarly, differentiating (3.18) in t and setting $\omega := \phi_{+,t}$, we have

$$\begin{cases} \omega_t = \dfrac{\omega_{xx}}{1+\phi_{+,x}^2} + \alpha(x,t,\phi_{+,x},\phi_{+,xx})\omega_x - b\Big[v_y\omega + g(0, v(x,\phi_+,t))\Big]\sqrt{1+\phi_{+,x}^2} \quad \text{in } Q_T, \\ \omega(x,0) = \phi_{+,t}(x,0), \quad x \in \mathbb{R}. \end{cases} \tag{3.23}$$

Using (3.21), we see that the coefficient of ω in (3.23), $bv_y\sqrt{1+\phi_{+,x}^2}$, is bounded in Q_{T^*}, where the bound can be made independent of ε. This allows us apply the maximum principle to obtain

$$\|\omega\|_{L^\infty(Q_{T^*})} \leq \hat{M}. \tag{3.24}$$

for some $\hat{M} = \hat{M} > 0$ depending only on M. The estimates (3.21) and (3.24) allows us to apply standard theory of quasilinear parabolic PDEs [21, Chapter 5] to obtain the existence and uniqueness of solutions of (3.1) for $t \in [0, T^*]$.

Next we shall show the lower estimate of ω. By **(A2)**, we have $\phi_{+,t}(x,0) \geq \zeta_0$. As similar in (3.20), there exists a constant $C = C(\varepsilon) > 0$ such that

$$\Big| - bv_y\omega\sqrt{1+\phi_{+,x}^2}\Big| \leq C(\varepsilon), \quad t \in [0, T^*].$$

It is easily seen from $g(0, v(x,\phi_+,t)) < 0$ that

$$\omega_-(t) := e^{-C(\varepsilon)t}\zeta_0$$

becomes a subsolution of (3.23). Hence we obtain

$$\phi_{+,t}(x,t) \geq e^{-C(\varepsilon)T^*}\zeta_0 := \zeta_+, \quad t \in [0, T^*]. \tag{3.25}$$

Finally, we shall show that the solution ϕ_+ can be extended for all $t \geq T^*$. For this, by (3.22), we can take $\varepsilon > 0$ small enough such that

$$T^*(\varepsilon) > \frac{|\eta| + k_2}{c}.$$

It follows from Proposition 3.5 that

$$\phi_+(x, T^*) > k_2, \quad x \in \mathbb{R}.$$

By (3.6), we have $v(x, \phi_+(x,t), t) = 0$ for all $x \in \mathbb{R}$ and $t \geq T^*$. Hence we see that after time T^*, ϕ_+ satisfies

$$\phi_{+,t} = \frac{\phi_{+,xx}}{1+\phi_{+,x}^2} + a\sqrt{1+\phi_{+,x}^2},\ x \in \mathbb{R},\ t > T^*. \tag{3.26}$$

It is well known (cf. [27]) that for any $T > T^*$, the problem (3.26) with initial data $\phi_+(x, T^*)$ has a unique classical solution for all $t \in [T^*, T]$ with

$$\|\phi_{+,x}\|_{L^\infty(\mathbb{R}\times[T^*,T])} \leq \sup_{x\in\mathbb{R}} |\phi_{+,x}(x, T^*)|.$$

Differentiating (3.26) in t, we easily obtain that

$$\|\phi_{+,t}\|_{L^\infty(\mathbb{R}\times[T^*,T])} \leq \sup_{x\in\mathbb{R}} |\phi_{+,t}(x, T^*)|.$$

Together with uniform estimates (3.21) and (3.24), we have

$$\|\phi_{+,x}\|_{L^\infty(Q_T)} + \|\phi_{+,t}\|_{L^\infty(Q_T)} \le M^+.$$

Note that M^+ is independent of T and ε. Combining this estimate and (3.15), we can apply the standard theory of quasilinear parabolic PDEs [21, Chapter 5] to show that ϕ_+ exists globally in time and part (i) follows.

For part (ii), we have already known (3.25) for $t \in [0, T^*]$. For $t > T^*$, since we have the uniform bound $\|\phi_{+,x}\|_{L^\infty(Q_T)}$, differentiating (3.26) in t and using the maximum principle gives $\phi_{+,t}(x,t) \ge \zeta_+$ for all $t \ge 0$ where ζ_+ is given in (3.25), which implies (ii). This completes the proof. □

We now move to Step 2. Namely, the global existence and uniqueness of the back. We need to investigate arrival time T_+ along the moving coordinate.

Lemma 3.7. *The arrival time $T_+(x,y)$ of the front belongs to $C^1(\Omega^+(0)) \cap C^1(\mathbb{R}^2 \setminus \Omega^+(0))$ and is globally Lipschitz in $\mathbb{R}^2$.*

Proof. By Proposition 3.6 (ii), we see that $T_+(x,\cdot)$ is strictly increasing in $[\phi_{+,0}(x), \infty)$. By the implicit function theorem implies that T_+ is of $C^1(\Omega^+(0))$. Moreover,

$$\left|\frac{\partial T_+}{\partial x}\right| = \frac{|\phi_{+,x}|}{\phi_{+,t}} \le \frac{M^+}{\zeta_+}, \quad 0 < \frac{\partial T_+}{\partial y} = \frac{1}{\phi_{+,t}} \le \frac{1}{\zeta_+}.$$

These facts imply that T_+ is globally Lipschitz continuous in $\Omega^+(0)$.

Recall that $T_+(x,y) = 0$ for $(x,y) \in \mathbb{R}^2 \setminus \Omega^+(0)$. Hence Lemma 3.7 follows. □

Lemma 3.8. *Let $\delta_i \in (0,1)$ $(i = 1,2)$ be defined in Proposition 3.5. Then the following estimates hold:*

$$\begin{aligned}&\left| T_+(x,y) - \left(\frac{y - \widehat{\phi}^*(x;c)}{c}\right)_+ \right| \\ \le\ & \min\left\{\frac{1}{c}\max\left\{\frac{|\eta|\delta_2(\varepsilon)}{1-\delta_2(\varepsilon)}, \frac{|\eta|\delta_1(\varepsilon)}{1+\delta_1(\varepsilon)}\right\}, \hat{M}|x|e^{-\frac{c}{a}\sqrt{c^2-a^2}(1-\delta_2(\varepsilon))|x|}\right\} := E_0^\varepsilon(x) \end{aligned} \tag{3.27}$$

for $(x,y) \in \mathbb{R}^2$, where $\hat{M}$ is a positive constant independent of ε.

Proof. Substituting $t = T_+(x,y)$ into (3.14) and noting that $\phi_+(x, T_+(x,y)) = y$, we have

$$\frac{1}{c}\left[y - \frac{\widehat{\phi}^*((1-\delta_2)x;c)}{1-\delta_2} - \frac{|\eta|\delta_2}{1-\delta_2}\right] \le T_+(x,y) \le \frac{1}{c}\left[y - \frac{\widehat{\phi}^*((1+\delta_1)x;c)}{1+\delta_1} + \frac{|\eta|\delta_1}{1+\delta_1}\right] \tag{3.28}$$

for all $(x,y) \in \mathbb{R}^2$. It follows that

$$\begin{aligned}&\left| T_+(x,y) - \left(\frac{y - \widehat{\phi}^*(x;c)}{c}\right)_+ \right| \\ \le\ & \frac{1}{c}\max\left\{-\widehat{\phi}^*(x;c) + \frac{\widehat{\phi}^*((1-\delta_2)x;c)}{1-\delta_2} + \frac{|\eta|\delta_2}{1-\delta_2}, \widehat{\phi}^*(x;c) - \frac{\widehat{\phi}^*((1+\delta_1)x;c)}{1+\delta_1} + \frac{|\eta|\delta_1}{1+\delta_1}\right\}.\end{aligned}$$

Together with Lemma 2.2(4) we have

$$\left| T_+(x,y) - \left(\frac{y - \widehat{\phi}^*(x;c)}{c}\right)_+ \right| \le \frac{1}{c}\max\left\{\frac{|\eta|\delta_2}{1-\delta_2}, \frac{|\eta|\delta_1}{1+\delta_1}\right\}, \quad (x,y) \in \mathbb{R}^2.$$

Moreover, using Lemma 2.2(1), we have

$$-\widehat{\phi}^*(x;c)+\frac{\widehat{\phi}^*((1-\delta_2)x;c)}{1-\delta_2}+\frac{|\eta|\delta_2}{1-\delta_2}=O(|x|e^{-\frac{c}{a}\sqrt{c^2-a^2}(1-\delta_2)|x|})\text{ as }x\to\pm\infty;$$
$$\widehat{\phi}^*(x;c)-\frac{\widehat{\phi}^*((1+\delta_1)x;c)}{1+\delta_1}+\frac{|\eta|\delta_1}{1+\delta_1}=O(|x|e^{-\frac{c}{a}\sqrt{c^2-a^2}|x|})\text{ as }x\to\pm\infty,$$

from which (3.27) follows. This completes the proof. □

Due to Lemma 3.8, we can investigate v along the moving coordinate.

