Compact traveling waves for anisotropic curvature flow with driving force

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Abstract

To study the dynamics of an anisotropic curvature flow with an external driving force depending only on the normal vector, we focus on traveling waves composed of Jordan curves in $\mathbb{R}^2$. Here we call them compact traveling waves. The objective of this study is to investigate thoroughly the condition of the driving force for the existence of compact traveling waves to the anisotropic curvature flow. It is shown that all traveling waves are strictly convex and unstable, and that a compact traveling wave is unique, if they exist. To determine the existence of compact traveling waves, three cases are considered: if the driving force is positive, there exists a compact traveling wave; if it is negative, there is no traveling wave; if it is sign-changing, a positive answer is obtained under the assumption called "*admissible condition*". We also obtain a necessary and sufficient condition for the existence of axisymmetric compact traveling waves. Lastly, we make reference to the inverse problem and non-convex compact traveling waves.

1 Introduction

Several physical/chemical phases coexist in various phenomena such as growth of a snow crystal, propagation of action potential in a nerve axon, soap films, and cell locomotion. Interfaces are located at the boundaries between the multiple phases, and interfacial dynamics plays an important role in understanding such phenomena. Interfacial motion in such phenomena is often described by the parabolic free boundary problem, which is abbreviated as FBP for simplicity. More precisely, the evolution of interfaces of the phases can be regarded as a family of $\Gamma(t)$ which represents an embedded hypersurface in $\mathbb{R}^N (N \geq 2)$ at time t and is governed by

$$\beta(\mathbf{n})V = \gamma\left(u, \nabla u, \mathbf{n}, C(t)\right) - \alpha(\mathbf{n})\kappa, \tag{1.1}$$

where $\mathbf{n}$, V, and κ denote the outer normal vector, normal velocity, and mean curvature of $\Gamma(t)$, respectively, and α, β, γ are given functions. Furthermore, $C(t)$ is a non-local function, e.g., the perimeter of $\Gamma(t)$ or the volume of the phase $\Omega(t)$, and u is an unknown function defined in the domain $\Omega(t)$ adjacent to $\Gamma(t)$ or in $\mathbb{R}^N$. The above interface equation includes the Stefan condition, the modified Gibbs-Thomson relation in some FBPs such as the Stefan problem and the Mullins-Sekerka problem (or the Hele-shaw problem) [31]. Furthermore it is seen in the free problem related to incompressible viscous two-phase fluid flow [27]. We can confirm that (1.1) also includes the volume-preserving mean curvature flows and the singular limit problem derived from FitzHugh-Nagumo equation [8].

In some physical and biological problems such as oil-droplet motion, cell locomotion and dynamics in excitable media [38, 39, 32], self-propelled localized patterns similar to traveling waves are observed. Thus it is important to study the traveling waves whose boundary are composed of Jordan curves in $\mathbb{R}^2$. We call such a traveling wave "compact traveling wave" (the precise definition will be given in Definition 1.1). To describe such phenomena, (1.1) is often used. In recent years, the existence of compact traveling waves to some FBPs were reported. Choi and Lui [10] studied a FBP, describing cell locomotion, such that the evolutional equation of free boundary is composed of (1.1) with $\alpha \equiv \beta \equiv 1$ and $\gamma \equiv \gamma(\mathbf{n}, C(t))$. In their work, the non-local function $M(t)$ is composed of the perimeter of $\Gamma(t)$. They succeeded to show the existence of convex compact traveling wave to the problem in $\mathbb{R}^2$. Independently, for the same motivation, Monobe and Ninomiya [33, 34] also showed the multiple-existence of convex compact traveling waves to another FBP with (1.1). The driving force γ in (1.1) also has the form $\gamma \equiv \gamma(\mathbf{n}, C(t))$, but $M(t)$ is composed of the total mass of u over the phase $\Omega(t)$. In the proof of the above two results, it is essential to consider the existence of compact traveling waves to (1.1) with $\alpha \equiv \beta \equiv 1$ and $\gamma \equiv \gamma(\mathbf{n})$, because $C(t)$ is a constant when the solution is a traveling wave. Meanwhile, Chen, Kohsaka and Ninomiya [8] considered a singular limit of a FitzHugh-Nagumo-type reaction-diffusion system. They obtained a FBP with (1.1) satisfying

$\gamma \equiv \gamma(u)$, as the limit problem, and showed that there exists a compact traveling wave to the FBP in $\mathbb{R}^2$. It was reported that either the shape of $\Gamma(t)$ is convex or non-convex depending on the speed of the traveling wave.

As seen in the above examples, we can confirm that some FBPs with (1.1) have compact traveling waves. From this, we are naturally interested in the relation between the distribution of α, β, γ and the existence of compact traveling waves to (1.1). However, in general, α, β and γ are complicated due to the coupling of u, which is often unknown function depending on the evolutionary equation in phase $\Omega(t)$. In this study, we start with a simple case where the coefficients of (1.1) depend only on the normal vector $\mathbf{n}$ by considering a homogeneous distribution of u and ignoring the non-local function $C(t)$. In addition, we assume that α and β are positive given functions to guarantee that (1.1) is well-posed. By dividing by α, the anisotropic curvature flow (1.1) can be simplified as the following curvature flow with a driving force $\gamma(\mathbf{n})$:

$$\beta(\mathbf{n})V = \gamma(\mathbf{n}) - \kappa. \tag{1.2}$$

Before stating our results, we discuss the historical background of the interface equation (1.2) from a mathematical perspective, in particular, the relation between the driving force $\gamma(\mathbf{n})$ and traveling waves. The simplest form of (1.2) is the so-called *mean-curvature equation* given by

$$V = -\kappa. \tag{1.3}$$

The equation (1.3) arises from the first variation of the surface area functional. Furthermore, (1.3) was proposed by Mullins [30] as a mathematical model for describing the motion of grain boundaries. The short-time existence and uniqueness of classical solutions to (1.3) has been demonstrated by Gage-Hamilton [20], Evans-Spruck [15, 16] and Huskin and Polden [24]. For more general setting, we refer the reader to Chen-Giga-Goto [9] and Brakke [6]. Note that their results are applicable to (1.2) under some restrictions(see [11]). The mathematical results for the dynamics of this geometric flow in $\mathbb{R}^2$ have been obtained, for example, by Gage-Hamilton [20] and Grayson [22]. Briefly, their assertion is that any Jordan curve in $\mathbb{R}^2$ shrinks to a single point in a self-similar manner after the curve becomes convex. This claim is valid only when $N = 2$. In practice, it was shown that certain initial-closed surfaces in $N \geq 3$, such as "*Angenent torus*" and "*dumbbell-shaped surfaces*", result in a topological change in a finite time (see [2], [4] and the references therein). However, if we allow interface $\Gamma(t)$ to be a non-compact curve in

$\mathbb{R}^2$, then the scenario is entirely different; the line corresponds to stationary solution of (1.3), and moreover, there are some special solutions which does not shrink to a single point, such as a traveling wave called the "*Grim Reaper* (or *hair pin*)" (cf. [3]), and a spiral curve called "*Yin-Yang curve*" pointed out in Altschulter [1].

Another well-known example of an interface equation (1.2) is the *eikonal-curvature equation* expressed as

$$V = A - \kappa, \tag{1.4}$$

where A is a positive constant. The equation (1.4) is an approximation of some reaction-diffusion systems describing the wave propagation in excitable media, e.g., Belousov-Zhabotinsky (B-Z) reaction [25, 26, 37]. Similarly, using singular perturbation arguments, we see that (1.4) is related to some reaction-diffusion systems such as the Allen-Cahn-Nagumo equation [17], the Lotka-Volterra equations [14] and so on. Note that (1.4) is also related to crystal growth [7] and laminar flame propagation [28]. It is easily seen that, if $\Gamma(t)$ is a Jordan curve, (1.4) has a unique disk-shaped stationary solution of radius $1/A$, where the solution is unstable [13]. In contrast with the mean-curvature flow, not every Jordan curve satisfying (1.4) in $\mathbb{R}^2$ shrinks to a single point; for instance, it is clear that a disk-shaped solution with radius $R(t)$, where $R(0) > 1/A$, expands to infinity as t approaches infinity. In case of a non-compact curve, the line is not a stationary solution but a traveling wave. In addition, it was reported by Ninomiya and Taniguchi [35, 36] that there exists a traveling wave combining two lines asymptotically in $\mathbb{R}^2$, known as the "*V-shaped traveling front*", and in higher dimension by Hamel, Monneau and Roquejoffre [23].

Interface equations with anisotropic curvature are also frequently studied. For instance, considering the interfacial energy and difference between bulk energies, we obtain the following anisotropic interface equation in $\mathbb{R}^2$ [21]:

$$b(\theta)V = A - \Big(f(\theta) + f''(\theta)\Big)\,\kappa, \tag{1.5}$$

where θ is the angle between the x-axis and $\mathbf{n}$ at $\Gamma(t)$, A is a positive constant describing the difference between the phases in terms of bulk energy, $b(\theta)$ is a positive function, and $f(\theta)$ is the interfacial energy. We remark that (1.5) corresponds to (1.2) if $f(\theta) + f''(\theta)$ is positive. For (1.5), there are various results related to well-posedness and convex properties. In particular, if

$b, f \in C^\infty([0,2\pi])$ and $f + f'' > 0$, there exists a maximal time $T_M > 0$ such that $\Gamma(t)$ is a smooth Jordan curve in $(0,T)$, and as t approaches T_M, $\Gamma(t)$ shrinks to a single point, develops a kink or has self-intersections. Moreover, there is a unique bounded stationary interface for any $f+f'' > 0$ (see Gurtin's book [21] and the references therein). We will discuss this topic later in Section 4.4. In case of non-compact curve, we refer to [29].

1.1 Problem setting

From these typical examples, although the research of traveling waves is well done in some cases, almost of them are discussed in non-compact region, i.e., $\mathbb{R}^N$. Thus, we can not apply the analysis method for non-compact traveling wave to compact one directly. In this study, our objective is to answer the following question :

Q. Under what condition of $\gamma(\mathbf{n})$ does (1.2) have a traveling wave solution composed of Jordan curves?

We now formulate a traveling wave for (1.2). Suppose that $\Gamma(t)$ is a Jordan curve traveling with constant shape Γ_0 and velocity $\mathbf{c} \in \mathbb{R}^2$, i.e.,

$$\Gamma(t) = \Gamma_0 + t\mathbf{c}$$

for $t \geq 0$, where Γ_0 is a Jordan curve that is positively oriented (counterclockwise direction). Let $x(s)$ and $y(s)$ be functions satisfying $\Gamma_0 = \{(x(s), y(s)) \mid s \in [0, L)\}$, where L and s are the perimeter and arc length of Γ_0, respectively. We denote the angle between the x-axis and outer normal vector $\mathbf{n} = \mathbf{n}(s)$ at $(x(s), y(s))$ by $\theta = \theta(s)$, as in (1.5). Then, normal unit vector $\mathbf{n}$ and unit tangent vector $\mathbf{t} = (x'(s), y'(s))$ at $(x(s), y(s))$ satisfy

$$\mathbf{n} = (\cos\theta, \sin\theta), \quad \mathbf{t} = (-\sin\theta, \cos\theta). \tag{1.6}$$

Since $\mathbf{n}$ is determined by θ, we can regard β and γ as functions of θ, i.e., there exist functions $\tilde{\beta}$ and $\tilde{\gamma}$ such that

$$\beta(\mathbf{n}) = \tilde{\beta}(\theta), \quad \gamma(\mathbf{n}) = \tilde{\gamma}(\theta).$$

For simplicity of notation, we omit the tilde of $\tilde{\beta}(\theta)$ and $\tilde{\gamma}(\theta)$. Let $\mathbf{e}(\theta)$ be defined by

$$\mathbf{e}(\theta) := (\cos\theta, \sin\theta).$$

Then, curvature κ of Γ_0 and normal velocity V at Γ_0 are represented by

$$\kappa = \theta_s, \quad V = \mathbf{c} \cdot \mathbf{n} = \mathbf{c} \cdot \mathbf{e}(\theta), \tag{1.7}$$

respectively, because $\mathbf{n} = \mathbf{e}(\theta)$. Note that as for vector $\mathbf{c}$, there is a point $(c, \eta) \in [0, \infty) \times [0, 2\pi)$ satisfying $\mathbf{c} = c\,\mathbf{e}(\eta)$. If $\mathbf{c} = \mathbf{0}$, η can be chosen arbitrarily. Thus, if $\Gamma(t)$ is traveling with velocity $\mathbf{c} = c\,\mathbf{e}(\eta)$, it follows from (1.6) and (1.7) that (1.2) is reduced to the following ordinary differential equations (ODEs) :

$$\begin{cases} \theta_s = \gamma(\theta) - c\beta(\theta)\cos(\theta - \eta) & \text{in } (0, \ell), \\ x_s = -\sin\theta & \text{in } (0, \ell), \\ y_s = \cos\theta & \text{in } (0, \ell), \\ \theta(0) = \theta_0,\ x(0) = x_0,\ y(0) = y_0, \end{cases} \tag{1.8}$$

where θ_0, x_0, and y_0 are the initial data and ℓ is a positive constant. Recall that Γ_0 is a closed curve. Hence, as ℓ is equal to perimeter L of $\Gamma(t)$, angle $\theta(s)$ and point $(x(s), y(s))$ must satisfy the boundary condition

$$(\theta(L) - 2\pi, x(L), y(L)) = (\theta_0, x_0, y_0). \tag{1.9}$$

Consequently, we obtain the following problem of determining (c, η) such that it satisfies

$$\begin{cases} \theta_s = \gamma(\theta) - c\beta(\theta)\cos(\theta - \eta) & \text{in } (0, L), \\ x_s = -\sin\theta & \text{in } (0, L), \\ y_s = \cos\theta & \text{in } (0, L), \\ \theta(0) = \theta(L) - 2\pi = \theta_0, & \\ x(0) = x(L) = x_0, & \\ y(0) = y(L) = y_0. & \end{cases} \tag{1.10}$$

Since our purpose is to find a classical solution to (1.10), we impose the continuity on β and γ as follows:

(A) β is always positive. Moreover, β and γ are Lipschitz continuous with 2π-periodic.