Lemma 3.9. *Let $\delta_i\in(0,1)$ be defined in Proposition 3.5 and $(\phi_\pm,v)$ be a solution of* (3.1)-(3.2). *Then the following estimates hold:*

$$\left|v(x,y,t)-\widehat{v}^{\widehat{\phi}^*}(x,y-ct)\right|\le E_1^\varepsilon(x) \tag{3.29}$$

for all $t>T_+(x,y)$, where $\widehat{v}^{\widehat{\phi}^*}(x,y)$ are given in (1.12) *and*

$$E_1^\varepsilon(x):=g_1G_1^{-1}(\varepsilon)\chi_{[-k_1,k_1]}(x)+g_1E_0^\varepsilon(x) \tag{3.30}$$

with $\sup_{x\in\mathbb{R}}|E_1^\varepsilon(x)|\to0$ as $\varepsilon\to0$.

Proof. By Lemma 3.2,

$$\begin{aligned}
&|v(x,y,t)-\widehat{v}^{\widehat{\phi}^*}(x,y-ct)| \qquad (3.31)\\
=&\left|G_1\left(G_1^{-1}(v(x,y,T_+(x,y)))+t-T_+(x,y)\right)-G_1\left(\left(\frac{\widehat{\phi}^*(x;c)-y+ct}{c}\right)_+\right)\right|\\
\le&\ g_1\left|G_1^{-1}(v(x,y,T_+))+t-T_+-\left(\frac{\widehat{\phi}^*(x;c)-y+ct}{c}\right)_+\right|,
\end{aligned}$$

where we have used $\max_{v\ge0}|G_1'(v)|=g_1$.

To continue the above estimate we divide our discussion into two parts:

$$\text{(i) }\widehat{\phi}^*(x;c)-y+ct\ge0;\quad\text{(ii) }\widehat{\phi}^*(x;c)-y+ct<0.$$

For (i),

$$|v(x,y,t)-\widehat{v}^{\widehat{\phi}^*}(x,y-ct)|\ \le\ g_1G_1^{-1}(\varepsilon)\chi_{[-k_1,k_1]}(x)+g_1\left|T_+-\left(\frac{y-\widehat{\phi}^*(x;c)}{c}\right)_+\right|.$$

For (ii), using Proposition 3.5 we have

$$ct<y-\widehat{\phi}^*(x;c)=\phi_+(x,T_+(x,y))-\widehat{\phi}^*(x;c)\le\frac{\widehat{\phi}^*((1-\delta_2)x;c)}{1-\delta_2}+\frac{|\eta|\delta_2}{1-\delta_2}+cT_+(x,y)-\widehat{\phi}^*(x;c).$$

Hence we have

$$\begin{aligned}
&|v(x,y,t)-\widehat{v}^{\widehat{\phi}^*}(x,y-ct)|\\
&\le\ g_1\left|G_1^{-1}(v(x,y,T_+))+t-T_+\right|\\
&\le\ g_1G_1^{-1}(\varepsilon)\chi_{[-k_1,k_1]}(x)+\frac{g_1}{c}\left(-\widehat{\phi}^*(x;c)+\frac{\widehat{\phi}^*((1-\delta_2)x;c)}{1-\delta_2}+\frac{|\eta|\delta_2}{1-\delta_2}\right)\\
&\le\ g_1G_1^{-1}(\varepsilon)\chi_{[-k_1,k_1]}(x)+g_1E_0^\varepsilon(x),
\end{aligned}$$

where $E_0^\varepsilon(x)$ is defined in (3.27). By taking $E_1^\varepsilon(x)$ defined in (3.30), then (3.29) follows from Lemma 3.8 and combining two cases (i) and (ii). Moreover, we see that $\sup_{x\in\mathbb{R}}|E_1^\varepsilon(x)|\to0$ as $\varepsilon\to0$ since $G_1^{-1}(\varepsilon)\to0$ and $\delta_i(\varepsilon)\to0$ $(i=1,2)$ as $\varepsilon\to0$. This completes the proof. □

By the help of Lemma 3.8 and Lemma 3.9, we can establish the following result which is parallel to Proposition 3.5.

Proposition 3.10. *Let $(\phi_\pm, v)$ be a solution of (3.1)-(3.2) and $\delta_i \in (0,1)$, $i = 1, 2$, be defined in Proposition 3.5 and $T > 0$. Moreover, assume that $\phi_+(x,t) > \phi_-(x,t)$ for $x \in \mathbb{R}$ and $0 \le t \le T$ and that there exists a positive constant K_1 such that*

$$\sup_{x\in\mathbb{R}, t\in[0,T]} |\phi_{-,x}(x,t)| \le K_1 \quad \text{for some } K_1 > 0 \text{ independent of } \varepsilon.$$

Then there exist $\delta_i = \delta_i(\varepsilon) \in (0,1)$ $(i = 4, 5)$ such that $\lim_{\varepsilon\to 0} \delta_i(\varepsilon) = 0$ and

$$\frac{\widehat{\phi}^*((1+\delta_4)x; c)}{1+\delta_4} - \frac{|\eta|\delta_4}{1+\delta_4} \le \phi_-(x,t) - ct + c\,G_1^{-1}(\frac{2a}{b}) \le \frac{\widehat{\phi}^*((1-\delta_5)x; c)}{1-\delta_5} + \frac{|\eta|\delta_5}{1-\delta_5} \tag{3.32}$$

for all $x \in \mathbb{R}$ and $t \in [0,T]$, where $\eta < 0$ and $\widehat{\phi}^*(x; c)$ are defined in Proposition 1.2.

Proof. First recall that v in the first equation of (3.2) is given by $v^{\phi_+,0,T_+}$ as in (3.4). Plugging ϕ_- into L, we have

$$\begin{aligned} L[\phi_-] &= \Big(bv^{\phi_+,0,T_+}(x, \phi_-(x,t), t) - 2a\Big)\sqrt{1+\phi_{-,x}^2} \\ &= b\Big(v^{\phi_+,0,T_+}(x, \phi_-(x,t), t) - \widehat{v}^{\widehat{\phi}^*}(x, \phi_-(x,t) - ct)\Big)\sqrt{1+\phi_{-,x}^2} \\ &\quad + b\left\{G_1\left(\Big(\frac{\widehat{\phi}^*(x;c) + ct - \phi_-(x,t)}{c}\Big)_+\right) - G_1\Big(G_1^{-1}\Big(\frac{2a}{b}\Big)\Big)\right\}\sqrt{1+\phi_{-,x}^2}. \end{aligned}$$

Set

$$\begin{aligned} w(x,t) &:= \frac{\widehat{\phi}^*((1-\delta_5)x; c)}{1-\delta_5} + \frac{|\eta|\delta_5}{1-\delta_5} + ct - \phi_-(x,t) - c\,G_1^{-1}\Big(\frac{2a}{b}\Big), \\ v_2(x,t) &:= G_1\left(\left(\frac{w(x,t) + c\,G_1^{-1}\Big(\frac{2a}{b}\Big)}{c}\right)_+\right) \end{aligned}$$

where $\delta_5(\varepsilon) \in (0,1)$ is to be determined.

By direct computations, we get

$$L\left[w + \phi_-\right] = \delta_5 \left\{c - a\sqrt{1 + \widehat{\phi}_x^*((1-\delta_5)x)^2}\right\},$$

which yields that

$$L\left[w + \phi_-\right] - L[\phi_-] = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &:= b\Big(\widehat{v}^{\widehat{\phi}^*}(x, \phi_-(x,t) - ct) - v^{\phi_+,0,T_+}(x, \phi_-(x,t), t)\Big)\sqrt{1+\phi_{-,x}^2}, \\ I_2 &:= b\Big(v_2(x,t) - \widehat{v}^{\widehat{\phi}^*}(x, \phi_-(x,t) - ct)\Big)\sqrt{1+\phi_{-,x}^2}, \\ I_3 &:= b\left\{G_1\Big(G_1^{-1}\Big(\frac{2a}{b}\Big)\Big) - v_2(x,t)\right\}\sqrt{1+\phi_{-,x}^2}, \\ I_4 &:= \delta_5\left\{c - a\sqrt{1 + \widehat{\phi}_x^*((1-\delta_5)x; c)^2}\right\}. \end{aligned}$$

It can be seen that $I_2 \geq 0$ for all $x \in \mathbb{R}$ and $t \geq 0$. Indeed, by Lemma 2.2 (4), we see that

$$\frac{\widehat{\phi}^*((1-\delta_5)x;c)}{1-\delta_5} + \frac{|\eta|\delta_5}{1-\delta_5} + ct > \widehat{\phi}^*(x;c) + ct \quad \text{for all } \delta_5 \in (0,1),$$

which implies that

$$\begin{aligned} \widehat{v}^{\widehat{\phi}^*}(x, \phi_-(x,t) - ct) &= G_1\left(\left(\frac{\widehat{\phi}^*(x;c) + ct - \phi_-(x,t)}{c}\right)_+\right) \\ &\leq G_1\left(\left(\frac{w(x,t) + c\, G_1^{-1}\left(\frac{2a}{b}\right)}{c}\right)_+\right) = v_2(x,t) \end{aligned}$$

for all $\delta_5 \in (0,1)$.