To provide a definition of traveling waves in (1.2), we introduce two function spaces. We say that $x \in C^{m,1}([0, \ell])$ if and only if, for any $s \in [0, \ell]$, $x(s)$

is continuously differentiable for m-times and m-th order derivative $x^{(m)}(s)$ is Lipschitz continuous. In addition, Γ_0 belongs to $C^{2,1}$ if there exist functions $x, y \in C^{2,1}(-\ell, \ell)$ such that

$$\{(x(s), y(s)) \in \mathbb{R}^2 \mid s \in (-\ell, \ell)\} = \Gamma_0 \cap U, \quad x(0) = x_0,\ y(0) = y_0\}$$

for any $(x_0, y_0) \in \Gamma_0$, where U is a suitable open set in $\mathbb{R}^2$.

Definition 1.1. Suppose that $(\theta, x, y) \in C^{1,1}([0, L]) \times C^{2,1}([0, L]) \times C^{2,1}([0, L])$ satisfies (1.10) for suitable parameters c and η. Then, we define $(\Gamma_0, \mathbf{c})$ a *compact traveling wave* of (1.2) (or simply a *compact traveling wave*), where $\Gamma_0 := \{(x(s), y(s)) \mid s \in [0, L)\}$ and $\mathbf{c} := c\,\mathbf{e}(\eta)$. To specify the speed and angle of $\mathbf{c}$, we occasionally use notation $(\Gamma_0, c\,\mathbf{e}(\eta))$ instead of $(\Gamma_0, \mathbf{c})$ for $c \geq 0$ and $\eta \in [0, 2\pi)$.

Definition 1.2. We say that traveling wave $(\Gamma_0, \mathbf{c})$ of (1.2) is *strictly convex* (*concave*) if curvature κ of Γ_0 is positive (negative) for any point of Γ_0.

Definition 1.3. We define a compact traveling wave $(\Gamma_0, c\,\mathbf{e}(\eta))$ of (1.10) an *axisymmetric compact traveling wave with respect to* $\mathbf{e}(\zeta)$ for $\zeta \in [0, \pi)$, if Γ_0 is axisymmetric with respect to a line that is parallel to $\mathbf{e}(\zeta)$. In particular, when $\eta = \zeta$ or $\eta = \zeta + \pi$, we say that traveling wave $(\Gamma_0, \mathbf{c})$ is an *axisymmetric compact traveling wave with respect to the traveling direction.*

We remark that the uniqueness of a compact traveling wave is up to the shift. Actually, $\Gamma(t) + (x_1, y_1)$ is also a compact traveling wave for any $(x_1, y_1) \in \mathbb{R}^2$, if $\Gamma(t)$ is a compact traveling wave. To explain our main results, we utilize the following definition for γ:

Definition 1.4. We state that γ is *positive* (*non-positive*) if $\gamma(\theta) > 0$ ($\gamma(\theta) \leq 0$) for any θ. In addition, γ is *sign-changing* if there exist θ_1 and θ_2 such that $\gamma(\theta_1) > 0$ and $\gamma(\theta_2) \leq 0$.

Any continuous function γ can be classified into one of the above three cases. We also remark that the constant function $\gamma \equiv 0$ is regarded as non-positive one.

1.2 Main results

Our main results are as follows:

(a) Every compact traveling wave of (1.2) is strictly convex and unstable. Refer to Theorems 2.2 and 2.5 in Section 2.

(b) If γ is positive, then there exists a unique compact traveling wave of (1.2). In addition, velocity vector $\mathbf{c}$ satisfies that $\mathbf{c} \cdot \mathbf{e}_\gamma < 0$, where

$$\mathbf{e}_\gamma := \left(\int_0^{2\pi} \frac{\cos\theta}{\gamma(\theta)}\, d\theta, \int_0^{2\pi} \frac{\sin\theta}{\gamma(\theta)}\, d\theta \right). \tag{1.11}$$

Meanwhile, there is no compact traveling wave of (1.2) if γ is non-positive. Refer to Theorem 3.1 in Subsection 3.1 and Theorem **??** in Subsection 3.2.

(c) If γ is sign-changing, then a compact traveling wave is unique if it exists. In contrast with the positive case in (b), a compact traveling wave does not always exist for any sign-changing function γ. Indeed, a sign-changing driving force such that (1.2) does not possess a compact traveling wave is proposed in Section 3.2. Therefore, we need to impose a additional condition on $\gamma(\theta)$. It will be shown that (1.2) includes a compact traveling wave, if γ satisfies the *admissible condition* that is defined in Subsection 3.2. As for the velocity $\mathbf{c}$, we have that $\mathbf{c} \cdot \mathbf{e}(\theta) < 0$ for all $\theta \in \{\theta \in [0, 2\pi) \mid \gamma(\theta) \leq 0\}$. Refer to Theorems 3.5 and 3.8 in Subsection 3.2.

(d) When γ is symmetric to an angle, a necessary and sufficient condition for having compact traveling waves will be given by Theorem 4.4 in Section 3.

(e) There exists a driving force γ such that for any Jordan curve Γ_0^* and vector $\mathbf{c}^*$, (1.2) includes compact traveling wave $(\Gamma_0^*, \mathbf{c}^*)$. Refer to Subsection 4.5.

In order to show the existence of compact traveling waves to (1.2) in $\mathbb{R}^2$, we solve the first equation of (1.10) that is an ODE of θ. Note that x and y are determined by the second and the third equations, if θ is obtained. First we check a condition for c, η, and γ where θ can be solved over $[0, 2\pi)$ without the boundary condition (1.9). Then we know that if there is a compact traveling wave, the curvature θ_s is positive. Thus, the main difficulty of this problem is how to find a pair (c, η) satisfying $\theta_s > 0$ and (1.9). Our strategy is that we first find a set $\mathcal{S}$ of (c, η) satisfying $\theta_s > 0$ and the boundary condition of

y and θ in (1.9), and we next seek a point (c, η) with the boundary condition of x in (1.9). In our proof, the information of the boundary of the set $\mathcal{S}$ is essential. When γ is positive, we fortunately obtain a precise information from the boundary. However, for sign-changing case, we could not obtain a sufficient information from the boundary due to the degeneracy of the curvature θ_s. More precisely, we need to find a pair $(c, \eta) \in \mathcal{S}$ such that $x(L) = x(0)$. Since $x(L) - x(0)$ is represented by

$$x(L) - x(0) = -\int_0^{2\pi} \frac{\sin(\theta - \eta)}{\theta_s}\, d\theta,$$

we need to give careful attention to the zeros of the curvature $\kappa = \theta_s$. In general, it is not easy to confirm that $x(L) - x(0) = 0$ near zeros of θ_s. To overcome the difficulty, we will impose an additional condition on γ, which specifies the position of zero points of θ_s. On the other hand, considering axisymmetric compact traveling waves, we can obtain a necessary and sufficient condition of γ for having compact traveling waves.

In this study, we investigate our problem as $N = 2$. Our strategy is not directly applicable to the case where $N \geq 3$. One of the reasons is that, when $N \geq 3$, in general, (1.2) is not represented by a system of ODEs, but by a second order partial differential equation. Another reason is to specify the condition to be a closed hypersurface Γ_0. As $N = 2$, the closed condition of Γ_0 is simple. However, the condition for $N \geq 3$ is complicated. For instance, we consider the condition to be a convex and closed hypersurface Γ_0 satisfying (1.2). By the definition of compact traveling waves, we obtain that $V = \mathbf{c} \cdot \mathbf{n}$ as seen in (1.7). Hence, setting $\phi(\mathbf{n}) = \gamma(\mathbf{n}) - \beta(\mathbf{n})\mathbf{c} \cdot \mathbf{n}$, (1.2) is written by

$$\kappa = \phi(\mathbf{n}).$$

Thus, our problem is deeply related to the problem looking for a condition of $\phi(\mathbf{n})$ for which Γ_0 is convex and closed. Actually, this problem is called "Christoffel problem" in differential geometry and studied for a long time. About sixty years ago, this problem was completed by Berg [5] and Firey [18, 19]. As mentioned in their papers, a necessary and sufficient condition for the existence of a convex closed hypersurface Γ_0 is related to the non-degeneracy and symmetry of a Jacobi matrix composed of $\phi(\mathbf{n})$. Our problem is an extension of the Christoffel problem, because a velocity vector $\mathbf{c}$ also needs to be determined depending on ϕ. It is still open when $N \geq 3$.

This paper is organized as follows: In Section 2, we will show that if there exist traveling waves, every traveling wave is convex and unstable. Moreover, to exhibit the existence and uniqueness of compact traveling waves, we prepare some useful lemmas. The existence and uniqueness of compact traveling waves are shown in Section 3. We consider three cases as in Definition 1.4 and show the uniqueness of compact traveling waves for all the cases. We show the existence of compact traveling waves for a positive external force γ. For the sign-changing case, we introduce the new concept of *admissible condition*. Under this assumption, we show the existence of compact traveling waves. We discuss other related problems in Section 4, such as the inverse problem and existence of non-convex compact traveling waves.

2 Key properties

In this section, we first prove the positivity of the curvature. Since $\theta_s = K(\theta; c, \eta)$, the positivity of $K(\theta; c, \eta)$ is equivalent to the convexity of the traveling waves. Secondly, we introduce a set of (c, η) in which the curvature is positive, and investigate its properties. To investigate (c, η) that satisfies $(x(0), y(0)) = (x(L), y(L))$, we examine the monotonicity of function $y(L)$ with respect to c. Finally, it will be shown that there exists at most one compact traveling wave.

2.1 Convexity and stability

In this subsection, we will show the convexity of traveling waves. For simplicity of notation, we introduce a function

$$K(\theta; c, \eta) := \gamma(\theta) - c\beta(\theta)\cos(\theta - \eta),$$

where $c \geq 0$ and $\eta \in \mathbb{R}$.

Before proving the convexity of traveling waves, we prepare an auxiliary lemma.

Lemma 2.1. *Suppose that* (c, η) *in* (1.8) *are given in* $[0, \infty) \times [0, 2\pi)$*. Then, for any* $\ell > 0$*,* (1.8) *has a unique solution* $(\theta, x, y) \in C^{1,1}([0, \ell]) \times C^{2,1}([0, \ell]) \times C^{2,1}([0, \ell])$*. Moreover, the sign of* $\theta_s(s)$ *is definite for* $s \in [0, \ell]$ *if* $\theta_s(0) \neq 0$*.*

Proof. The existence and uniqueness of solutions (θ, x, y) to (1.8) for a short interval $[0, \ell]$ immediately follows from the standard argument of the theory of ODE. Let s_0 satisfy $\theta_s(s_0) = 0$. Then $(\theta, x, y) = (\theta(s_0), x(s_0) -$

$s\sin\theta(s_0), y(s_0)+s\cos\theta(s_0))$ satisfies (1.8) for any $s\in\mathbb{R}$. On account of the uniqueness of solutions (θ,x,y), $\theta_s(s)$ never attains 0 at a finite s, namely, the sign of $\theta_s(s)$ is definite for $s\in[0,\ell]$. Moreover the Lipschitz continuity of $K(\theta;c,\eta)$ guarantees the solvability of (1.8) for any interval $[0,\ell]$. This completes the proof. □ □

We now discuss the convexity of traveling waves.

Theorem 2.2. *Every compact traveling wave of* (1.2) *is strictly convex.*

Proof. Let $(\Gamma_0,\mathbf{c})$ be a compact traveling wave of (1.2). Then, it follows that

$$\int_0^L \theta_s\,ds = 2\pi$$

owing to the orientation of Γ_0. Hence, θ_s must be positive in some region. From Lemma 2.1, θ_s is positive in $[0,L]$. This implies that a compact traveling wave is strictly convex, which is our assertion. □ □

Remark 2.3. Lemma 2.1 can also ensure the unbounded convexity of traveling waves defined in the entire space, e.g., *Grim Repear* and *V-shaped traveling front* in $\mathbb{R}^2$. Incidentally, their shapes have been already studied (cf. [3] and [36]). However, the assertion of Lemma 2.1 does not always hold for the higher dimensional space. As a simple counter-example, we recall a *catenoid* that is a non-convex minimal surface satisfying (1.2) with $\beta\equiv 1$ and $\gamma\equiv 0$ in $\mathbb{R}^3$. Furthermore, the existence of non-compact and non-convex traveling waves of a mean curvature flow in $\mathbb{R}^N (N\geq 3)$ is shown in [12].

Remark 2.4. When we replace α, β and γ in (1.2) by some functions depending on positions x and y, Theorem 2.2 does not hold true for such a more general interface equation. We provide an example for non-convex compact traveling waves in Subsection 4.5.

As noted in Section 1, (1.4) has a unique stationary solution, and the solution is unstable. Note that the stationary solution is regarded as compact traveling wave with $\mathbf{c}=\mathbf{0}$ in our setting. Thus every compact traveling wave to (1.10) is unstable whenever γ is a positive constant. This claim is also true for our problem.

Theorem 2.5 (Ei-Yanagida [13])**.** *Every compact traveling wave of* (1.2) *is unstable.*

In practice, Ei and Yanagida [13] showed that any bounded stationary solutions of $V = F(\kappa, \mathbf{n})$ are unstable (see Corollary 2.1 in [13]). By taking

$$F(\kappa, \mathbf{n}) = \frac{\gamma(\mathbf{n})}{\beta(\mathbf{n})} - \frac{1}{\beta(\mathbf{n})}\kappa - \mathbf{c} \cdot \mathbf{n},$$

it follows that the compact traveling wave $(\Gamma_0, \mathbf{c})$ of (1.2) is a stationary solution of $V = F(\kappa, \mathbf{n})$ and unstable. To stabilize the compact traveling waves, we require more components related to the system (1.2).

2.2 An auxiliary set $\mathcal{S}$ and its properties

By Theorem 2.2, every traveling wave of (1.2) is convex, i.e., $K(\theta; c, \eta)$ is positive in $[0, 2\pi)$. Thus we focus our attention on a set of (c, η) with $K(\cdot; c, \eta) > 0$ in $[0, 2\pi)$. Define $\mathcal{S}$ and c_M by

$$\begin{aligned} \mathcal{S} &:= \left\{ (c, \eta) \in [0, \infty) \times [0, 2\pi) \,\middle|\, \inf_{\theta \in [0, 2\pi)} K(\theta; c, \eta) > 0 \right\}, \\ c_M &:= \sup_{\theta \in [0, 2\pi)} \frac{\gamma(\theta)}{\beta(\theta)}. \end{aligned}$$

Note that $\inf_{\theta \in [0,2\pi)} K(\theta; c, \eta) = \min_{\theta \in [0, 2\pi]} K(\theta; c, \eta)$ since K is 2π-periodic and continuous. In this subsection, we investigate the properties of $\mathcal{S}$.

Lemma 2.6. *Assume that $\mathcal{S} \neq \emptyset$. If (c_1, η) and (c_2, η) are contained in $\mathcal{S}$, where $0 \leq c_1 < c_2$, then $[c_1, c_2] \times \{\eta\} \subset \mathcal{S}$. Moreover, $\mathcal{S} \subset [0, c_M) \times [0, 2\pi)$.*

Proof. Take (c_1, η) and (c_2, η) in $\mathcal{S}$. Let t be a positive constant in $[0, 1]$. By simple calculations, we obtain

$$K(\theta; tc_2 + (1-t)c_1, \eta) \geq tK(\theta; c_2, \eta) + (1-t)K(\theta; c_1, \eta).$$

Since $K(\theta; c_2, \eta) > 0$ and $K(\theta; c_1, \eta) > 0$, it follows that $(tc_2 + (1-t)c_1, \eta) \in \mathcal{S}$ for any $t \in [0, 1]$.

Let $(c, \eta) \in \mathcal{S}$. Since $K(\theta; c, \eta) > 0$ for any $\theta \in [0, 2\pi)$, it follows that

$$K(\eta; c, \eta) = \gamma(\eta) - c\beta(\eta) > 0.$$

By the positivity of β, we obtain $c < c_M$, which completes the proof. □ □

Lemma 2.7. *If γ is positive, then $\{(0,\eta)\,|\,\eta \in [0,2\pi)\} \subset \mathcal{S}$. If γ is non-positive, then $\mathcal{S} = \emptyset$. In particular, if γ is sign-changing, then $\{(0,\eta)\,|\,\eta \in [0,2\pi)\} \cap \mathcal{S} = \emptyset$, and there exists $\eta_* \in [0,\pi)$ satisfying*

$$\{(c,\eta)\,|\,c \geq 0,\ \eta \in [\eta_*, \eta_* + \pi]\} \cap \mathcal{S} = \emptyset$$

or

$$\{(c,\eta)\,|\,c \geq 0,\ \eta \in [0,\eta_*] \cup [\eta_* + \pi, 2\pi)\} \cap \mathcal{S} = \emptyset.$$

Proof. Assume that γ is positive. Then it is easily seen that

$$\inf_{\theta \in [0,2\pi)} K(\theta; 0, \eta) = \inf_{\theta \in [0,2\pi)} \gamma(\theta) > 0$$

for any η. This implies that $\{(0,\eta) \mid \eta \in [0,2\pi)\} \subset \mathcal{S}$. As γ is non-positive, i.e., $\gamma(\theta) \leq 0$ for any θ, it is easily seen that $\inf_{\theta \in [0,2\pi)} K(\theta; c, \eta) < 0$ for any $(c,\eta) \in [0,\infty) \times [0,2\pi)$. Thus $\mathcal{S}$ is empty.

Suppose that γ is sign-changing. There is an angle $\theta_* \in [0,2\pi)$ satisfying $\gamma(\theta_*) \leq 0$. Thus we have

$$\inf_{\theta \in [0,2\pi)} K(\theta; 0, \eta) \leq \gamma(\theta_*) \leq 0$$

for any $\eta \in [0,2\pi)$, which implies that $\{(0,\eta)\,|\,\eta \in [0,2\pi)\} \cap \mathcal{S} = \phi$. Furthermore,

$$\inf_{\theta \in [0,2\pi)} K(\theta; c, \eta) \leq \gamma(\theta_*) - c\beta(\eta)\cos(\theta_* - \eta) \leq 0$$

for any $c \geq 0$ and $\eta \in (\theta_* - \pi/2, \theta_* + \pi/2)$. If $\theta_* - \pi/2 \in [0,\pi)$, we set $\eta_* = \theta_* - \pi/2$, and then we obtain $\{(c,\eta)\,|\,c \geq 0,\ \eta \in [\eta_*, \eta_* + \pi]\} \cap \mathcal{S} = \emptyset$. If $\theta_* - \pi/2 \in [\pi, 2\pi)$, we reset $\eta_* = \theta_* - 3\pi/2$ and obtain $\{(c,\eta)\,|\,c \geq 0,\ \eta \in [0,\eta_*] \cup [\eta_* + \pi, 2\pi)\} \cap \mathcal{S} = \emptyset$. This completes the proof. $\square$ $\square$

Remark 2.8. Considering Lemma 2.7, without loss of generality, we hereafter assume that, if γ is non-positive or sign-changing,

$$\{(c,\eta)\,|\,c \geq 0,\ \eta \in [0,\eta_*] \cup [\eta_* + \pi, 2\pi)\} \cap \mathcal{S} = \emptyset,$$

if necessary, after an appropriate rotation. Here η_* stands for the symbol shown in Lemma 2.7. From this, if γ is non-positive or sign-changing, and $\mathcal{S} \neq \emptyset$, the set $\mathcal{S}$ satisfies

$$\mathcal{S} \subset (0, c_M) \times (\eta_*, \eta_* + \pi) \subset [0,\infty) \times [0,2\pi). \tag{2.1}$$

Proposition 2.9. *Let $\mathcal{S}$ be not empty. Then it is bounded and simply connected. In particular, if γ is sing-changing, then $\mathcal{S}$ is open.*

Proof. Lemma 2.6 immediately implies that $\mathcal{S}$ is bounded. We show that $\mathcal{S}$ is simply connected.

We first consider the case that γ is positive. Let (c_1, η_1) and (c_2, η_2) be arbitrary points in $\mathcal{S}$. If $\eta_1 = \eta_2$, then a line segment $C_1 := [c_1, c_2] \times \{\eta_1\}$ connecting (c_1, η_1) with (c_2, η_2) is a subset of $\mathcal{S}$ due to Lemma 2.6. If $\eta_1 \neq \eta_2$, then we define the set C_2 by

$$C_2 = ([0, c_1] \times \{\eta_1\}) \cup (\{0\} \times [\eta_1, \eta_2]) \cup ([0, c_2] \times \{\eta_2\}).$$

The set C_2 is a polygonal line which connects (c_1, η_1) with (c_2, η_2). It follows from Lemma 2.6 and Lemma 2.7 that $C_2 \subset \mathcal{S}$, that is, $\mathcal{S}$ is connected. Applying Lemma 2.6 again, we know that $\mathcal{S}$ is simply connected.

To consider the case where γ is non-negative or sign-changing, we set

$$\begin{aligned} K(\theta; \mathbf{c}) &:= \gamma(\theta) - \beta(\theta)\mathbf{c} \cdot \mathbf{e}(\theta), \\ \mathcal{S}_* &:= \left\{ \mathbf{c} \in \mathbb{R}^2 \mid \inf_{\theta \in [0, 2\pi)} K(\theta; \mathbf{c}) > 0 \right\}. \end{aligned}$$

By the continuity of $K(\theta; \mathbf{c})$ with respect to $\mathbf{c}$, it is easy to check that the set $\mathcal{S}_*$ is open. Let $\mathbf{c}_1,\ \mathbf{c}_2$ be two arbitrary points of $\mathcal{S}_*$, namely, $K(\theta; \mathbf{c}_i) > 0$ $(i = 1, 2)$. For any $t \in [0, t]$, we have

$$\begin{aligned} K(\theta; (1-t)\mathbf{c}_1 + t\mathbf{c}_2) &= \gamma(\theta) - \beta(\theta)\{(1-t)\mathbf{c}_1 + t\mathbf{c}_2\} \cdot \mathbf{e}(\theta) \\ &= (1-t)K(\theta; \mathbf{c}_1) + tK(\theta; \mathbf{c}_2) \end{aligned}$$

for any $\mathbf{c}_i \in \mathcal{S}$ $(i = 1, 2)$. Thus we see that $(1-t)\mathbf{c}_1 + t\mathbf{c}_2 \in \mathcal{S}$ for any $t \in [0, 1]$, which implies that $\mathcal{S}_*$ is strictly convex.

If γ is positive, then $\mathbf{0} \in \mathcal{S}_*$ because

$$\inf_{\theta \in [0, 2\pi)} K(\theta; \mathbf{0}) = \inf_{\theta \in [0, 2\pi)} \gamma(\theta) > 0.$$

Otherwise, it follows from Lemma 2.7 that $\mathbf{0} \notin \mathcal{S}_*$.

Finally, we show that $\mathcal{S}$ is simply connected as γ is sign-changing. Since $\mathcal{S}_*$ is strictly convex, it is immediately know that $\mathcal{S}_*$ is simply connected. Moreover, as γ is non-negative or sign-changing, it holds $\mathbf{0} \notin \mathcal{S}_*$. Thus $\mathcal{S}_*$ is homeomorphic to $\mathcal{S}$ with the help of the polar coordinate mapping, which means that $\mathcal{S}$ is simply connected and open. This is our assertion. □ □

As seen in Lemma 2.7, if γ is positive, $\mathcal{S}$ is not empty; otherwise it is not certain. We here refer to the condition of γ as $\mathcal{S}$ is not empty. Define $c_-(\eta)$ and $c_+(\eta)$ by

$$c_-(\eta) := \max\left\{0, \sup_{|\xi-\pi|<\pi/2} \frac{\gamma(\xi+\eta)}{\beta(\xi+\eta)\cos\xi}\right\},$$
$$c_+(\eta) := \inf_{|\xi|<\pi/2} \frac{\gamma(\xi+\eta)}{\beta(\xi+\eta)\cos\xi}.$$

Next lemma gives us the geometrical meaning of $c_\pm(\eta)$.

Proposition 2.10. *If $\mathcal{S}$ is not empty, then for any $(c,\eta) \in \mathcal{S}$, it follows that*

$$c_-(\eta) < c_+(\eta). \tag{2.2}$$

Moreover,

$$\{c \in \mathbb{R} \mid (c,\eta) \in \mathcal{S}\} = (c_-(\eta), c_+(\eta)).$$

Proof. Assume that $\mathcal{S}$ is not empty. Take an arbitrary point (c_0,η_0) of $\mathcal{S}$. Then $K(\theta; c_0,\eta_0) > 0$ in $[0,2\pi]$ implies that

$$\frac{\gamma(\theta+\pi+\eta_0)}{\beta(\theta+\pi+\eta_0)\cos(\pi+\theta)} < c_0 < \frac{\gamma(\theta+\eta_0)}{\beta(\theta+\eta_0)\cos\theta}$$

for any $\theta \in (-\pi/2, \pi/2)$. Since $c_0 \geq 0$, we obtain that $c_-(\eta_0) \leq c_0 \leq c_+(\eta_0)$. Moreover we can exclude the possibility that $c_0 = c_+(\eta_0)$. In fact, if $(c_0,\eta_0) = (c_+(\eta_0),\eta_0) \in \mathcal{S}$, then

$$\begin{aligned} K(\theta; c_0,\eta_0) &= K(\theta; c_+(\eta_0),\eta_0) \\ &= \beta(\theta)\cos(\theta-\eta_0)\left(\frac{\gamma(\theta)}{\beta(\theta)\cos(\theta-\eta_0)} - c_+(\eta_0)\right). \end{aligned}$$

as $\cos(\theta-\eta_0) \neq 0$. Note that $\gamma(\eta_0 \pm \pi/2) > 0$ due to $K(\theta; c_0,\eta_0) > 0$. Then there is $\theta_0 \in (-\pi/2,\pi/2)$ such that $c_+(\eta_0) = \gamma(\theta_0)/(\beta(\theta_0)\cos(\theta_0-\eta_0))$, namely, $K(\theta_0; c_0,\eta_0) = 0$ because

$$\lim_{\xi\uparrow\frac{\pi}{2}} \frac{\gamma(\xi+\eta_0)}{\beta(\xi+\eta_0)\cos\xi} = \lim_{\xi\downarrow-\frac{\pi}{2}} \frac{\gamma(\xi+\eta_0)}{\beta(\xi+\eta_0)\cos\xi} = +\infty.$$

This contradicts the fact that $(c_0,\eta_0) \in \mathcal{S}$. Hence $c_-(\eta_0) \leq c_0 < c_+(\eta_0)$ for any $(c_0,\eta_0) \in \mathcal{S}$.