On the other hand, set

$$A := \frac{b}{c}\sqrt{1+K_1^2}\left(\sup_{s\geq 0}\left|G_1'(s)\right|\right).$$

It follows that $Aw + I_3 \geq 0$ for all $x \in \mathbb{R}$ and $t \in [0,T]$. Hence we have

$$L\left[w + \phi_-\right] - L[\phi_-] + Aw \geq I_1 + I_4, \quad x \in \mathbb{R} \text{ and } t \in (0,T].$$

By the assumption, we see that $t = T_-(x, \phi_-(x,t)) > T_+(x, \phi_-(x,t))$ for $t \in (0,T]$. Hence Lemma 3.9 is available to ensure

$$I_1 \geq -bE_1^\varepsilon(x)\sqrt{1+K_1^2}, \quad x \in \mathbb{R},\ t \in (0,T]. \tag{3.33}$$

By Lemma 2.3, (3.33) and the assumption that K_1 is independent of ε, for any sufficiently small ε, we can take a positive constant $\delta_5 = \delta_5(\varepsilon) \in (\delta_2(\varepsilon), 1)$ such that

$$\begin{aligned} &I_1 + I_4 \geq \delta_5(\varepsilon)\left\{c - a\sqrt{1 + \widehat{\phi}_x^*((1-\delta_5)x;c)^2}\right\} - bE_1^\varepsilon(x)\sqrt{1+K_1^2} \geq 0,\ t \in (0,T] \\ &w(x,0) = \frac{\widehat{\phi}^*((1-\delta_5)x;c)}{1-\delta_5} + \frac{|\eta|\delta_5}{1-\delta_5} > \phi_-(x,0), \\ &\lim_{\varepsilon\to 0}\delta_5(\varepsilon) = 0. \end{aligned}$$

By Lemma 3.3, $w(x,t) \geq 0$ for $x \in \mathbb{R}$ and $t \in [0,T]$. Namely, we have obtained the right-hand inequality of (3.14).

Using the similar argument as above, we can show the left-hand inequality of (3.14) with some $\delta_4(\varepsilon)$ satisfying $\lim_{\varepsilon\to 0}\delta_4(\varepsilon) = 0$. Hence the proof of Proposition 3.10 is completed. □

Remark 3.11. The uniform in time gradient estimates is important to ensure the back cannot catch up with the front. Suppose $(\phi_\pm, v)$ is a solution of (3.1)-(3.2). If the gradient estimates for $\phi_\pm$ are uniform in time, i.e., K_0 and K_1 given in Proposition 3.5 and Proposition 3.10, respectively, are independent of time, then we have

$$\phi_+(x,t) - \phi_-(x,t) = c\, G_1^{-1}(2a/b) + O(\varepsilon) \quad \text{for all } x \in \mathbb{R} \text{ and } t > 0$$

as long as the solution $(\phi_\pm, v)$ exists.

Lemma 3.12. *Let $\varepsilon > 0$ be defined in (1.10) and $(\phi_\pm, v)$ be as in Proposition 3.10. Then there exists a positive constant $\delta_6(\varepsilon)$ such that $\lim_{\varepsilon\to 0}\delta_6(\varepsilon) = 0$ and*

$$bv^{\phi_+,0,T_+}(x, \phi_-(x,t), t) - 2a \geq -\delta_6(\varepsilon)$$

if $t > T_+(x, \phi_-(x,t))$ for all $x \in \mathbb{R}$.

Proof. Lemma 3.2 and the monotonicity of G_1 imply

$$\begin{aligned} bv^{\phi_{+,0},T_+}(x,\phi_-(x,t),t)-2a &= bG_1\Big(G_1^{-1}(v^{\phi_{+,0},T_+}(x,\phi_-,T_+(x,\phi_-)))+t-T_+(x,\phi_-)\Big)-2a \\ &\geq bG_1\Big(t-T_+(x,\phi_-)\Big)-2a. \end{aligned}$$

From (3.28), we have

$$\begin{aligned} t-T_+(x,\phi_-(x,t)) &\geq t-\frac{1}{c}\left[\phi_-(x,t)-\frac{\widehat{\phi}^*((1+\delta_1)x;c)}{1+\delta_1}+\frac{|\eta|\delta_1}{1+\delta_1}\right] \\ &= -\frac{1}{c}\left[\phi_-(x,t)-ct-\frac{\widehat{\phi}^*((1+\delta_1)x;c)}{1+\delta_1}+\frac{|\eta|\delta_1}{1+\delta_1}\right]. \end{aligned}$$

Moreover, using (3.32) and Lemma 2.2 (4), we have

$$\begin{aligned} t-T_+(x,\phi_-(x,t)) &\geq G_1^{-1}(\frac{2a}{b})-\frac{1}{c}\left[\frac{\widehat{\phi}^*((1-\delta_5)x;c)}{1-\delta_5}+\frac{|\eta|\delta_5}{1-\delta_5}-\frac{\widehat{\phi}^*((1+\delta_1)x;c)}{1+\delta_1}+\frac{|\eta|\delta_1}{1+\delta_1}\right] \\ &\geq G_1^{-1}(\frac{2a}{b})-\frac{1}{c}\left[\frac{|\eta|\delta_5}{1-\delta_5}+\frac{|\eta|\delta_1}{1+\delta_1}\right]. \end{aligned}$$

Lemma 3.12 follows from the above inequalities. □

We are ready to establish the existence and uniqueness of the back.

Proposition 3.13. *The problem* (3.2) *with initial data* $(v_0,\phi_{-,0})$ *has a unique classical solution* (ϕ_-,v) *in* $Q_\infty:=\mathbb{R}\times[0,\infty)$*. Furthermore, the following hold:*

(i) $\|\phi_{-,x}\|_{L^\infty(Q_\infty)}+\|\phi_{-,t}\|_{L^\infty(Q_\infty)}\leq M^-$ *for some* M^- *depending only on* M_0*.*
(ii) $\phi_{-,t}(\cdot,t)\geq e^{-\zeta_- t}\zeta_0$ *in* Q_∞*, where* $\zeta_-=\zeta_-(\zeta_0,\varepsilon,M_0)$ *is a positive constant and* ζ_0 *is given in* **(A2)**.

Proof. For any given $T>0$, we assume that (ϕ_-,v) is a solution of (3.2) for $t\in[0,T]$ with

$$\phi_+(x,t)-\phi_-(x,t)>\frac{1}{3}c\,G^{-1}(2a/b). \tag{3.34}$$

Due to (3.34), v in the first equation of (3.2) can be represented by $v^{\phi_{+,0},T_+}(x,\phi_-,t)$ given in (3.4). Note that $T_+(x,y)=0$ for all $x\in\mathbb{R}$ and $y\leq\phi_{+,0}(x)$. Hence from (3.4) we have

$$v^{\phi_{+,0},T_+}(x,y,t)=G_1(G_1^{-1}(v_0(x,y))+t),\quad x\in\mathbb{R},\ y\leq\phi_{+,0}(x).$$

The direct computation implies

$$v_y^{\phi_{+,0},T_+}(x,y,t) = \frac{g(1,G_1(G_1^{-1}(v_0(x,y))+t))}{g(1,v_0(x,y))}v_{0,y}(x,y),\quad x\in\mathbb{R},\ y<\phi_{+,0}(x). \tag{3.35}$$

Note that $v_y^{\phi_{+,0},T_+}$ is not defined for $y=\phi_{+,0}(x)$, which leads to some complexity in establishing a priori uniform in time estimates $\|\phi_{-,x}\|_{L^\infty(Q_T)}$.

Different from the proof of Proposition 3.6, here we first estimate $\|\phi_{-,t}\|_{Q_T}$, where $Q_T:=\mathbb{R}\times(0,T)$. For small $h_0>0$ such that $\phi_-(x,t)<\phi_{+,0}(x)$ for $x\in\mathbb{R}$ and $t\in[0,h_0)$, differentiating the first equation of (3.2) with respect to t and setting $\bar{\omega}:=\phi_{-,t}$, we have

$$\begin{cases} \bar{\omega}_t=\dfrac{\bar{\omega}_{xx}}{1+\phi_{-,x}^2}+\alpha(x,t,\phi_{-,x},\phi_{-,xx})\bar{\omega}_x+b\Big[v_y\bar{\omega}+g(1,v^{\phi_{+,0},T_+}(x,\phi_-,t))\Big]\sqrt{1+\phi_{-,x}^2} & \text{in } Q_{h_0}, \\ \bar{\omega}(x,0)=\phi_{-,t}(x,0),\quad x\in\mathbb{R}, \end{cases} \tag{3.36}$$

where $v_y = v_y^{\phi_{+,0},T_+}(x,\phi_-(x,t),t)$. Note that $g(1,v^{\phi_{+,0},T_+}(x,\phi_-,t)) > 0$ and $\bar{\omega}(x,0) > 0$ by **(A2)**. It follows that from the maximum principle that

$$\bar{\omega} := \phi_{-,t} > 0, \quad x \in \mathbb{R},\ t \in (0,h_0). \tag{3.37}$$

Hence we have $\phi_{-,0}(x) < \phi_-(x,h) < \phi_{+,0}(x)$ for $x \in \mathbb{R}$ and $0 < h < h_0$.

Since v_y appearing (3.36) is not defined when $y = \phi_{+,0}(x)$, instead of considering (3.36) we shall estimate $\omega := (\phi_-(x,t+h) - \phi_-(x,t))/h$ for all small $h > 0$ to get an upper estimate for $\|\phi_{-,t}\|_{Q_T}$. By some simple computations, we have

$$\begin{aligned} -\Big\{ & W(v(x,\phi_-(x,t+h),t+h)) - W(v(x,\phi_-(x,t),t))\Big\} \\ = \ & b\left[\frac{v(x,\phi_-(x,t+h),t) - v(x,\phi_-(x,t),t)}{\phi_-(x,t+h)-\phi_-(x,t)}\right] h\omega \\ & + b\Big\{v(x,\phi_-(x,t+h),t+h) - v(x,\phi_-(x,t+h),t)\Big\}. \end{aligned}$$

Hence ω satisfies

$$\begin{cases} \omega_t = \dfrac{\omega_{xx}}{1+\phi_{-,x}^2} + \alpha\omega_x + b(-\beta_1\omega + \beta_2)\sqrt{1+\phi_{-,x}^2} & \text{in } Q_T, \\ \omega(x,0) = \dfrac{\phi_-(x,h) - \phi_-(x,0)}{h}, \quad x \in \mathbb{R}, \end{cases} \tag{3.38}$$

where

$$\begin{aligned} \phi_{-,x} &= \phi_{-,x}(x,t), \\ \alpha &= \frac{\phi_{-,xx}(x,t+h)\{\phi_{-,x}(x,t) + \phi_{-,x}(x,t+h)\}}{\{1+\phi_{-,x}(x,t)^2\}\{1+\phi_{-,x}(x,t+h)^2\}}, \\ \beta_1 &= -\frac{v(x,\phi_-(x,t+h),t) - v(x,\phi_-(x,t),t)}{\phi_-(x,t+h)-\phi_-(x,t)}, \\ \beta_2 &= \frac{1}{h}\int_t^{t+h} g(1,v(x,\phi_-(x,t+h),s))ds. \end{aligned}$$

It follows that from the maximum principle that

$$\omega > 0, \quad x \in \mathbb{R},\ t \in [0,T] \tag{3.39}$$

for any $h \in (0,h_0)$.

In order to get the upper bound of ω, we shall construct a supersolution. For this, we show that there is a positive constant ν independent of T and all small ε such that

$$\beta_1 > \nu > 0 \quad \text{for all } x \in \mathbb{R},\ t \in (0,T) \text{ and for all small } h > 0. \tag{3.40}$$

We divide our discussion into three cases:

(i) $\phi_-(x,t+h) \le \phi_{+,0}(x)$; (ii) $\phi_-(x,t) > \phi_{+,0}(x)$; (iii) $\phi_-(x,t) \le \phi_{+,0}(x) < \phi_-(x,t+h)$.

For the case (i), it suffices to estimate v_y for $y < \phi_{+,0}(x)$. By the definition of $g(1,\cdot)$ and the assumption **(H)**, we have

$$\frac{g(1,G_1(G_1^{-1}(v_0(x,y))+t))}{g(1,v_0(x,y))} \ge \frac{g_1 - \frac{g_2}{g_3}}{g_1} > 0.$$

By $y < \phi_{+,0}(x)$ and (1.10), there exists a positive constant ν_1 independent of all small ε such that $v_{0,y}(x,y) \le -\nu_1$. By (3.35), we have

$$v_y^{\phi_{+,0},T_+}(x,y,t) < -\Big(\frac{g_1 - \frac{g_2}{g_3}}{2g_1}\Big)\nu_1 \tag{3.41}$$

for any $y < \phi_{+,0}(x)$. By (3.41) and the mean value theorem, there exists $\xi < \phi_-(x,t+h) \leq \phi_{+,0}(x)$ such that

$$\beta_1 = -v_y^{\phi_{+,0},T_+}(x,\xi,t) > \Big(\frac{g_1 - \frac{g_2}{g_3}}{2g_1}\Big)\nu_1.$$

Hence (3.40) follows in the case (i).

To consider the case (ii), it suffices to estimate v_y for $y > \phi_{+,0}(x)$ and $t > T_+(x,y)$. From (3.4) we have

$$v^{\phi_{+,0},T_+}(x,y,t) = G_1\Big(G_1^{-1}(v^{-\infty,\infty}(x,y,T_+(x,y))) + t - T_+(x,y)\Big).$$

Set $\widetilde{v} := v^{-\infty,\infty}(x,y,T_+(x,y))$. Then direct computation gives

$$v_y^{\phi_{+,0},T_+}(x,y,t) \quad = \quad I + J, \tag{3.42}$$

where

$$\begin{aligned} I &:= \frac{g(1, G_1(G_1^{-1}(\widetilde{v}) + t - T_+))}{g(1,\widetilde{v})}\frac{g(0, G_0(G_0^{-1}(v_0(x,y)) + T_+))}{g(0, v_0(x,y))}v_{0,y}(x,y), \\ J &:= \frac{g(1, G_1(G_1^{-1}(\widetilde{v}) + t - T_+))}{g(1,\widetilde{v})}g(0,\widetilde{v})T_{+,y} - g(1, v^{\phi_{+,0},T_+})T_{+,y}. \end{aligned}$$

We now estimate I. By $y > \phi_{+,0}(x)$ and (1.10), there exists $\delta_{13}(\varepsilon) > 0$ such that $|v_{0,y}(x,y)| \leq \delta_{13}(\varepsilon)$ and $\lim_{\varepsilon\to 0}\delta_{13}(\varepsilon) = 0$. Also, by the definition of $g(1,\cdot)$ and $g(0,\cdot)$, it follows that

$$|I| \leq \frac{g_1}{g_1 - \frac{g_2}{g_3}}\delta_{13}(\varepsilon). \tag{3.43}$$

To estimate J, we see from Lemma 3.7 that

$$T_{+,y}(x,y(x,t)) = \frac{1}{\phi_{+,t}(x, T_+(x,y(x,t)))}.$$

By Proposition 3.6 (i),

$$T_{+,y}(x,y(x,t)) \geq \frac{1}{M^+}, \tag{3.44}$$

where M^+ is independent of ε. By (3.44) and the fact that $g(0,\widetilde{v}) < 0$, we have

$$J \leq -\frac{g_1 - \frac{g_2}{g_3}}{g_1 M^+}. \tag{3.45}$$

By (3.42), (3.43) and (3.45) and note that $\lim_{\varepsilon\to 0}\delta_{13}(\varepsilon) = 0$ and that M^+ is independent of ε, it follows that there exists $\nu_2 > 0$ independent of T and all small ε such that

$$v_y^{\phi_{+,0},T_+}(x,y,t) < -\nu_2, \qquad x \in \mathbb{R},\ y > \phi_{+,0}(x),\ t \in (0,T). \tag{3.46}$$

Again, as in the case (i), the mean value theorem implies that $\beta_1 > \nu_2$ in the case (ii).

Next we consider the case (iii). If $\phi_-(x,t) < \phi_{+,0}(x)$, then the mean value theorem implies

$$\begin{aligned} \beta_1 &= -\frac{v(x,\phi_-(x,t+h),t) - v(x,\phi_{+,0}(x),t) + v(x,\phi_{+,0}(x),t) - v(x,\phi_-(x,t),t)}{\phi_-(x,t+h) - \phi_-(x,t)} \\ &= -\frac{v_y(x,y_1,t)(\phi_-(x,t+h) - \phi_{+,0}(x)) + v_y(x,y_2,t)(\phi_{+,0}(x) - \phi_-(x,t))}{\phi_-(x,t+h) - \phi_-(x,t)} \\ &\geq \nu_2 + \Big(\frac{g_1 - \frac{g_2}{g_3}}{2g_1}\Big)\nu_1 \end{aligned}$$

by (3.41) and (3.46), where $\phi_{+,0}(x) < y_1 < \phi_-(x,t+h)$ and $\phi_-(x,t) < y_2 < \phi_{+,0}(x)$. If $\phi_-(x,t) = \phi_{+,0}(x)$, then the mean value theorem implies

$$\beta_1 = -\frac{v(x,\phi_-(x,t+h),t) - v(x,\phi_{+,0}(x),t)}{\phi_-(x,t+h) - \phi_{+,0}(x)} \geq \nu_2.$$

Combining the above discussion, we have shown (3.40) for all cases.

By (3.40) and the fact that $\beta_2 < g_1$, it is easy to check that

$$\omega^+ := \max\left\{\frac{g_1}{\nu}, \|\phi_{-,t}(\cdot,0)\|_{L^\infty(\mathbb{R})}\right\}$$

is a supersolution of (3.38). Together with (3.37), we have

$$0 < \phi_{-,t}(x,t) \leq \omega^+, \quad x \in \mathbb{R},\ t \in [0,T]. \tag{3.47}$$

We now use (3.47) to derive an estimate of $\phi_{-,x}$ for $t \in [0,T]$. To do so, we consider an auxiliary function (cf. [10])

$$Q(x,t) := [\phi_{-,x}(x,t)]^2.$$

If there exists $x_0 \in \mathbb{R}$ and $t_0 \in [0,T]$ such that $Q(x_0,t_0)$ attains a local maximum which must be positive, we have

$$0 = Q_x(x_0,t_0) = 2\phi_{-,x}(x_0,t_0)\phi_{-,xx}(x_0,t_0).$$

Since $\phi_{-,x}(x_0,t_0) \neq 0$, it follows that $\phi_{-,xx}(x_0,t_0) = 0$. By Lemma 3.12,

$$\begin{aligned} \omega^+ &\geq \phi_{-,t}(x_0,t_0) \\ &= \Big[bv^{\phi_{+,0},T_+}(x_0,\phi_-(x_0,t_0),t_0) - a\Big]\sqrt{1+Q_x(x_0,t_0)} \\ &\geq \Big[a - \delta_6(\varepsilon)\Big]\sqrt{1+Q_x(x_0,t_0)}. \end{aligned}$$

Since $\delta_6(\varepsilon) \to 0$ as $\varepsilon \to 0$, and ω^+ depends only on M_0, there exists $\widetilde{M} = \widetilde{M}(M_0) > 0$ such that $Q(x_0,t_0) \leq \widetilde{M}^2$. In other words, for any local maximum point $(x_0,t_0) \in \mathbb{R}\times[0,T]$ of $|u_x|$, we have $|\phi_{-,x}(x_0,t_0)| \leq \widetilde{M}$. If such (x_0,t_0) does not exist for all large x, it means that $|\phi_{-,x}|$ is increasing for all large $|x|$. In this case, we can see that $|\phi_{-,x}| \leq 2m_*$ for all large $|x|$, where $m_* > 0$ is given in Lemma 1.2. Otherwise, it will contradict to Proposition 3.10. From above discussions, we know that

$$|\phi_{-,x}(x,t)| \leq \max\{\widetilde{M}, 2m_*\}, \quad x \in \mathbb{R},\ t \in (0,T). \tag{3.48}$$

From Proposition 3.6 (i) and (3.48), we know that the estimate for $\phi_{\pm,x}$ is uniform in time. From Remark 3.11 we see that for any small $\varepsilon > 0$,

$$\phi_+(x,t) - \phi_-(x,t) > \frac{2}{3}c\,G^{-1}(2a/b), \quad x \in \mathbb{R},\ t \in (0,T), \tag{3.49}$$

which is a better estimate than (3.34). Combining (3.34) and (3.49), we see that for all small $\varepsilon > 0$, any solution of (3.2) must satisfy $\phi_-(x,t) < \phi_+(x,t)$. This means that if (ϕ_-,v) is a solution of (3.2), v in the first equation of (3.2) is always represented by $v^{\phi_{+,0},T_+}(x,\phi_-,t)$.