We next check that $(c_-(\eta_0),\eta_0)$ and $(c_+(\eta_0),\eta_0)$ are part of the boundary of $\mathcal{S}$. By the above discussion, it follows that $\inf_{\theta\in[0,2\pi)} K(\theta;c_+(\eta_0),\eta_0)=0$, i.e., $(c_+(\eta_0),\eta_0)\notin\mathcal{S}$. Furthermore Lemma 2.6 leads to $(c_+(\eta_0)+\varepsilon,\eta_0)\notin\mathcal{S}$ for any $\varepsilon>0$. As for $c_-(\eta_0)$, it depends on the sign of $\gamma(\theta)$. Let us consider the case $\gamma>0$. Then $c_-(\eta_0)=0$ and $(c_-(\eta_0),\eta_0)\in\mathcal{S}$. By the definition of $\mathcal{S}$, we see that $\inf_{\theta\in[0,2\pi)} K(\theta;c_-(\eta_0),\eta_0)=0$, i.e., $(c_-(\eta_0),\eta_0)\notin\mathcal{S}$. Next we consider the case that γ is sign-changing. Repeating the same argument as $c_+(\eta_0)$, we confirm that there exists a point $\theta_0\in[0,2\pi)$ satisfying $K(\theta_0;c_-(\eta_0),\eta_0)=0$. This yields $(c_-(\eta_0)-\varepsilon,\eta_0)\notin\mathcal{S}$ for any $\varepsilon\geq 0$.

Next we show that $(c,\eta_0)\in\mathcal{S}$ for any $c\in(c_-(\eta_0),c_+(\eta_0))$ when $c_-(\eta_0)<c_+(\eta_0)$. We here remark that for any $\theta\in[0,2\pi)$,

$$K(\theta;c_-(\eta_0),\eta_0)+K(\theta;c_+(\eta_0),\eta_0)>0. \tag{2.3}$$

Indeed, if there exists θ_0 satisfying $K(\theta_0;c_-(\eta_0),\eta_0)=K(\theta_0;c_+(\eta_0),\eta_0)=0$, then we have

$$\gamma(\theta_0)=c_-(\eta_0)\beta(\theta_0)\cos(\theta_0-\eta_0)=c_+(\eta_0)\beta(\theta_0)\cos(\theta_0-\eta_0).$$

As $\gamma(\theta_0)\neq 0$, it must be $c_-(\eta_0)=c_+(\eta_0)$. This contradict the fact $c_-(\eta_0)<c_+(\eta_0)$ as shown in the above. If $\gamma(\theta_0)=0$, then $\theta_0=\eta_0\pm\pi/2$ because of $c_+(\eta_0)\beta(\theta_0)>0$. However, this contradicts $\gamma(\eta_0\pm\pi/2)>0$ by the property of (c_0,η_0). As a result, we obtain (2.3). From this property and

$$\inf_{\theta\in[0,2\pi)} K(\theta;c_-(\eta_0),\eta_0)=\inf_{\theta\in[0,2\pi)} K(\theta;c_+(\eta_0),\eta_0)=0,$$

we see that, for $t\in(0,1)$,

$$\begin{aligned}&K(\theta;tc_-(\eta)+(1-t)c_+(\eta),\eta_0)\\&=tK(\theta;c_-(\eta_0),\eta_0)+(1-t)K(\theta;c_+(\eta_0),\eta_0)\\&>0.\end{aligned}$$

Thus if $c\in(c_+(\eta_0),c_+(\eta_0))$, then $(c,\eta_0)\in\mathcal{S}$, which implies that $(c_-(\eta_0),\eta_0)$ and $(c_+(\eta_0),\eta_0)$ are part of the boundary of $\mathcal{S}$. This is our assertion. □ □

2.3 The nullcline of Y and its properties

By Lemma 2.1, for an arbitrary $(c,\eta)\in[0,\infty)\times[0,2\pi)$, there exists a unique constant L such that the solution θ of (1.8) satisfies $\theta(L)=2\pi$. Thus, we only have to check that the solutions (x, y) of (1.8) satisfy $x(L) = x_0$ and $y(L) = y_0$ for a ceratin (c, η). We now prepare auxiliary two functions $X(c, \eta)$ and $Y(c, \eta)$ defined by

$$X(c,\eta) := \int_0^{2\pi} \frac{-\sin(\xi-\eta)}{K(\xi;c,\eta)}d\xi, \quad Y(c,\eta) := \int_0^{2\pi} \frac{\cos(\xi-\eta)}{K(\xi;c,\eta)}d\xi$$

for any $(c, \eta) \in \mathcal{S}$. As seen in the following lemma, in order to show the existence of compact traveling wave $(\Gamma_0, c\,\mathbf{e}(\eta))$ of (1.2), we have only to find a $(c, \eta) \in \mathcal{S}$ such that $X(c, \eta) = Y(c, \eta) = 0$.

Lemma 2.11. *The following two are equivalent:*

(i) (1.2) *has a compact traveling wave* $(\Gamma_0, c\,\mathbf{e}(\eta))$.

(ii) (c, η) *belongs to* $\mathcal{S}$ *and satisfies*

$$X(c,\eta) = Y(c,\eta) = 0. \tag{2.4}$$

In particular, as $c = 0$, (2.4) *holds for any* $\eta \in [0, 2\pi)$.

Proof. Let (θ, x, y) be a solution of (1.8) with $(c, \eta) \in \mathcal{S}$. If solution (θ, x, y) satisfies (1.10), then it follows from the matching condition (1.9) that

$$0 = x(L) - x(0) = \int_0^{2\pi} \frac{-\sin\xi}{K(\xi;c,\eta)}d\xi, \tag{2.5}$$

$$0 = y(L) - y(0) = \int_0^{2\pi} \frac{\cos\xi}{K(\xi;c,\eta)}d\xi. \tag{2.6}$$

Thus, if there is a suitable $(c, \eta) \in \mathcal{S}$ satisfying (2.5) and (2.6), (1.2) has a compact traveling wave $(\Gamma_0, c\,\mathbf{e}(\eta))$. By using the standard rotation matrix, we obtain

$$\begin{pmatrix} \displaystyle\int_0^{2\pi} \frac{-\sin\xi}{K(\xi;c,\eta)}d\xi \\ \displaystyle\int_0^{2\pi} \frac{\cos\xi}{K(\xi;c,\eta)}d\xi \end{pmatrix} = \begin{pmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{pmatrix} \begin{pmatrix} X(c,\eta) \\ Y(c,\eta) \end{pmatrix}.$$

This leads the statement (2.4) immediately. We note that $X(0, \eta) = Y(0, \eta) = 0$ for any $\eta \in [0, 2\pi)$ as $c = 0$, and the proof is completed. $\square$ $\square$

The advantage of finding the zero points of X and Y, instead of (2.5) and (2.6), is that the function Y is monotone increasing with respect to c in $\mathcal{S}$ as seen in the following lemma. Hereafter, Λ denotes all sets η satisfying $(c,\eta) \in \mathcal{S}$, namely,

$$\Lambda := \{\eta \in [0, 2\pi) \mid (c,\eta) \in \mathcal{S}\}.$$

Lemma 2.12. *Assume that $\mathcal{S}$ is not empty. Then, for any $\eta \in \Lambda$, $Y(c,\eta)$ is monotone increasing in $c \in (c_-(\eta), c_+(\eta))$ and*

$$\lim_{c \uparrow c_+(\eta)} Y(c,\eta) = \infty, \qquad \lim_{c \downarrow c_-(\eta)} Y(c,\eta) = \begin{cases} Y(0,\eta) & \text{if } \gamma \text{ is positive,} \\ -\infty & \text{otherwise.} \end{cases} \tag{2.7}$$

Proof. Recall that $c_-(\eta) < c_+(\eta)$ from Proposition 2.10. An easy computation shows that

$$Y_c(c,\eta) = \int_0^{2\pi} \frac{\beta(\xi)\cos^2(\xi-\eta)}{K(\xi;c,\eta)^2} d\xi > 0$$

for any $c \in (c_-(\eta), c_+(\eta))$. Thus, we obtain the monotonicity.

We next consider the limit of $Y(c,\eta)$ as c tends to $c_+(\eta)$. let $(c,\eta) \in \mathcal{S}$. It is obvious that, for any $\eta \in \Lambda$, $Y(c,\eta)$ is finite in $(c_-(\eta), c_+(\eta)) \times \{\eta\}$ since $K(\theta;c,\eta) > 0$. Thus $K(\theta;c_+(\eta),\eta) \geq 0$ for any $\theta \in [0,2\pi)$, and there exists a point θ_* such that $K(\theta_*;c_+(\eta),\eta) = 0$. On account of the Lipschitz continuity of K, there exists a positive constant C_K such that $|K(\theta;c,\eta)| \leq C_K|\theta-\theta_*|$ in an appropriate neighborhood of θ_*. Thus

$$\lim_{c \uparrow c_+(\eta)} \int_0^{2\pi} \frac{1}{K(\xi;c,\eta)} d\xi = \infty.$$

Next we show that $\cos(\theta_* - \eta) \neq 0$. If $\cos(\theta_* - \eta) = 0$, then we have $\gamma(\theta_*) = K(\theta_*;c_+(\eta),\eta) = 0$. However this means that $(c_-(\eta), c_+(\eta)) \times \{\eta\} \cap \mathcal{S} = \emptyset$ because $K(\theta_*;c,\eta) = \gamma(\theta_*) - c\beta(\theta_*)\cos(\theta_* - \eta) = 0$ for any $c \geq 0$. This contradicts the fact $(c,\eta) \in \mathcal{S}$. Thus $\cos(\theta_* - \eta) \neq 0$. According to the monotonicity, $Y(c,\eta)$ goes to ∞ as c tends to $c_+(\eta)$.

Finally we consider the limit of $Y(c,\eta)$ as c tends to $c_-(\eta)$. Note that $c_-(\eta) = 0$ if γ is positive. Since $K(\theta;0,\eta) = \gamma(\theta) > 0$, as c tends to $c_-(\eta)$, the integrand of $Y(c,\eta)$ does not have any singularity, namely, $Y(c,\eta)$ goes to $Y(0,\eta)$, simply. If γ is sign-changing, then $c_-(\eta) > 0$. As discussed in the above, $K(\theta; c_-(\eta), \eta) \geq 0$ for any $\theta \in [0, 2\pi)$, and there exists a point θ_* such that $K(\theta_*; c_-(\eta), \eta) = 0$. Repeating the same argument and considering the monotonicity, we see that $Y(c, \eta)$ goes to $-\infty$ as c tends to $c_-(\eta)$. The proof is completed. □ □

We now focus our attention on the set (c, η) satisfying $Y(c, \eta) = 0$. Define $\mathcal{Y}$ and Λ_- by

$$\begin{aligned}
\mathcal{Y} &:= \{(c, \eta) \in \mathcal{S} \mid Y(c, \eta) = 0\}, \\
\Lambda_- &:= \{\eta \in [0, 2\pi) \mid Y(c, \eta) < 0,\ (c, \eta) \in \mathcal{S}\}.
\end{aligned}$$

Clearly, $\Lambda_- \subset \Lambda$ is satisfied.

Proposition 2.13. *Suppose that Λ_- is non-empty and connected. Then, $\mathcal{Y}$ is a C^1 simple curve that is represented by the graph $c = \varphi(\eta)$ from Λ_-. Moreover, there is at most one $(c, \eta) \in \mathcal{S}$ satisfying $X(c, \eta) = Y(c, \eta) = 0$.*

Proof. Lemmas 2.6 and 2.12 imply that for any $\eta \in \Lambda_-$, there exists a unique point $c = c(\eta) \in (c_-(\eta), c_+(\eta))$ such that $Y(c, \eta) = 0$. Recall that $Y_c(c, \eta) > 0$ in $\mathcal{Y}$ by Lemma 2.12. By the implicit function theorem for $Y(c, \eta)$, there exists a unique function $\varphi \in C^1(\Lambda_-)$ such that $Y(\varphi(\eta), \eta) = 0$, and $c = \varphi(\eta) > 0$ for any $\eta \in \Lambda_-$. In contrast, there is no point c satisfying $Y(c, \eta) = 0$ as $\eta \notin \Lambda_-$ because $Y(c, \eta) \geq 0$ and $Y_c(c, \eta) > 0$ in $\mathcal{S}$. Consequently, $\mathcal{Y}$ is a C^1 simple curve.