Now we can apply standard theory of quasilinear parabolic PDEs [21] for the first equation of (3.2). As similar to Step 1, we set

$$w(x,t) = \phi_-(x,t) - \widehat{\phi}^*(x;c) - ct,$$

where $\widehat{\phi}^*(x;c)$ is defined in Proposition 1.2. Then w satisfies

$$\begin{aligned} w_t &= \Big(\arctan(w_x + \widehat{\phi}^*_x)\Big)_x - c \\ &\quad -W(v^{\phi_{+,0},T_+}(x,\phi_-,t))\sqrt{1+(w_x+\widehat{\phi}^*_x)^2}. \end{aligned}$$

Due to Proposition 3.10 and the estimate (3.48), by [21, Theorem 8.1, Chapter 5], we obtain the existence and uniqueness for w for $t \in (0,T)$ for all $T>0$, and then so does ϕ_-. Hence (3.1) has a unique solution for all $t>0$. Note that the estimates (3.47) and (3.48) do not depend on T. Hence we obtain the first statement (i).

The second statement (ii) follows from a similar argument as in deriving (3.25). Hence we complete the proof of Proposition 3.13. □

We are ready to show Theorem 1.7.

Proof of Theorem 1.7. By Proposition 3.6 and Proposition 3.13, we have already shown the existence and uniqueness of solutions of (1.2) and (1.3) except for v in $\Omega_-(t)$. To finish the proof, it suffices to show Step 3. In fact, for any $(x,y) \in \Omega_-(t)$, we can solve (1.3) by

$$v(x,y,t) = G_0(G_0^{-1}(v(x,y,T_-)) + t - T_-), \quad t > T_-(x,y), \tag{3.50}$$

using the value $v(x,y,T_-(x,y))$ on the back given in (3.4). Hence the proof of Theorem 1.7 is completed. □

3.2. **Asymptotic stability of traveling curved waves.** In this subsection, we deal with the asymptotic stability of the traveling curved wave. We always assume that the initial data $(v_0, \phi_{\pm,0})$ satisfying (1.10)-(1.13) and **(A1)-(A2)**.

To show the convergence of the front, we construct a supersolution and a subsolution as the following form which used in [28]:

$$w(x,t) := \frac{1}{\alpha(t)}\widehat{\phi}^*(\alpha(t)x;c) + ct + \beta(t)$$

with some suitable functions $\alpha(t)$ and $\beta(t)$. We call $w_+(x,t)$ (resp. $w_-(x,t)$) a supersolution (resp. a subsolution) if $L[w_+(x,t)] \geq 0$ (resp. $L[w_+(x,t)] \leq 0$) for $x \in \mathbb{R}$ and $t>0$.

Let us recall some results of [28].

Lemma 3.14 (Lemmas 2.1 and 2.2 in [28]). *Set*

$$\begin{aligned}
\alpha_\pm(t) &:= 1 \mp \delta e^{-\gamma t}, \\
\beta_\pm(t) &:= \sigma\left(\frac{1}{\alpha_\pm(t)} - 1\right) \pm \frac{(|\eta| - \sigma)\delta}{1 \mp \delta}, \\
w_\pm(x,t) &:= \frac{1}{\alpha_\pm(t)}\widehat{\phi}^*(\alpha_\pm(t)x;c) + \beta_\pm(t) + ct.
\end{aligned}$$

If σ, γ, δ satisfy

$$\gamma > 0, \quad 0 < \delta < 1, \quad 0 < \sigma < \Sigma(\gamma,\delta),$$

where

$$\Sigma(\gamma,\delta) := \inf_{-\infty<z<\infty}\left\{z\widehat{\phi}^*_z(z;c) - \widehat{\phi}^*(z;c) + \frac{(1-\delta)^2}{\gamma}\left(c - a\sqrt{1+\widehat{\phi}^*_z(z;c)^2}\right)\right\},$$

then w_+ (resp. w_-) is a supersolution (resp. subsolution) to $L[w]=0$. Moreover,

$$w_-(x,0) < \widehat{\phi}^*(x;c) < w_+(x,0), \qquad x \in \mathbb{R}$$

and

$$\lim_{|x|\to\infty} |w_\pm(x,0) - \widehat{\phi}^*(x;c)| = 0.$$

For the readers' convenience, we recall the outline of the proof in [28]. We set

$$z = \alpha(t)x.$$

By direct calculations,

$$\begin{aligned} L[w] &= w_t - \frac{w_{xx}}{1+w_x^2} - a\sqrt{1+w_x^2} \\ &= \frac{\alpha_t}{\alpha^2}(z\widehat{\phi}_z^*(z;c) - \widehat{\phi}^*(z;c)) + c + \beta_t - \frac{\alpha\widehat{\phi}_{zz}^*(z;c)}{1+\widehat{\phi}_z^*(z;c)^2} - a\sqrt{1+\widehat{\phi}_z^*(z;c)^2} \\ &= \frac{\alpha_t}{\alpha^2}\Big\{z\widehat{\phi}_z^*(z;c) - \widehat{\phi}^*(z;c) + \frac{(1-\alpha)\alpha^2}{\alpha_t}(c - a\sqrt{1+\widehat{\phi}_z^*(z;c)^2}) + \frac{\beta_t\alpha^2}{\alpha_t}\Big\}. \end{aligned}$$

Hence we have

$$L[w_\pm] = \pm\frac{\gamma\delta e^{-\gamma t}}{\alpha_\pm(t)^2}\Big\{z\widehat{\phi}_z^*(z;c) - \widehat{\phi}^*(z;c) + \frac{\alpha_\pm(t)^2}{\gamma}(c - a\sqrt{1+\widehat{\phi}_z^*(z;c)^2}) - \sigma\Big\}. \tag{3.51}$$

By Lemma 2.2 (2) and (3), we have $\Sigma(\gamma,\delta) > 0$. If $0 < \sigma < \Sigma(\gamma,\delta)$,

$$z\widehat{\phi}_z^*(z;c) - \widehat{\phi}^*(z;c) + \frac{\alpha_\pm(t)^2}{\gamma}\left(c - a\sqrt{1+\widehat{\phi}_z^*(z;c)^2}\right) - \sigma \geq \Sigma(\gamma,\delta) - \sigma > 0. \tag{3.52}$$

Thus $w_\pm$ are a supersolution and a subsolution, respectively. We also note that

$$\Sigma(\gamma,\delta) \uparrow |\eta| \quad \text{as } \gamma \to 0 \text{ uniformly for } \delta \in (0,\beta] \text{ for any } 0 < \beta < 1. \tag{3.53}$$

Thanks to Proposition 3.5 and Lemma 3.14, we can apply the comparison principle to show the convergence of the front.

Proposition 3.15. *For any small $\varepsilon' > 0$, there is a positive constant T such that*

$$0 \leq \sup_{(x,y)\in\Omega_+(t)} v(x,y,t) \leq \varepsilon', \quad t \geq T \tag{3.54}$$

$$\sup_{x\in\mathbb{R}} |\phi_+(x,t) - \widehat{\phi}^*(x;c) - ct| \leq \varepsilon', \quad t \geq T. \tag{3.55}$$

Proof. From (3.12) we see that there exists such $T > 0$ such that (3.54) follows. We now prove (3.55). By (3.53), we can choose $\gamma \in (0,\gamma_0)$ sufficiently small such that

$$|\eta| - \varepsilon' < \Sigma(\gamma,\delta_i) \leq |\eta| \quad \text{for } i = 1,2,$$

where Σ is defined in Lemma 3.14 and $\delta_i \in (0,1)$ is defined in Proposition 3.5 ($i = 1,2$). Thus we can choose $\sigma \in (|\eta| - 3\varepsilon', |\eta| - 2\varepsilon')$ and very close to $|\eta| - 2\varepsilon'$ such that

$$\varepsilon' < \Sigma(\gamma,\delta_i) - \sigma < 3\varepsilon', \ i = 1,2. \tag{3.56}$$

Moreover, since $\gamma_0 > \gamma$ we can choose $T_0 \gg 1$ such that

$$\frac{b\varepsilon(1+\delta_1)^2\sqrt{1+(M^+)^2}}{\gamma\delta_1}e^{-(\gamma_0-\gamma)T_0} < \varepsilon', \tag{3.57}$$

where M^+ is given in Proposition 3.6.