Next, we show that $(c, \eta) \in \mathcal{S}$ with $X(c, \eta) = Y(c, \eta) = 0$ is at most one. Since $Y(\varphi(\eta), \eta) = 0$ in Λ_-, we observe that

$$Y_c(\varphi(\eta), \eta)\, \varphi'(\eta) + Y_\eta(\varphi(\eta), \eta) = 0 \quad \text{in } \Lambda_-.$$

Let $Z(\eta)$ be given by $Z(\eta) := X(\varphi(\eta), \eta)$. Then, $Z(\eta)$ satisfies

$$\frac{d}{d\eta} Z \quad = \quad X_c \varphi'(\eta) + X_\eta = \frac{X_\eta Y_c - X_c Y_\eta}{Y_c}. \tag{2.8}$$

A simple calculation yields

$$\begin{aligned}
X_c(c,\eta) &= -\int_0^{2\pi} \frac{\beta(\xi)\sin(\xi-\eta)\cos(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi,\\
X_\eta(c,\eta) &= Y - \int_0^{2\pi} \frac{c\beta(\xi)\sin^2(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi,\\
Y_c(c,\eta) &= \int_0^{2\pi} \frac{\beta(\xi)\cos^2(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi,\\
Y_\eta(c,\eta) &= -X + \int_0^{2\pi} \frac{c\beta(\xi)\cos(\xi-\eta)\sin(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi.
\end{aligned}$$

Using the above inequalities, we obtain

$$\begin{aligned}
&X_\eta Y_c - X_c Y_\eta\\
&\quad= \left(Y - \int_0^{2\pi} \frac{c\beta(\xi)\sin^2(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi\right)\int_0^{2\pi} \frac{\beta(\xi)\cos^2(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi\\
&\qquad+\int_0^{2\pi} \frac{\beta(\xi)\sin(\xi-\eta)\cos(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi\\
&\qquad\times\left(-X + \int_0^{2\pi} \frac{c\beta(\xi)\cos(\xi-\eta)\sin(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi\right)\\
&\quad= -X\int_0^{2\pi} \frac{\beta(\xi)\sin(\xi-\eta)\cos(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi\\
&\qquad-\int_0^{2\pi} \frac{c\beta(\xi)\sin^2(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi\int_0^{2\pi} \frac{\beta(\xi)\cos^2(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi\\
&\qquad+\int_0^{2\pi} \frac{\beta(\xi)\sin(\xi-\eta)\cos(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi\int_0^{2\pi} \frac{c\beta(\xi)\cos(\xi-\eta)\sin(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi.
\end{aligned}$$

Thus, (2.8) satisfies

$$\frac{d}{d\eta}Z = \frac{X_\eta Y_c - X_c Y_\eta}{Y_c} = -\frac{|X_c|}{Y_c}Z - c\frac{Y_c W - (X_c)^2}{Y_c},$$

where

$$W := \int_0^{2\pi} \frac{\beta(\xi)\sin^2(\xi-\eta)}{K(\xi;c,\eta)^2}d\xi.$$

By the Cauchy-Schwarz inequality and $\sin(\xi - \eta) \neq \cos(\xi - \eta)$ in $[0, 2\pi)$, we have $Y_c W - (X_c)^2 > 0$. This implies that if there exists a point η_0 satisfying $Z(\eta_0) = 0$, Z is always strictly decreasing at η_0. Consequently, there is at most one $(c, \eta) \in \mathcal{Y}$ satisfying $X(c, \eta) = 0$, which completes the proof. $\square$ $\square$

3 Existence and uniqueness of compact traveling waves

In this section, we show the existence and uniqueness of traveling waves under various driving force γ. To this end, we first introduce the result for the case γ is non-positive or positive. After that, it will be shown the result of sign-changing case.

3.1 Definite driving force

In this subsection we consider the cases where γ is non-positive or positive. Now we state the main result.

Theorem 3.1. *If γ is non-positive, then there is no compact traveling wave for* (1.2)*. If γ is positive, then there exists a unique traveling wave $(\Gamma_0, \mathbf{c})$ of* (1.2)*.*

Proof. By Lemma 2.7, $\mathcal{S}$ is empty if γ is non-positive. Thus we immediately know non-existence of compact traveling waves to (1.2).

Suppose that that γ is positive. Let η, η_0 be arbitrary points in $[0, 2\pi)$. Then the function Y satisfies

$$\begin{aligned} Y(0, \eta + \eta_0) &= \int_0^{2\pi} \frac{\cos(\xi - \eta_0 - \eta)}{\gamma(\xi)} d\xi \\ &= \int_0^{2\pi} \frac{\cos(\xi - \eta_0)\cos\eta + \sin(\xi - \eta_0)\sin\eta}{\gamma(\xi)} d\xi \\ &= Y(0, \eta_0)\cos\eta - X(0, \eta_0)\sin\eta. \end{aligned}$$

The same argument implies

$$\begin{pmatrix} X(0, \eta + \eta_0) \\ Y(0, \eta + \eta_0) \end{pmatrix} = \begin{pmatrix} \cos\eta & \sin\eta \\ -\sin\eta & \cos\eta \end{pmatrix} \begin{pmatrix} X(0, \eta_0) \\ Y(0, \eta_0) \end{pmatrix} \tag{3.1}$$

for any $\eta \in [0, 2\pi)$.

Assume that $(X(0,\eta_0), Y(0,\eta_0)) = (0,0)$ with some $\eta_0 \in [0,2\pi)$. Then $(X(0,\cdot), Y(0,\cdot)) \equiv (0,0)$. From Lemma 2.11, there exists a compact traveling wave solution with $c = 0$. Lemma 2.12 leads to $Y(c,\eta) > 0$ for any $c > 0$ and $(c,\eta) \in \mathcal{S}$. This implies that there are no other compact traveling waves.

Assume that $(X(0,\eta_0), Y(0,\eta_0)) \neq (0,0)$ with some $\eta_0 \in [0,2\pi)$. Since the matrix in right hand side of (3.1) is a rotation matrix, we can assume that

$$X(0,\eta_0) = 0, \quad Y(0,\eta_0) > 0.$$

Since

$$Y(0,\eta) = Y(0,\eta_0)\cos(\eta - \eta_0) = -X(0, \eta - \pi/2) \tag{3.2}$$

for any $\eta \in [0,2\pi)$, we can assume that $\eta_0 \in (0,\pi/2)$ after the appropriate rotation, if necessary. Substituting $\eta = \eta_0 + \pi$ and $\eta = \eta_0$ into (3.2), we have

$$\begin{aligned} X\left(0, \eta_0 + \frac{\pi}{2}\right) &= Y(0,\eta_0) > 0, \\ X\left(0, \eta_0 + \frac{3\pi}{2}\right) &= X\left(0, \eta_0 - \frac{\pi}{2}\right) = -Y(0,\eta_0) < 0. \end{aligned}$$

Furthermore (3.2) leads to $\Lambda_- = (\eta_0 + \pi/2, \eta_0 + 3\pi/2)$. By Proposition 2.13, there exists a C^1 function φ such that $\mathcal{Y} = \{(\varphi(\eta), \eta) \mid \eta \in \Lambda_-\}$. Since $Z(\eta_0 + \pi/2) = X(0, \eta_0 + \pi/2) > 0$ and $Z(\eta_0 + 3\pi/2) = \eta_0 + 3\pi/2 < 0$, there is at least one zero point η. By Proposition 2.13, Z has a unique zero point $\eta \in [0,2\pi)$. Therefore we complete the proof. □ □

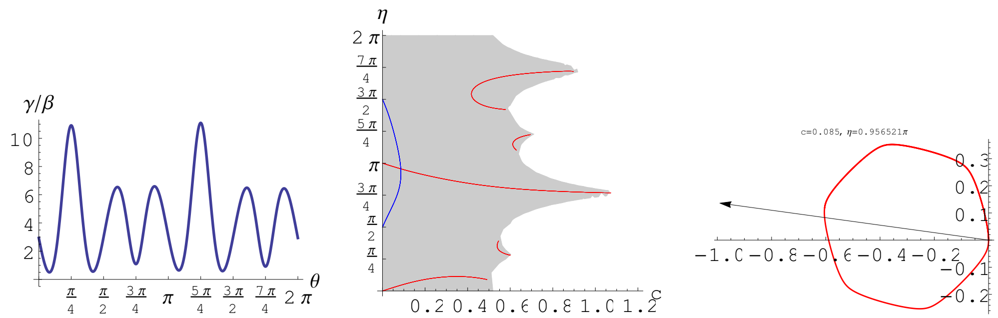
Remark 3.2. As seen in the introduction, the uniqueness is up to the shift. Non-existence of compact traveling waves in Theorem 3.1 is guaranteed will be extended to Corollary 3.11 in Subsection 3.2.

Moreover, this theorem implies that two traveling waves do not coexist. Even if there are two or more local maxima of γ, the speed and the direction are uniquely determined.

Example 3.3. Let γ and β be given by

$$\begin{cases} \gamma(\theta) &:= \; 1 + 10\cos^2\left(3\theta + \dfrac{\pi}{4}\right) - \dfrac{\cos\theta}{8}, \\ \beta(\theta) &:= \; 1 + \cos^2\theta. \end{cases} \tag{3.3}$$

Note that γ is positive. Then the corresponding set $\mathcal{S}$ and the nullclines of X and Y are given in Fig. 3.1.



(a) (b) (c)

Figure 3.1: Numerical results for a positive driving force γ given by (3.3). (a) the graph of γ/β. (b) the nullclines of X and Y, indicated by red and blue, respectively, on $\mathcal{S}$. (c) the corresponding compact traveling wave $(\Gamma_0, \mathbf{c})$. Here Γ_0 is a hexagonal shape indicated by red and $\mathbf{c}$ is represented by the arrow.

Now we give an estimate of the velocity $\mathbf{c}$ of compact traveling waves. The vector $\mathbf{e}_\gamma$ defined in (1.11) is significant in our next lemma. For the convenience of the reader we write $\mathbf{e}_\gamma$ here again,

$$\mathbf{e}_\gamma = \left(\int_0^{2\pi} \frac{\cos\theta}{\gamma(\theta)}\, d\theta, \int_0^{2\pi} \frac{\sin\theta}{\gamma(\theta)}\, d\theta \right).$$

Proposition 3.4. *Suppose that there exists a compact traveling wave* $(\Gamma_0, \mathbf{c})$ *of* (1.2) *under a positive driving force* γ. *Then it follows that*

(i) *if* $\mathbf{e}_\gamma = \mathbf{0}$, *then* $\mathbf{c} = \mathbf{0}$,

(ii) *if* $\mathbf{e}_\gamma \neq \mathbf{0}$, *then* $\mathbf{c} \neq \mathbf{0}$ *with* $0 < |\mathbf{c}| < c_M$ *and* $\mathbf{c} \cdot \mathbf{e}_\gamma < 0$.

Proof. Let (c, η) be a polar coordinate of the velocity $\mathbf{c}$, i.e., $\mathbf{c} = c\,\mathbf{e}(\eta)$. If $\mathbf{e}_\gamma = \mathbf{0}$, then $X(0,0) = Y(0,0) = 0$. It follows from Lemma 2.11 that $(\Gamma_0, \mathbf{0})$ is a compact traveling wave of (1.2). The uniqueness of solutions leads to $\mathbf{c} = \mathbf{0}$.

Conversely, if $\mathbf{c} = \mathbf{0}$, then $\mathbf{e}_\gamma = \mathbf{0}$ by the proof of Theorem 3.1. If $\mathbf{e}_\gamma \neq \mathbf{0}$, then $\mathbf{c} \neq \mathbf{0}$ and Lemma 2.6 guarantees $0 < |\mathbf{c}| < c_M$. Moreover we see $\eta \in \Lambda_-$ by Proposition 2.13. It follows from the monotonicity of Y_c in Lemma 2.12 that $\Lambda_- = \{\eta \in [0, 2\pi) \mid Y(0,\eta) < 0\}$. Then

$$\mathbf{c} \cdot \mathbf{e}_\gamma = c(\cos\eta, \sin\eta) \cdot (X(0,0), Y(0,0)) = cY(0,\eta) < 0.$$

From this, we obtain our assertion. □ □

3.2 Sign-changing driving force

When γ is sign-changing, the existence of compact traveling waves is not guaranteed in general. For example, setting

$$\gamma(\theta) = \cos\theta - \frac{1}{2}, \tag{3.4}$$

we see that γ is sign-changing and $\gamma(\theta) \leq 0$ in $[\pi/3, 5\pi/3]$. Due to the length of negative region of γ, $\inf_{\theta\in[0,2\pi)} K(\theta; c, \eta) \leq 0$, which leads to $\mathcal{S} = \emptyset$. In consequence, there is no traveling wave to (1.2) under (3.4). From this consideration, every sign-changing γ does not always satisfy $\mathcal{S} \neq \emptyset$. Furthermore it is not easy to show the existence of compact traveling waves for sign-changing case even if $\mathcal{S} \neq \emptyset$, because $X(c, \eta)$ and $Y(c, \eta)$ does not have a good property as (3.2) in the proof of Theorem 3.1, which is useful in distinguishing the signs of X and Y. Thus we here change the strategy and use the information of the signs of X and Y from the boundary of $\mathcal{S}$.