Define $w_+(x,t)$ given in Lemma 3.14 with $\delta := \delta_2$. By Lemma 3.4, $L[w_+] - L[\phi_+] \geq L[w_+]$. Using (3.51), (3.52) and (3.56), we have

$$L[w_+] - L[\phi_+] \geq \frac{\gamma\delta e^{-\gamma t}}{\alpha_+(t)^2}\Big\{\Sigma(\gamma,\delta_2) - \sigma\Big\} > 0, \quad x \in \mathbb{R} \text{ and } t \geq 0.$$

Also, it follows from Proposition 3.5 that $w_+(x,0) \geq \phi_+(x,0)$ for all $x \in \mathbb{R}$. By comparison (Lemma 3.3), we conclude that

$$w_+(x,t) \geq \phi_+(x,t), \quad x \in \mathbb{R} \text{ and } t \geq 0. \tag{3.58}$$

Next, we set $w_-(x,t)$ given in Lemma 3.14 with $\delta := \delta_1$. Then Proposition 3.5 gives

$$w_-(x,0) + cT_0 \leq \phi_+(x,T_0), \quad x \in \mathbb{R}.$$

Using Lemma 3.4, (3.51), (3.57) and (3.56), we have

$$\begin{aligned} & L[w_-(x,t)+cT_0]-L[\phi_+(x,t+T_0)] \\ \leq\ & -\frac{\gamma\delta_1 e^{-\gamma t}}{\alpha_-(t)^2}\Big\{\Sigma(\gamma,\delta_1)-\sigma-\frac{b\varepsilon(1+\delta_1)^2}{\gamma\delta_1}e^{-(\gamma_0-\gamma)T_0}\sqrt{1+{M_+}^2}\Big\} \\ \leq\ & -\frac{\gamma\delta_1 e^{-\gamma t}}{\alpha_-(t)^2}\Big\{\Sigma(\gamma,\delta_1)-\sigma-\varepsilon'\Big\}<0, \quad x\in\mathbb{R} \text{ and } t\geq T_0. \end{aligned}$$

By comparison (Lemma 3.3), we have

$$w_-(x,t)+cT_0\leq \phi_+(x,t+T_0), \quad x\in\mathbb{R} \text{ and } t\geq 0. \tag{3.59}$$

By (3.58) and (3.59), it follows that

$$w_-(x,t)-ct\leq\phi_+(x,t+T_0)-c(t+T_0)\leq w_+(x,t+T_0)-c(t+T_0),$$

for all $x\in\mathbb{R}$ and $t\geq 0$. Then there exists $T'>0$ such that

$$\widehat{\phi}^*(x;c)-\frac{(|\eta|-\sigma)\delta_1}{1+\delta_1}\leq\phi_+(x,t+T_0)-c(t+T_0)\leq\widehat{\phi}^*(x;c)+\frac{(|\eta|-\sigma)\delta_2}{1-\delta_2},$$

for all $x\in\mathbb{R}$ and $t\geq T'$. Since $|\eta|-\sigma<3\varepsilon'$, we have

$$|\phi_+(x,t)-(\widehat{\phi}^*+ct)|\leq 3\varepsilon'\max\Big\{\frac{\delta_1}{1+\delta_1},\frac{\delta_2}{1-\delta_2}\Big\}$$

for all $x\in\mathbb{R}$ and $t\geq T_0+T'$. This completes the proof. □

Remark 3.16. We remark that in Proposition 3.15, the choice of $\delta_i(\varepsilon)$ $(i=1,2)$ does not depend on ε'.

Due to the convergence of the front, we can derive the convergence of the back by constructing suitable super- and subsolutions. The convergence of v over $\Omega(t)\cup\Omega_-(t)$ can be done by using a similar argument as in Lemma 3.9.

Proposition 3.17. *Let $(\phi_\pm,v)$ be a solution of (3.1)-(3.2) Then for any small $\varepsilon'>0$, there is a positive constant T' such that*

$$\sup_{(x,y)\in\Omega(t)\cup\Omega_-(t)}|v(x,y,t)-\widehat{v}^*(x,y-ct)|\leq\varepsilon' \quad \textit{for all } t\geq T' \tag{3.60}$$

$$\sup_{x\in\mathbb{R}}\left|\phi_-(x,t)-\left(\widehat{\phi}^*(x;c)+ct-c\,G_1^{-1}\Big(\frac{2a}{b}\Big)\right)\right|\leq\varepsilon' \quad \textit{for all } t\geq T'. \tag{3.61}$$

Proof. Remember that a positive constant ε in (1.10) is fixed and so $\delta_i(\varepsilon)$ $(i=4,5)$ defined in Proposition 3.10 is also fixed.

We first prove (3.61). For any given $0<\varepsilon'<2|\eta|\delta_5(\varepsilon)/[1-\delta_5(\varepsilon)]$, we construct a subsolution ψ_- and a supersolution ψ_+ of the equation for the back ϕ_-

$$\phi_{-,t} \;=\; \frac{\phi_{-,xx}}{1+\phi_{-,x}^2}-W\big(v^{\phi_+,0,T_+}(x,\phi_-(x,t),t)\big)\sqrt{1+\phi_{-,x}^2}$$

as

$$\psi_\pm(x,t):=\widehat{\phi}^*(x;c)+ct-c\,G_1^{-1}\Big(\frac{2a}{b}\Big)\pm\beta_\pm(t),$$

where $\beta_\pm(t)$ satisfies the following ordinary differential equation:

$$\beta_\pm'(t)=-M_\pm\big(\beta_\pm(t)-\varepsilon'\big),\ t>T_1,\ \beta_\pm(T_1)=\frac{2|\eta|\delta_5(\varepsilon)}{1-\delta_5(\varepsilon)}>\varepsilon',$$

where $M_\pm>0$ and T_1 will be determined later.

Let $T_1 \gg 1$ such that $\phi_-(x, T_1) > k_1$ for $x \in \mathbb{R}$, where k_1 is given in (1.13). If follows from Lemma 3.2 and (1.13) that

$$v(x, y, T_+(x, y)) = 0, \quad x \in \mathbb{R},\ y \geq \phi_-(x, T_1).$$

Using Lemma 3.2 again, we have

$$\begin{aligned} v(x, y, t) &= G_1(G_1^{-1}(v(x, y, T_+(x, y))) + t - T_+(x, y)) \\ &= G_1(t - T_+(x, y)), \qquad x \in \mathbb{R},\ y \geq \phi_-(x, t) \text{ and } t \geq T_1. \end{aligned}$$

Let us define

$$T^*(x, y) := \left(\frac{y - \widehat{\phi}^*(x; c)}{c} \right)_+ . \tag{3.62}$$

Since we have shown the convergence of the front (Proposition 3.15), there exists $k^* > k_1$ such that

$$\Big| T_+(x, y) - T^*(x, y) \Big| < \frac{\varepsilon'}{c}, \quad x \in \mathbb{R} \text{ and } y \geq \widehat{\phi}^*(x; c) + k^*. \tag{3.63}$$

If necessary we take T_1 larger such that $\psi_+(x, t) \geq \widehat{\phi}^*(x; c) + k^*$ for all $x \in \mathbb{R}$ and $t \geq T_1$. Hence by (3.63) and (3.62),

$$\begin{aligned} & t - T_+(x, \psi_+) \\ & \quad = t - T^*(x, \psi_+) + T^*(x, \psi_+) - T_+(x, \psi_+) \\ & \quad \leq t - \left(\frac{ct - c\, G_1^{-1}(2a/b) + \beta_+}{c} \right)_+ + \frac{\varepsilon'}{c} \\ & \quad \leq G_1^{-1}(\frac{2a}{b}) - \frac{\beta_+}{c} + \frac{\varepsilon'}{c} \end{aligned}$$

for all $x \in \mathbb{R}$ and $t \geq T_1$.

Recall that

$$\beta_+(T_1) = \frac{2|\eta|\delta_5(\varepsilon)}{1 - \delta_5(\varepsilon)}, \tag{3.64}$$

where $\eta > 0$ is given in Proposition 1.2. By the monotonicity of G_1 and the mean value theorem, there exists $M_1 > 0$ such that

$$\begin{aligned} G_1(t - T_+(x, \psi_+)) - \frac{2a}{b} &\leq G_1 \left(G_1^{-1}(\frac{2a}{b}) - \frac{\beta_+}{c} + \frac{\varepsilon'}{c} \right) - G_1(G_1^{-1}(\frac{2a}{b})) \\ &\leq M_1(-\beta_+(t) + \varepsilon') \end{aligned}$$

Taking $M_+ := bM_1$, direct computations give

$$\begin{aligned} & \psi_{+,t} - \frac{\psi_{+,xx}}{1 + \psi_{+,x}^2} - b \left(G_1(t - T_+(x, \psi_+)) - \frac{a}{b} \right) \sqrt{1 + \psi_{+,x}^2} \\ & \quad \geq \beta_+'(t) - b \left(G_1(t - T_+(x, \psi_+)) - \frac{2a}{b} \right) \\ & \quad \geq \beta_+'(t) + M_+(\beta_+(t) - \varepsilon') \\ & \quad = 0 \end{aligned}$$

for all $x \in \mathbb{R}$, and $t \geq T_1$. Hence ψ_+ is a supersolution for all $t \geq T_1$. By Proposition 3.10, Lemma 2.2(4) and (3.64),

$$\begin{aligned}
\phi_-(x, T_1) &\leq \widehat{\phi}^*(x; c) + cT_1 - c\, G_1^{-1}\Big(\frac{2a}{b}\Big) + \frac{2|\eta|\delta_5(\varepsilon)}{1-\delta_5(\varepsilon)} \\
&= \widehat{\phi}^*(x; c) + cT_1 - c\, G_1^{-1}\Big(\frac{2a}{b}\Big) + \beta_+(T_1) \\
&= \psi_+(x, T_1), \quad x \in \mathbb{R}.
\end{aligned}$$

In order to apply Lemma 2.5, we define

$$F(x, t, u, p) := b\left(G_1(t - T_+(x, u)) - \frac{a}{b}\right)\sqrt{1+p^2}$$

Then we have

$$\begin{aligned}
F_u(x, t, u, p) &= -b\sqrt{1+p^2}G_1'(t - T_+(x, p))\frac{\partial T_+(x, p)}{\partial p}, \\
F_p(x, t, u, p) &= b\left(G_1(t - T_+(x, u)) - \frac{a}{b}\right)\frac{p}{\sqrt{1+p^2}}.
\end{aligned}$$

It is easy to check that $\psi_{+,x}, \psi_{+,xx} \in L^\infty$. By Lemma 3.7, we can apply Lemma 2.5 to guarantee

$$\phi_-(x, t) \leq \psi_+(x, t) = \widehat{\phi}^*(x; c) + ct - c\, G_1^{-1}\Big(\frac{2a}{b}\Big) + \beta_+(t), \quad x \in \mathbb{R},\ t \geq T_1.$$

Similarly, we can take $T_2 \gg 1$ and $M_- \gg 1$ such that

$$\phi_-(x, t) \geq \psi_-(x, t) = \widehat{\phi}^*(x; c) + ct - c\, G_1^{-1}\Big(\frac{2a}{b}\Big) - \beta_-(t)$$

for all $x \in \mathbb{R}$ and $t \geq T_2$.

Since $\beta_\pm(t)$ decays to ε' as $t \to \infty$, we obtain (3.61).

We now derive (3.60) for $(x, y) \in \Omega(t)$ by a similar proof as in Lemma 3.9. To do so, we shall estimate (3.31) via ε' instead of ε. It can be done along the proof of Lemma 3.9 with minor modifications. As in Lemma 3.9 we divide our discussion into two parts:

$$\text{(i) } \widehat{\phi}^*(x; c) - y + ct \geq 0; \quad \text{(ii) } \widehat{\phi}^*(x; c) - y + ct < 0.$$

For (i), using (3.54) and (3.63), the term (3.31) can be estimated as

$$|v(x, y, t) - \widehat{v}^{\widehat{\phi}^*}(x, y - ct)| \leq g_1\left[G_1^{-1}(\varepsilon')\chi_{[-k_1, k_1]}(x) + \frac{\varepsilon'}{c}\right], \ (x, y) \in \Omega(t)$$

for all large t. For (ii), using (3.55) we have

$$\widehat{\phi}^*(x; c) + ct < y = \phi_+(x, T_+(x, y)) \leq \widehat{\phi}^*(x; c) + cT_+(x, y) + \varepsilon'$$

for all large t. Hence (3.31) can be estimated as

$$\begin{aligned}
&|v(x, y, t) - \widehat{v}^{\widehat{\phi}^*}(x, y - ct)| \\
&\leq g_1\left|G_1^{-1}(v(x, y, T_+)) + t - T_+\right| \\
&\leq g_1\left[G_1^{-1}(\varepsilon')\chi_{[-k_1, k_1]}(x) + \frac{\varepsilon'}{c}\right], \ (x, y) \in \Omega(t)
\end{aligned}$$

for all large t. Combining the estimates in (i) and (ii), we obtain (3.60) for $(x, y) \in \Omega(t)$.

Finally, we can use a similar process as the above to derive (3.60) for $(x, y) \in \Omega_-(t)$ by using (3.50). This completes the proof. □

Proof of Theorem 1.8. Combining Proposition 3.15 and Proposition 3.17, we see that Theorem 1.8 follows. □

4. Gradient blow-up

In this section, we give an example to illustrate that the gradient blowup can take place for the solution ϕ_+ to the system (3.1) if the initial data is far from the traveling curved waves. To do so, let $g_i > 0$ $(i = 1, .., 4)$ such that **(H)** holds and $2/3 < a/b$. Next, we choose $(v_0, \phi_{+,0})$ satisfying

$$\phi_{+,0} - \widehat{\phi}^* \in L^\infty(\mathbb{R}) \cap C^2(\mathbb{R}),$$

$$v_0 \in C^1(\mathbb{R}^2), \quad v_0(x,y) = \begin{cases} 0 & (0 \le y), \\ \dfrac{1}{3} & (\xi_1 \le y \le \xi_0), \\ \dfrac{2}{3} & (y \le \xi_2) \end{cases}$$

where ξ_i $(i = 0, 1, 2)$ are constants $(\xi_2 < \xi_1 < \xi_0 < 0)$ specified later and $\widehat{\phi}^*$ is defined in Proposition 1.2. Then we look for a gradient blowup solution ϕ_+ satisfying

$$\phi_{+,t} \quad = \quad \frac{\phi_{+,xx}}{1+\phi_{+,x}^2} + \Big(a - bv(x,\phi_+,t)\Big)\sqrt{1+\phi_{+,x}^2}, \quad x \in \mathbb{R},\ t > 0 \tag{4.1}$$

where $v(x, y, t) := G_0(G_0^{-1}(v_0(x, y)) + t)$.

The gradient blowup of the equation (4.1) essentially follows from [12] using a geometric approach, though the time t is not included explicitly in his equation. We give an outline of proof briefly. The equation (4.1) can be rewritten as the curvature equation with driving force depending on x, y and t:

$$\mathcal{V} = -\kappa + a - bv(x, y, t). \tag{4.2}$$

By the choice of the initial data v_0, $v(x, y, t) = v_1(t)$ (resp. $v_2(t)$) in the region $\mathbb{R} \times [\xi_1, \xi_0]$ (resp. $\mathbb{R} \times (-\infty, \xi_2]$), where

$$v_i(t) = G_0(G_0^{-1}(i/3) + t), \quad i = 1, 2.$$

Then there is a positive time T satisfying

$$0 \le v_1(t) \le \frac{1}{3} < v_2(T) \le v_2(t) \le \frac{2}{3} < \frac{a}{b}, \quad 0 \le t \le T. \tag{4.3}$$

To show the gradient blowup, we consider two circles $C_i(t)$ $(i = 1, 2)$ with radii $R_i(t)$ centered at (x_i, y_i), respectively, where

$$x_1 < x_2, \quad y_2 < \xi_2 < \xi_1 < y_1 < \xi_0 < 0.$$

The normal of C_1 is taken as outward, while that of C_2 is as inward. We will choose x_i, y_i $(i = 1, 2)$ and ξ_i $(i = 0, 1, 2)$ such that

$$C_1(t) \subset \mathbb{R} \times [\xi_1, \xi_0], \qquad C_2(t) \subset \mathbb{R} \times (-\infty, \xi_2]$$

for $0 \le t \le T$. Let R_i be defined by

$$\begin{aligned} R_{1,t} \quad &= \quad -\frac{1}{R_1} + a - \frac{b}{3}, \quad R_1(0) = \frac{12}{b(3v_2(T)-1)}, \\ R_{2,t} \quad &= \quad -\frac{1}{R_2} - a + bv_2(T), \quad R_2(0) = \frac{12}{b(3v_2(T)-1)}. \end{aligned}$$

Note that by (4.3), $R_1(t)$ increases to ∞ as $t \to \infty$ and $R_2(t)$ shrinks to a point at some time. By (4.3) and some simple calculations, it can be confirmed that

$$R_{1,t} - \left(-\frac{1}{R_1} + a - bv(x,y,t)\right) \le 0, \quad 0 \le t < T,$$
$$-R_{2,t} - \left(\frac{1}{R_2} + a - bv(x,y,t)\right) \ge 0, \quad 0 \le t < T.$$

This implies that $C_1(t)$ is a subsolution of (4.2), while $C_2(t)$ is a supersolution for $0 \le t < T$.