In this section, as seen in Remark 2.8, without loss of generality, we assume $\Lambda \Subset [0, 2\pi)$ when γ is sign-changing. The uniqueness of compact traveling waves for sign-changing case is obtained as well as positive case.

Theorem 3.5. *Assume that γ is sign-changing and $\mathcal{S}$ is not empty. Then $\Lambda_- = \Lambda$. Moreover, if* (1.2) *has compact traveling waves, then there exists at most one compact traveling wave.*

Proof. We first show $\Lambda_- = \Lambda$. We see at once that $\Lambda_- \subset \Lambda$ by definition. For any $\eta \in \Lambda$, it follows from (2.7) in Lemma 2.12 that there exists a constant $c \in (c_-(\eta), c_+(\eta))$ satisfying $Y(c, \eta) < 0$. Thus we obtain $\Lambda \subset \Lambda_-$.

Next we claim the uniqueness of solutions. By Proposition 2.9, $\mathcal{S}$ is simply connected, and then Λ is connected. Since $\Lambda = \Lambda_-$, it follows that Λ_- is not empty and connected. Applying Proposition 2.13, we complete the proof. □ □

Before stating an existence theorem, we investigate some properties for the boundary of $\mathcal{S}$. As seen in Lemma 2.6, $\mathcal{S}$ is bounded when γ is sign-changing. In what follows, η_+ and η_- denote the supremum and infimum of Λ, respectively, that is,

$$\eta_+ := \sup\Lambda = \sup_{(c,\eta)\in\mathcal{S}} \eta, \quad \eta_- := \inf\Lambda = \inf_{(c,\eta)\in\mathcal{S}} \eta,$$

and $\partial\mathcal{S}_+$ (resp. $\partial\mathcal{S}_-$) also stand for the set of $(c, \eta_+) \in \partial\mathcal{S}$ (resp. $(c, \eta_-) \in \partial\mathcal{S}$), where $\partial\mathcal{S}$ stands for the boundary of $\mathcal{S}$, i.e., $\partial\mathcal{S} := \mathrm{Cl}(\mathcal{S}) \setminus \mathrm{Int}(\mathcal{S})$. As mentioned above, in general, (1.2) might have no compact traveling wave when γ is sign-changing. We here give an additional condition for γ and investigate the distribution of θ satisfying $K(\theta; c, \eta) = 0$ on $\partial\mathcal{S}_+$ and $\partial\mathcal{S}_-$.

Definition 3.6. We say that $\mathcal{S}$ is *admissible*, if there is a positive constant δ such that

$$\{c^*\} \times (\eta_+ - \delta, \eta_+) \subset \mathcal{S}, \quad \{c_*\} \times (\eta_-, \eta_- + \delta) \subset \mathcal{S}$$

for every $(c^*, \eta_+) \in \partial\mathcal{S}_+$ and $(c_*, \eta_-) \in \partial\mathcal{S}_-$.

Next lemma give a relation between admissible condition of $\mathcal{S}$ and the position of θ satisfying $K(\theta; c, \eta) = 0$.

Lemma 3.7. *Let $\mathcal{S}$ be admissible. Then, for any $(c^*, \eta_+) \in \partial\mathcal{S}_+$ (resp. $(c_*, \eta_-) \in \partial\mathcal{S}_-$), $K(\theta; c^*, \eta_+) > 0$ in $[\eta_+ + \pi, \eta_+ + 2\pi)$ (resp. $K(\theta; c_*, \eta_-) > 0$ in $(\eta_+, \eta_+ + \pi]$). Moreover, $\partial\mathcal{S}_\pm$ is composed of a single point or a line segment. If $\partial\mathcal{S}_+$ (resp. $\partial\mathcal{S}_-$) is a single point (c^*, η_+) (resp. (c_*, η_-)), there exist at least two points $\theta_+^* \in [\eta_+, \eta_+ + \pi/2)$ and $\theta_-^* \in (\eta_+ + \pi/2, \eta_+ + \pi)$ (resp. $\theta_+^* \in (\eta_- + \pi, \eta_- + 3\pi/2]$ and $\theta_-^* \in [\eta_- + 3\pi/2, \eta_- + 2\pi]$) such that $K(\theta_+^*; c^*, \eta_+) = K(\theta_-^*; c^*, \eta_+) = 0$ (resp. $K(\theta_+^*; c_*, \eta_-) = K(\theta_-^*; c_*, \eta_-) = 0$). Otherwise it holds that $K(\eta_+ + \pi/2; c^*, \eta_+) = 0$ (resp. $K(\eta_- + 3\pi/2; c_*, \eta_-) = 0$).*

Proof. Let θ^* satisfy $K(\theta^*; c^*, \eta_+) = 0$. Since $\mathcal{S}$ is admissible, we have

$$K(\theta^*; c^*, \eta_+ - \varepsilon) > 0$$

for every $\varepsilon \in (0, \delta]$. From $\beta > 0$ and $c^* > 0$, we see

$$\begin{aligned} 0 &> K(\theta^*; c^*, \eta_+) - K(\theta^*; c^*, \eta_+ - \varepsilon) \\ &= c^*\beta(\theta^*)\left(\cos(\theta^* - \eta_+ + \varepsilon) - \cos(\theta^* - \eta_+)\right). \end{aligned}$$

This inequality says that $\theta^* \in [\eta_+, \eta_+ + \pi)$. It follows from

$$\inf_{\theta \in [0, 2\pi)} K(\theta; c^*, \eta_+) = 0$$

and the admissible property of $\mathcal{S}$ that

$$K(\theta; c^*, \eta_+) > 0 \quad \text{in } [\eta_+ + \pi, \eta_+ + 2\pi).$$

Next we investigate the set of $\partial\mathcal{S}_\pm$. Assume that $(c_1^*, \eta_+), (c_2^*, \eta_+) \in \partial\mathcal{S}_+$. Then $(c_1^*, \eta_+ - \varepsilon), (c_2^*, \eta_+ - \varepsilon) \in \mathcal{S}$ for any $\varepsilon \in (0, \delta)$, and then $[c_1^*, c_2^*] \times [\eta_+ - \varepsilon, \eta_+) \subset \mathcal{S}$. This leads to $(tc_1^* + (1-t)c_2^*, \eta_+) \in \partial\mathcal{S}_+$ for $t \in [0, 1]$, which implies that $\partial\mathcal{S}_\pm$ is composed of a single point or a line segment. This assertion also holds for $\partial\mathcal{S}_-$,

Let $\partial\mathcal{S}_+$ be a single point. From the definition of $\partial\mathcal{S}_+$, it follows that

$$\inf_{\theta\in[0,2\pi)} K(\theta; c^* \pm \varepsilon, \eta_+) < 0$$

for small $\varepsilon > 0$. This implies that there are some points $\theta_+^*, \theta_-^* \in [\eta_+, \eta_+ + \pi)$ satisfying

$$K(\theta_\pm^*; c^*, \eta_+) = 0 \quad \text{and} \quad K(\theta_\pm^*; c^* \pm \varepsilon, \eta_+) < 0.$$

If $K(\theta_+^*; c^* + \varepsilon, \eta_+) < 0$, then

$$0 < K(\theta_+^*; c^*, \eta_+) - K(\theta_+^*; c^* + \varepsilon, \eta_+) = -\varepsilon\beta(\theta_+^*)\cos(\theta_+^* - \eta_+).$$

Thus $\theta_+^* \in (\eta_+ + \pi/2, \eta_+ + \pi]$. As $K(\theta_+^*; c^* - \varepsilon, \eta_+) < 0$, it follows $\theta_-^* \in [\eta_+, \eta_+ + \pi/2)$.

If $\partial\mathcal{S}_+$ is a line segment,

$$\inf_{\theta\in[0,2\pi)} K(\theta; c^* \pm \varepsilon, \eta_+) = 0$$

for any small $\varepsilon > 0$. Since $\partial K/\partial c = -\beta(\theta)\cos(\theta - \eta)$, it is expected that at least one of $K(\eta_+ + \pi/2; c^*, \eta_+) = 0$ and $K(\eta_+ + 3\pi/2; c^*, \eta_+) = 0$ is guaranteed. If $K(\eta_+ + 3\pi/2; c^*, \eta_+) = 0$ is correct, then $\gamma(\eta_+ + \pi/2; c^*, \eta_+) = 0$. From this,

$$K\left(\eta_+ + \frac{3\pi}{2}; c^*, \eta_+ - \delta\right) = -c\beta\left(\eta_+ + \frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{2} + \delta\right) < 0.$$

This contradicts the fact that $\mathcal{S}$ is admissible. Thus it follows that $K(\eta_+ + \pi/2; c^*, \eta_+) = 0$. As for $\partial\mathcal{S}_-$, we can also show the desired claim. Therefore we complete the proof. $\square$ $\square$

Now we state the existence theorem.

Theorem 3.8. *If $\mathcal{S}$ is admissible, then there exists a unique compact traveling wave $(\Gamma_0, \mathbf{c})$ of (1.2).*

Proof. The uniqueness of traveling wave solutions is already established by Theorem 3.5. To show the existence of compact traveling waves, we need to find a point (c, η) satisfying $X(c, \eta) = 0$ on $\mathcal{Y}$. Recall that from the proof of Theorem 3.5 and Proposition 2.13, $\Lambda = \Lambda_-(\neq \emptyset)$ holds, and then $\mathcal{Y}$ is a C^1 curve which connected $\partial\mathcal{S}_+$ to $\partial\mathcal{S}_-$. Now we examine the sign of X on $\mathcal{Y}$ in the neighbourhood of $\partial\mathcal{S}_+$ and $\partial\mathcal{S}_-$.

We now focus attention on the neighbourhood of $\partial\mathcal{S}_+$. By Lemma 3.7, $\partial\mathcal{S}_+$ is composed of a single point or an interval. Assume that $\partial\mathcal{S}_+$ is a single point. According to Lemma 3.7, there are at least two points $\theta_+^* \in [\eta_+, \eta_+ + \pi/2)$ and $\theta_-^* \in (\eta_+ + \pi/2, \eta_+ + \pi)$ satisfying $K(\theta_+^*; c^*, \eta_+) = K(\theta_-^*; c^*, \eta_+) = 0$ for $(c^*, \eta_+) \in \partial\mathcal{S}_+$. As $\theta_+^* = \eta_+$, it is not easy to confirm the sign of $X(c, \eta)$ because it is primitive function $-\sin(\theta_+^* - \eta_+)/K(\theta_+^*; c^*, \eta_+)$ is an indeterminate form at θ_+^*. Moreover the right-sided limit (resp. the left-sided limit) of the primitive function is positive (resp. negative). To overcome the difficulty, we use a modified function $X^\sigma(c, \eta)$, instead of $X(c, \eta)$, which is given by

$$X^\sigma(c, \eta) = \int_0^{2\pi} \frac{-\sin(\theta - \eta + \sigma)}{K(\theta; c, \eta)}\, d\theta,$$

where σ is a small positive constant satisfying

$$\{\theta \in [0, 2\pi) \mid K(\theta; c^*, \eta_+) = 0\} \subset (\eta_+ - \sigma, \eta_+ - \sigma + \pi).$$

Note that the existence of σ is guaranteed by Lemma 3.7. A simple computation gives $X^\sigma(c, \eta) = (\cos\sigma) \cdot X(c, \eta) - (\sin\sigma) \cdot Y(c, \eta)$. We see at once that the sign of X is equivalent to that of X^σ on $\mathcal{Y}$ as $\cos\sigma > 0$.

From now, we will show that $X^\sigma(c, \eta) < 0$ in $\mathcal{Y} \cap B_\delta(c^*, \eta_+)$, where $B_\delta(c^*, \eta_+)$ is an open ball with the center (c^*, η_+) and the radius δ. To this end, we need to show that the sign of $X^\sigma(c, \eta)$ near $X^\sigma(c^*, \eta_+)$ is negative. Note that X^σ is divided into two parts as follows:

$$X^\sigma(c, \eta) = \int_\eta^{\eta - \sigma + \pi} \frac{-\sin(\theta - \eta + \sigma)}{K(\theta; c, \eta)}\, d\theta + \int_{\eta - \sigma + \pi}^{\eta + 2\pi} \frac{-\sin(\theta - \eta + \sigma)}{K(\theta; c, \eta)}\, d\theta.$$

Since $K(\theta; c, \eta)$ is continuous with respect to c and η, the first term satisfies

$$\lim_{(c,\eta)\to(c^*,\eta_+)} \int_\eta^{\eta - \sigma + \pi} \frac{-\sin(\theta - \eta + \sigma)}{K(\theta; c, \eta)}\, d\theta = -\infty \tag{3.5}$$

for every $(c, \eta) \in \mathcal{S}$. Define ε by

$$\varepsilon := \min_{\theta \in [\eta - \sigma + \pi, \eta + 2\pi]} K(\theta; c^*, \eta_+) > 0.$$

By the continuity of $K(\theta; c^*, \eta_+)$ in c and η, there is a positive constant δ_1 such that

$$\min_{\theta \in [\eta-\sigma+\pi, \eta+2\pi]} K(\theta; c, \eta) > \frac{\varepsilon}{2} \quad \text{in } \mathcal{S} \cap B_{\delta_1}(\mathrm{c}^*, \eta_+).$$

Thus the second term satisfies

$$\int_{\eta-\sigma+\pi}^{\eta+2\pi} \frac{-\sin(\theta - \eta + \sigma)}{K(\theta; c, \eta)} \, d\theta \le \frac{2}{\varepsilon}(1 + \cos\sigma) \tag{3.6}$$

for every $(c, \eta) \in \mathcal{S} \cap B_{\delta_1}(c^*, \eta_+)$, which means that the second term is bounded. Combining (3.5) with (3.6) yields that there exists a constant $0 < \delta_2 \le \delta_1$ such that $X^\sigma(c, \eta) < 0$ in $\mathcal{S} \cap B_{\delta_2}(c^*, \eta_+)$. Recall that $\mathcal{Y}$ is a C^1 curve which is connected between $\partial\mathcal{S}_+$ and $\partial\mathcal{S}_-$. Thus we obtain the desired assertion $X^\sigma(c, \eta) < 0$ in $\mathcal{Y} \cap B_{\delta_2}(c^*, \eta_+)$.