On the other hand, by (4.3),

$$\frac{d}{dt}(R_1 + R_2) = -\frac{1}{R_1} - \frac{1}{R_2} + b\left(v_2(T) - \frac{1}{3}\right),$$

which implies that $R_1(t) + R_2(t)$ is increasing in time near $t = 0$. Therefore we can choose x_1, x_2 and $t_0 \in (0, T]$ such that $R_2(t_0) > 0$ and

$$R_1(0) + R_2(0) < x_2 - x_1 < R_1(t_0) + R_2(t_0).$$

The above inequality shows the projections of $C_1(t)$ and $C_2(t)$ to the x-axis overlap each other before $t = t_0$. Let us fix y_1, y_2, ξ_0, ξ_1 and ξ_2 such that

$$y_2 + R_2(T) < \xi_2 < \xi_1 < y_1 - R_1(T) < y_1 + R_1(T) < \xi_0 < 0.$$

Hence we can choose $\phi_{+,0} \in C^2(\mathbb{R})$ with $\phi_{+,0} - \widehat{\phi}^* \in L^\infty(\mathbb{R})$ such that the graph $y = \phi_{+,0}(x)$ is above the graph of the upper semicircle of $C_1(0)$ for $x \in I_1(0)$ and is below the graph of the lower semicircle of $C_2(0)$ for $x \in I_2(0)$, where

$$I_i(t) := [x_i - R_i(t), x_i + R_i(t)], \quad i = 1, 2.$$

By the theory of [21], the problem (4.1) with such initial data $\phi_{+,0}(x)$ has a unique solution $\phi_+(x,t)$ for some time interval. By comparison, it is easy to see that

$$-c^*t - \|\phi_{+,0} - \widehat{\phi}^*\|_{L^\infty(\mathbb{R})} \le \phi_+(x,t) - \widehat{\phi}^*(x;c) \le c^*t + \|\phi_{+,0} - \widehat{\phi}^*\|_{L^\infty(\mathbb{R})}$$

for some $c^* \gg 1$ as long as $\phi_+(x,t)$ exists. Thus we have a priori estimate

$$\|\phi_+ - \widehat{\phi}^*\|_{L^\infty(\mathbb{R}\times[0,T])} \le C$$

for some positive constant $C = C(T, \|\phi_{+,0}\|_{L^\infty(\mathbb{R})})$. By the theory of [21], we see that the existence time of $\phi_+(x,t)$ can be extended until the gradient blow-up occurs. In fact, by comparison, we see that $y = \phi_+(x,t)$ is still above (resp. below) the graph of the upper (resp. lower) semicircle of $C_1(t)$ (resp. $C_2(t)$) for $x \in I_1(t)$ (resp. $x \in I_2(t)$) as long as ϕ_+ exists. Since the projections of $C_1(t)$ and $C_2(t)$ to the x-axis overlap each other before $t = t_0$, the gradient blowup for ϕ_+ occurs at some time $t_1 \in (0, t_0)$. Therefore the gradient blowup of the front equation may take place for the initial data $(\phi_{+,0}, v_0)$ which is far from the traveling curved wave. We also emphasize that this phenomenon is not observed for the curvature flow with a constant driving force.

From the above discussion, it means that the interface cannot be represented by a graph at some time $t = t_1$. However, it can be still extended after $t = t_1$ if we use (1.1) instead of (1.2) and (1.3). Because our discussion is not applicable to this situation, the asymptotic behavior of the solution to the free boundary problem (1.1) after $t = t_1$ still remains open.

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References

[1] P. K. Brazhnik, *Exact solutions for the kinematic model of autowaves in two-dimensional excitable media*, Physica D, 94, (1996) 205–220.

[2] P. K. Brazhnik, V. A. Davydov, *Non-spiral autowave structures in unrestricted excitable media*, Physics Letters, A **199** (1995), 40–44.

[3] X.Y. Chen, *Dynamics of interfaces in reaction diffusion systems*, Hiroshima Mathematical Journal **21** (1991), 47–83.

[4] X.F. Chen, *Generation and propagation of interfaces for reaction-diffusion systems*, Trans. Amer. Math. Soc. **334** (1992) 877–913.

[5] X.F. Chen, C. Gao, *Well-posedness of a free boundary problem in the limit of slow-diffusion fast-reaction systems*, J. Partial Differ. Equ **19** (2006) 48–79.

[6] Y.Y. Chen, J.S. Guo, H. Ninomiya, *Existence and uniqueness of rigidly rotating spiral waves by a wave front interaction model*, Physica D, 241, (2012) 1758–1766

[7] Y.-Y. Chen, Y. Kohsaka, H. Ninomiya, *Traveling spots and traveling fingers in singular limit problems of reaction-diffusion systems*, Discrete Cont. Dyn. Syst. (Ser. B) **19** (2014), 697–714.

[8] Y.Y. Chen, H. Ninomiya, R. Taguchi, *Traveling spots on multi-dimensional excitable media*, Journal of Elliptic and Parabolic Equations 1, (2015) 281–305.

[9] X. Chen, J-S. Guo, F. Hamel, H. Ninomiya, J.-M. Roquejoffre, *Traveling waves with paraboloid like interfaces for balanced bistable dynamics*, Ann. I. H. Poincare, AN **24**, (2007) 369–393.

[10] K. Deckelnick, C. M. Elliott, *Propagation of graphs in two-dimensional inhomogeneous media*, Appl. Numer. Math. **56** (2006), 1163–1178.

[11] K. Deckelnick, C. M. Elliott, G. Richardson, *Long time asymptotics for forced curvature ow with applications to the motion of a superconducting vortex*, Nonlinearity **10** (1997), 655–678.

[12] Y. Giga, *Interior derivative blow-up for quasilinear parabolic equations*, Discrete Conti. Dyn. Syst. **1** (1995), 449–461.

[13] Y. Giga, S. Goto, H. Ishii, *Global existence of weak solutions for interface equations coupled with diffusion equations*. SIAM journal on mathematical analysis **23** (1992), 821–835.

[14] J.-S. Guo, H. Ninomiya, J.-C. Tsai, *Existence and uniqueness of stabilized propagating wave segments in wave front interaction model*, Physica D: Nonlinear Phenomena **239** (2010), 230–239.

[15] J.S. Guo. H. Ninomiya, C.C. Wu, *Existence of a rotating wave pattern in a disk for a wave front interaction model*, Communications on Pure and Applied Analysis, **12** (2013), 1049–1063.

[16] F. Hamel, R. Monneau, J.-M. Roquejoffre, *Existence and qualitative properties of multidimensional conical bistable fronts*, Discrete Contin. Dyn. Syst., **13** (2005), 1069–1096.

[17] F. Hamel, R. Monneau, J.-M. Roquejoffre, *Asymptotic properties and classification of bistable fronts with Lipschitz level sets*, Discrete Contin. Dyn. Syst., **14** (2006), 75–92.

[18] D. Hilhorst, Y. Nishiura, M. Mimura, *A free boundary problem arising in some reacting-diffusing system*, Proc. Roy. Soc. Edinburgh Sect. A, **118** (1991), 355–378.

[19] J. P. Keener, J. Sneyd, *Mathematical Physiology*, Springer Science and Business Media, 2008.

[20] K. Krisher, A. Mikhailov, *Bifurcation to travelling spots in reaction–diffusion systems*, Phys. Rev. Lett. **73** (1994), 3165–3168.

[21] D.A. Ladyzenskaja, V.A. Solonnikov, N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type* American Mathematical Society., Providence, R.I. 1968.

[22] E. Mihaliuk, T. Sakurai, F. Chirila, K. Showalter, *Experimental and theoretical studies of feedback stabilization of propagating wave segments*, Faraday Discuss **120** (2001), 383–394.

[23] E. Mihaliuk, T. Sakurai, F. Chirila, K. Showalter, *Feedback stabilization of unstable propagating waves*, Phys. Review E. **65** (2002), 065602.

[24] A.S. Mikhailov, V.S. Zykov, *Kinematical theory of spiral waves in excitable media: comparison with numerical simulations*, Physica D **52** (1991), 379–397.

[25] J.D. Murray, *Mathematical biology. I: An introduction*, Springer-Verlag, New York, 2004.
[26] M. Nara, M. Taniguchi, *Stability of a traveling wave in curvature flows for spatially nondecaying perturbations*, Discrete Contin. Dyn. Syst. **14** (2006) 203–220.
[27] H. Ninomiya, M. Taniguchi, *Traveling curved fronts of a mean curvature flow with constant driving force*, in: Free Boundary Problems: Theory and Applications I, GAKUTO Internat. Ser. Math. Sci. Appl. **13** (2000) 206–221.
[28] H. Ninomiya, M. Taniguchi, *Stability of traveling curved fronts in a curvature flow with driving force*, Methods Appl. Anal. **8** (2) (2001) 429–450.
[29] H. Ninomiya, M. Taniguchi, *Existence and global stability of traveling curved fronts in the Allen-Cahn equations*, J. Differential Equations, **213** (2005) 204–233.
[30] H. Ninomiya, M. Taniguchi, *Global stability of traveling curved fronts in the Allen-Cahn equations*, Discrete Contin. Dyn. Syst., **15** (2006) 819–832.
[31] P. Pelcé, *Dynamics of Curved Fronts*, Academic Press, San Diego, 1988.
[32] P. Pelcé, J. Sun, *Wave front interaction in steadily rotating spirals*, Physica D: Nonlinear Phenomena **48** (1991) 353–366.
[33] V. Perez–Munuzuri, M. Gomez–Gesteira, A.P. Munuzuri, V.A. Davydov, V. Perez-Villar, *V-shaped stable nonspiral patterns*, Phys. Rev. E **51** (2) (1995) 845–847.
[34] M. Protter, H. Weinberger, *Maximum principles in differential equations*. Prentice-Hall 1967.
[35] M. Taniguchi, *Traveling fronts of pyramidal shapes in the Allen-Cahn equations*, SIAM J. Math. Anal., **39** (2007) 319–344.
[36] M. Taniguchi, *The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen-Cahn equations*, J. Differential Equations, **246** (2009) 2103–2130.
[37] V. K. Vanag, L. Yang, M. Dolnik, A. M. Zhabotinsky, I. R. Epstein, *Oscillatory cluster patterns in a homogeneous chemical system with global feedback*, Nature, **406** (2000) 389–391.
[38] A. T. Winfree, *When Time Breaks Down: The Three-Dimensional Dynamics of Electrochemical Waves and Cardiac Arrhythmias*, Princeton University Press, Princeton, NJ, 1987.
[39] V.S. Zykov, K. Showalter, *Wave front interaction model of stabilized propagating wave segments*, Phys. Review Letters **94** (2005), 068302.

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