Next let us check the sign of X in case $\partial\mathcal{S}_+$ is an interval. We write the interval of $\partial\mathcal{S}_+$ as $[c_1^*, c_2^*]$. The admissible property for $\mathcal{S}$ and the continuity of $K(\theta; c, \eta)$ in c and η leads to the existence of a constant $0 < \delta_3 \le \delta$ satisfying $X^\sigma(c, \eta) < 0$ in $[c_1^* + \delta_3, c_2^* - \delta_3] \times [\eta_+ - \delta_3, \eta_+]$. Remark that $K(\eta_+ + \pi/2; c^*, \eta_+) = 0$ and $\sin(\pi/2 + \sigma) > 0$. Repeating the same argument above, we know that X is negative in $\mathcal{S} \cap (B_{\delta_4}(c_1^*, \eta_+) \cup B_{\delta_4}(c_2^*, \eta_+))$ for $\delta_4 \in (0, \delta_2]$. As a result, X^σ satisfies

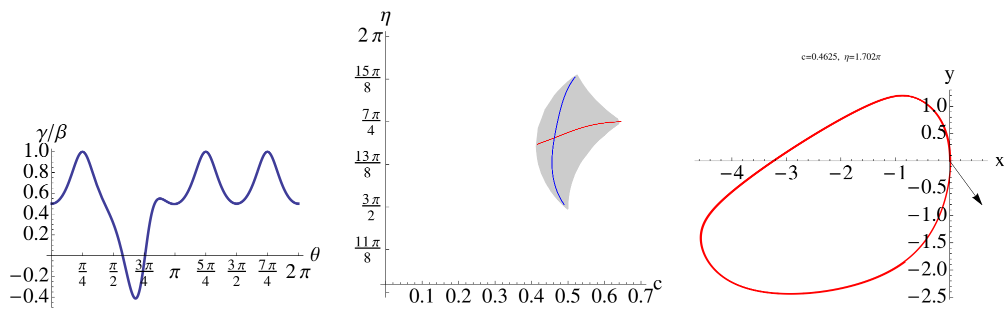
$$X^\sigma(c, \eta) < 0 \quad \text{in } \mathcal{S} \cap \mathcal{M},$$

where $\mathcal{M} := B_{\delta_4}(c_1^*, \eta_+) \cup \{[c_1^* + \delta_3, c_2^* - \delta_3] \times [\eta_+ - \delta_3, \eta_+]\} \cup B_{\delta_4}(c_2^*, \eta_+)$. By the property of $\mathcal{Y}$, $\mathcal{Y} \cap \mathcal{M} \ne \emptyset$ holds, which implies the existence of (c, η) satisfying $X(c, \eta) < 0$ on $\mathcal{Y}$.

As for $\partial\mathcal{S}_-$, we also repeat the same argument and can show that there is a (c, η) in the neighbourhood of $\partial\mathcal{S}_-$ satisfying $X(c, \eta) > 0$ on $\mathcal{Y}$. Finally, there exists a point $(c, \eta) \in \mathcal{Y}$ satisfying $X^\sigma(c, \eta) = X(c, \eta) = 0$ because X^σ is continuous on $\mathcal{Y}$. This is our assertion. □ □

Example 3.9. Here we give two numerical results for sign-changing driving forces γ. First, let γ and β be defined by

$$\begin{cases} \gamma(\theta) := \begin{cases} 1 - \dfrac{3}{2} \exp\left[-5\left(\theta - \dfrac{2\pi}{3}\right)^2\right] & \left(0 \le \theta < \dfrac{5\pi}{3}\right), \\ 1 - \dfrac{3}{2} \exp\left[-5\left(\theta - \dfrac{8\pi}{3}\right)^2\right] & \left(\dfrac{5\pi}{3} \le \theta < 2\pi\right), \end{cases} \\ \beta(\theta) := 1 + \cos^2 2\theta. \end{cases} \tag{3.7}$$



(a) (b) (c)

Figure 3.2: Numerical results for a sign-changing driving force γ and β given by (3.7). (a) the graph of γ/β. (b) the nullclines of X and Y, indicated by red and blue, respectively, on $\mathcal{S}$. (c) the corresponding compact traveling wave $(\Gamma_0, \mathbf{c})$. Here Γ_0 is a convex shape indicated by red and $\mathbf{c}$ is represented by the arrow.

Note that γ is sign-changing. Then the corresponding set $\mathcal{S}$ and the nullclines of X and Y are given in Fig.3.2.

Next let γ and β be given by

$$\begin{cases} \gamma(\theta) & := & \gamma_1(\theta) - \dfrac{\gamma_1(2\pi) - \gamma_1(0)}{2\pi}\theta, \\ \beta(\theta) & := & 1, \end{cases} \tag{3.8}$$

where

$$\gamma_1(\theta) := 1.2 - 1.3e^{-5(\theta - 2\pi/3 + 0.3)^2} - 0.9e^{-5(\theta - 3\pi/2 - 0.3)^2}.$$

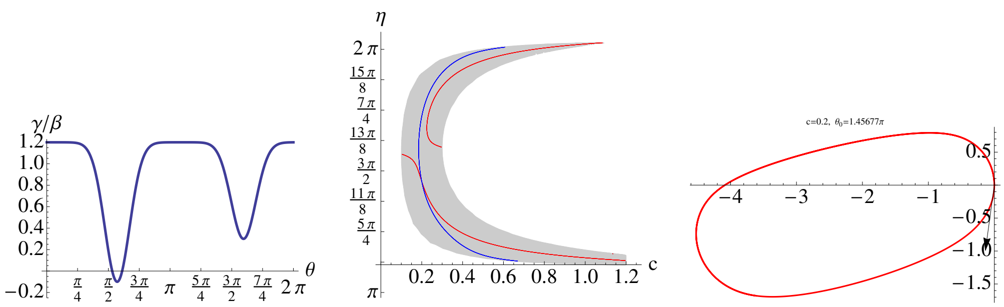
As well as (3.7), γ given by (3.8) is sign-changing. The $\mathcal{S}$ and nullclines of X and Y are given in Fig 3.3. As far as we confirm two numerical results, (3.7) satisfies admissible condition, but (3.8) does not satisfy admissible condition.

As mentioned in Proposition 3.4, we refer to the estimate of $\mathbf{c}$ under the sing-changing γ. The following cone set is helpful for understanding it :

$$\mathcal{C}_- := \left\{ c\,\mathbf{e}(\theta_-) \in \mathbb{R}^2 \mid \gamma(\theta_-) \le 0,\ c \ge 0 \right\}.$$

Lemma 3.10. *Suppose that there exists a compact traveling wave* $(\Gamma_0, \mathbf{c})$ *to* (1.2) *under a sign-changing* γ. *Then*

$$0 < -\inf_{\xi \in [0, 2\pi)} \frac{\gamma(\xi)}{\beta(\xi)} < |\mathbf{c}| < \sup_{\xi \in [0, 2\pi)} \frac{\gamma(\xi)}{\beta(\xi)}$$



(a) (b) (c)

Figure 3.3: Non-admissible example. (a) the graph of γ/β (b) the graph of $\mathcal{S}$ and the nullclines of X, Y, which are indicated by red and blue (c) the corresponding compact traveling wave with the velocity.

and $\mathbf{c}\cdot\widetilde{\mathbf{c}}<0$ *for any* $\widetilde{\mathbf{c}}\in\mathcal{C}_-\setminus\{\mathbf{0}\}$.

Proof. From Lemma 2.6, $0\le|\mathbf{c}|<c_M$. If $|\mathbf{c}|=c\le-\inf_{\xi\in[0,2\pi)}\gamma(\xi)/\beta(\xi)$, then

$$\inf_{\xi\in[0,2\pi)}K(\xi;c,\eta)\le\inf_{\xi\in[0,2\pi)}\beta(\xi)\left(\inf_{\xi\in[0,2\pi)}\frac{\gamma(\xi)}{\beta(\xi)}+c\right)\le 0, \tag{3.9}$$

which contradicts the fact that $(\Gamma_0,\mathbf{c})$ is a compact traveling wave to (1.2). Thus $|\mathbf{c}|>-\inf_{\xi\in[0,2\pi)}\gamma(\xi)/\beta(\xi)$. Let θ_- be a point satisfying $\gamma(\theta_-)\le 0$. A simple calculation shows that if $\eta\in[\theta_--\pi/2,\theta_-+\pi/2]$, then

$$K(\theta_-;c,\eta)=\gamma(\theta_-)-c\beta(\theta_-)\cos(\theta_--\eta)\le 0.$$

Thus $\eta\in(\theta_-+\pi/2,\theta_-+3\pi/2)$ for any $(c,\eta)\in\mathcal{S}$, which implies $\mathbf{e}(\eta)\cdot\mathbf{e}(\theta_-)<0$. □ □

According to Lemma 3.10, if (1.2) has a compact traveling wave $(\Gamma_0,\mathbf{c})$ under the sign-changing γ, the velocity $\mathbf{c}$ belongs to the polar set $\mathcal{C}_-^o$ of the cone set $\mathcal{C}_-$, where $\mathcal{C}_-^o$ is defined by

$$\mathcal{C}_-^o:=\left\{\mathbf{c}\in\mathbb{R}^2\;\middle|\;\sup_{\widetilde{\mathbf{c}}\in\mathcal{C}_-}\mathbf{c}\cdot\widetilde{\mathbf{c}}\le 0\right\}.$$

Here we give a sufficient condition of γ such that (1.2) has no compact traveling wave.

Corollary 3.11. *If $\mathcal{C}_-^o \setminus \{\mathbf{0}\}$ is empty, then there are no traveling waves.*

Proof. Let $\mathbf{c}$ be an arbitrary vector of $\mathbb{R}^2\setminus\{\mathbf{0}\}$ and assume that $\sup_{\widetilde{\mathbf{c}}\in\mathcal{C}_-} \mathbf{c}\cdot\widetilde{\mathbf{c}} > 0$. Then there is a vector $\widetilde{\mathbf{c}} \in \mathcal{C}_-$ with $\mathbf{c}\cdot\widetilde{\mathbf{c}} > 0$. By Lemma 3.10, $\mathbf{c}$ is not the velocity of compact traveling wave for (1.2), which completes the proof. □ □

4 Special cases

In this section, we discuss the topics related to compact traveling waves of (1.2) by using the obtained properties.

4.1 Symmetric driving force

In this subsection, we study a compact traveling wave with a symmetric property. Throughout this subsection, we say that β is symmetric to η if $\beta(\theta) = \beta(2\eta - \theta)$ holds for any $\theta \in [\eta, \eta + \pi]$. We note that if β is symmetric to η, then it is also symmetric to $\eta + \pi$. We also define the symmetry of γ in the same way as β. Under this setting, we can find a necessary and sufficient condition of γ to obtain a compact traveling wave $(\Gamma_0, \mathbf{c})$, completely.

Proposition 4.1. *Suppose that β is symmetric to $\eta_* \in [0, \pi)$ and (1.2) has a compact traveling wave $(\Gamma_0, c\,\mathbf{e}(\eta_*))$. Then the traveling wave is an axisymmetric compact traveling wave with respect to the traveling direction if and only if γ is symmetric to η_*, that is, $\gamma(\theta) = \gamma(2\eta_* - \theta)$ for $\theta \in [\eta_*, \eta_* + \pi]$.*

Proof. We first assume that a compact traveling wave $(\Gamma_0, c\,\mathbf{e}(\eta_*))$ is axisymmetric with respect to the traveling direction. Remember that (c, η_*) is a pair of (1.10) which has a solution $(\theta(s), x(s), y(s))$ in $[0, L]$. Without restriction of generality, we can assume that $\theta_0 = \eta_*$. According to Theorem 2.2 and assumptions, the traveling wave is strictly convex and axisymmetric, namely, $\theta_s(s) > 0$ and $\theta(L - s) = 2\eta_* + 2\pi - \theta(s)$. Hence we have $\theta_s(s) = \theta_s(L - s)$ and $\gamma(\theta(s)) = \gamma(\theta(L - s))$ due to (1.10). Thus $\gamma(\theta) = \gamma(2\eta_* - \theta)$ holds for $\theta \in [\theta(0), \theta(L/2)]$.

Next we assume that $\gamma(\theta)$ is symmetric to η_*. Since every traveling wave $(\Gamma_0, \mathbf{c})$ is strictly convex, Γ_0 is represented by $\Gamma_0 = \{(x(\theta), y(\theta)) \mid \theta \in [0, 2\pi)\}$, where

$$(x(\theta), y(\theta)) = \left(x_0 + \int_{\theta_0}^{\theta} \frac{-\sin\xi}{K(\xi; c, \eta_*)} d\xi,\ y_0 + \int_{\theta_0}^{\theta} \frac{\cos\xi}{K(\xi; c, \eta_*)} d\xi \right).$$

Without loss of generality we can assume that $(x_0, y_0, \theta_0) = (0, 0, \eta_*)$. Now we rotate Γ_0 as follows :

$$\begin{pmatrix} \bar{x}(\theta) \\ \bar{y}(\theta) \end{pmatrix} = \begin{pmatrix} \cos(-\eta_*) & -\sin(-\eta_*) \\ \sin(-\eta_*) & \cos(-\eta_*) \end{pmatrix} \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix}. \tag{4.1}$$

Let us denote by $\bar{\Gamma}_0$ the set of $(\bar{x}(\theta), \bar{y}(\theta))$ for $\theta \in [0, 2\pi)$. If $\bar{\Gamma}_0$ is axisymmetric with resect to x–axis, then Γ_0 is axisymmetric with respect to $\mathbf{e}(\eta_*)$. Thus our purpose is achieved by showing that $(\bar{x}(\theta), \bar{y}(\theta)) = (\bar{x}(2\pi + 2\eta_* - \theta), -\bar{y}(2\pi + 2\eta_* - \theta))$. Since (4.1) is represented by

$$(\bar{x}(\theta), \bar{y}(\theta)) = \left(\int_{\eta_*}^{\theta} \frac{-\sin(\xi - \eta_*)}{K(\xi; c, \eta_*)} d\xi, \ \int_{\eta_*}^{\theta} \frac{\cos(\xi - \eta_*)}{K(\xi; c, \eta_*)} d\xi \right),$$

we see

$$\begin{aligned}
&(\bar{x}(2\pi + 2\eta_* - \theta), -\bar{y}(2\pi + 2\eta_* - \theta)) \\
&= \left(\int_{\eta_*}^{2\pi+2\eta_*-\theta} \frac{-\sin(\xi - \eta_*)}{K(\xi; c, \eta_*)} d\xi, \ \int_{\eta_*}^{2\pi+2\eta_*-\theta} \frac{-\cos(\xi - \eta_*)}{K(\xi; c, \eta_*)} d\xi \right) \\
&= \left(\int_{\eta_*}^{\theta} \frac{\sin((2\eta_* - \xi) - \eta_*)}{K(2\eta - \xi; c, \eta_*)} d\xi, \ \int_{\eta_*}^{\theta} \frac{\cos((2\eta_* - \xi) - \eta_*)}{K(2\eta - \xi; c, \eta_*)} d\xi \right) \\
&= (\bar{x}(\theta), \bar{y}(\theta)).
\end{aligned}$$

The last equality is obtained by $K(2\eta_* - \xi; c, \eta_*) = K(\xi; c, \eta_*)$. As a result, we complete the proof. □ □

The assumption relating to β in Proposition 4.1 cannot be removed. We here give a simple counterexample.

Example 4.2. Let β and γ be given by

$$\beta(\theta) = \frac{2 + \sin\theta}{5}, \quad \gamma(\theta) = 1 + \frac{2 + \sin\theta}{5} \cos\theta.$$

Note that β is symmetric to $\pi/2$. Then θ_s is written by

$$\theta_s = K(\theta; 1, 0) = 1.$$

This implies that $(S^1, \mathbf{e}(0))$ is an axisymmetric compact traveling wave with respect to the traveling direction, but γ is not symmetric to 0.

From the definition of $\mathcal{S}$, if (1.2) has a traveling wave, then $\mathcal{S} \neq \emptyset$. However the converse of the statement is not always true. Indeed, as mentioned in the previous section, when γ is sign-changing, we imposed the admissible condition on $\mathcal{S}$. Next theorem asserts that if β and γ are symmetric to an angle η, the converse is true without admissible condition.

Theorem 4.3. *Assume that β and γ are symmetric to $\eta_* \in [0, \pi)$. Then* (1.2) *has a unique compact traveling wave $(\Gamma_0, \mathbf{c})$ if and only if $\mathcal{S} \neq \emptyset$. Moreover the compact traveling wave is an axisymmetric compact traveling wave with respect to the traveling direction.*

Proof. Suppose that there exists a unique compact traveling wave $(\Gamma_0, c\,\mathbf{e}(\eta))$. By Theorem 2.2, the traveling wave is strictly convex, that is, $(c, \eta) \in \mathcal{S}$. Thus $\mathcal{S} \neq \emptyset$.

Conversely, we suppose that $\mathcal{S} \neq \emptyset$. By Lemma 2.7, we only have to consider two cases: γ is positive or sign-changing. Let us check the existence of compact traveling waves. When γ is positive, Theorem 3.1 implies the existence and uniqueness of a compact traveling wave.

Next let γ be sign-changing. Without loss of generality, we here assume that $\mathcal{S}$ satisfies (2.1) as mentioned in Remark 2.8. We now show that if (c, η) is an arbitrary point of $\mathcal{S}$, then $(c, 2\eta_* - \eta) \in \mathcal{S}$ or $(c, 2\eta_* - \eta + 2\pi) \in \mathcal{S}$. By simple calculations, we have

$$\begin{aligned} K(\theta; c, 2\eta_* - \eta) &= \gamma(\theta) - c\beta(\theta)\cos(\theta - 2\eta_* + \eta) \\ &= \gamma(2\eta_* - \theta) - c\beta(2\eta_* - \theta)\cos(2\eta_* - \theta - \eta) \\ &= K(2\eta_* - \theta; c, \eta) \end{aligned}$$

Here we used the symmetric property of $\beta(\theta)$, $\gamma(\theta)$ and $\cos\theta$. Since

$$\inf_{\theta \in [0, 2\pi)} K(2\eta_* - \theta; c, \eta) = \inf_{\theta \in [0, 2\pi)} K(\theta; c, \eta) > 0,$$

we see that, if $0 < \eta \leq 2\eta_*$ (resp. $2\eta_* < \eta \leq 2\pi$), then $(c, 2\eta_* - \eta) \in \mathcal{S}$ (resp. $(c, 2\eta_* - \eta + 2\pi)$).

From now, we confirm that either

$$[0, \infty) \times \{\eta_*\} \cap \mathcal{S} \neq \emptyset \quad \text{and} \quad [0, \infty) \times \{\eta_* + \pi\} \cap \mathcal{S} = \emptyset \tag{4.2}$$

or

$$[0, \infty) \times \{\eta_*\} \cap \mathcal{S} = \emptyset \quad \text{and} \quad [0, \infty) \times \{\eta_* + \pi\} \cap \mathcal{S} \neq \emptyset \tag{4.3}$$

is satisfied. Let (c, η) be a point of $\mathcal{S}$. If η satisfies $0 < \eta \leq 2\eta_*$, $(c, 2\eta_* - \eta) \in \mathcal{S}$. We then remark that either $\eta < \eta_* < 2\eta_* - \eta$ or $2\eta_* - \eta < \eta_* < \eta$ holds. Since $\mathcal{S}$ is simply connected, there exists a constant $c_0 > 0$ satisfying $(c_0, \eta_*) \in \mathcal{S}$. As seen in (2.1), the range of η for $\mathcal{S}$ is smaller than π. Thus $(c, \eta_* + \pi) \notin \mathcal{S}$ for any $c \geq 0$. This implies (4.2). Also, if η satisfies $2\eta_* < \eta < 2\pi$, then $(c, 2\eta_* - \eta + 2\pi) \in \mathcal{S}$. Repeating the same argument, we obtain (4.3).

We now show the existence of compact traveling wave. If (4.2) is true, Proposition 2.10 guarantees $c_-(\eta_*) < c_+(\eta_*)$. Since $\gamma(\xi+\eta_*) - c\beta(\xi+\eta_*)\cos\xi$ is even in ξ due to the symmetric property for γ and β, we have

$$X(c, \eta_*) = -\int_{-\pi}^{\pi} \frac{\sin\xi}{\gamma(\xi + \eta_*) - c\beta(\xi + \eta_*)\cos\xi} d\xi = 0$$

for any $c \in (c_-(\eta_*), c_+(\eta_*))$. Meanwhile, it follows from Lemmas 2.12 that there is a unique point $c_* \in (c_-(\eta_*), c_+(\eta_*))$ with $Y(c_*, \eta_*) = 0$. Since (c_*, η_*) satisfies $X(c_*, \eta_*) = Y(c_*, \eta_*) = 0$, there is a unique compact traveling wave $(\Gamma_0, c_*\mathbf{e}(\eta_*))$. Similarly, if (4.3) is satisfied, we confirm the existence of compact traveling wave $(\Gamma_0, c_*\mathbf{e}(\eta_* + \pi))$.

As for the symmetry of Γ_0, Proposition 4.1 guarantees that a compact traveling wave is an axisymmetric compact traveling wave with respect to the traveling direction, if it exists. Therefore we complete the proof. □ □

Theorem 4.3 gives us a necessary and sufficient condition of the existence of traveling waves to (1.2), but we do not specify whether the traveling direction of $\mathbf{c}$ is $\mathbf{e}(\eta_*)$ or $\mathbf{e}(\eta_* + \pi)$. Next result provides a condition such that (1.2) has a unique axisymmetric compact traveling wave $(\Gamma_0, c\,\mathbf{e}(\eta_*))$ with respect to the traveling direction.

Theorem 4.4. *Assume that β is symmetric to $\eta_* \in [0, \pi)$. Then* (1.2) *has a unique axisymmetric compact traveling wave $(\Gamma_0, c\,\mathbf{e}(\eta_*))$ with respect to the traveling direction if and only if γ is symmetric to η_* and either the following assertions* (a) *or* (b) *holds :*

(a) *For the case where γ is positive,*

$$\int_{\eta_*}^{\eta_*+\pi} \frac{\cos(\xi - \eta_*)}{\gamma(\xi)} d\xi \leq 0 \tag{4.4}$$

(*the equality holds if and only if $c = 0$*);

(b) *For the case where γ is sign-changing, $c_-(\eta_*) < c_+(\eta_*)$, i.e.,*

$$\sup_{|\xi-\pi|<\pi/2} \frac{\gamma(\xi+\eta)}{\beta(\xi+\eta)\cos\xi} < \inf_{|\xi|<\pi/2} \frac{\gamma(\xi+\eta)}{\beta(\xi+\eta)\cos\xi} \tag{4.5}$$

Proof. Let $(\Gamma_0, c\,\mathbf{e}(\eta_*))$ be a unique axisymmetric compact traveling wave with respect to the traveling direction of (1.2). Proposition 4.1 leads to the symmetry of γ. We remark that $(c, \eta_*) \in \mathcal{Y}$ and $\eta_* \in \Lambda_-$ by Proposition 2.13.

If γ is positive, it follows from Lemma 2.12 and $Y(c, \eta_*) = 0$ that $Y(0, \eta_*) \leq 0$, which means that (4.4) holds. In particular, as $c = 0$, we see $Y(0, \eta_*) = 0$ by Lemma 2.11. Thus (a) holds. Let us assume that γ is sign-changing. Then $\mathcal{S} \neq \emptyset$ by Theorem 4.3. From Proposition 2.10, (2.2) holds for every $\eta \in \mathcal{S}$. Thus (4.5) is obtained, and (b) is true.

Conversely, we assume that γ is symmetric to η_*. By Theorem 4.3, there exists a unique axisymmetric compact traveling wave $(\Gamma_0, \mathbf{c})$ with respect to the traveling direction, where $\mathbf{c}$ is equal to $c\,\mathbf{e}(\eta_*)$ or $c\,\mathbf{e}(\eta_* + \pi)$. If (a) is satisfied, $Y(0, \eta_*) \leq 0$. According to (3.2), $Y(0, \eta_* + \pi) \geq 0$. Considering the monotonicity of Y with respect to c, we see $\mathbf{c} = c\,\mathbf{e}(\eta)$. When (b) is satisfied, then $(c, \eta_*) \in \mathcal{S}$ for any $c \in (c_-(\eta_*), c_+(\eta_*))$. By (2.1), it is easily seen that $(c, \eta_* + \pi) \notin \mathcal{S}$ for any $c \geq 0$, which implies that $\mathbf{c} = c\,\mathbf{e}(\eta)$. This is our assertion. □ □

The following corollary is a direct consequence of Theorem 4.4.

Corollary 4.5. *Assume that $\beta \equiv 1$. Then for any $\eta_* \in [0, 2\pi)$, (1.2) has a unique axisymmetric compact traveling wave $(\Gamma_0, c\,\mathbf{e}(\eta_*))$ with respect to the traveling direction if and only if γ is symmetric to η_* and either* (a) *or* (b) *in Theorem 4.4 holds.*

If β and γ have two symmetric axis η_1 and η_2, then the traveling wave solution is stationary even if there exists.

Corollary 4.6. *Suppose that γ and β are at least symmetric to η_1 and η_2, where $\eta_1, \eta_2 \in [0, \pi)$ and $\eta_1 \neq \eta_2$. If γ is positive, then there exists a unique axisymmetric compact traveling wave $(\Gamma_0, \mathbf{0})$ with respect to two vectors $\mathbf{e}(\eta_1)$ and $\mathbf{e}(\eta_2)$. If γ is sign-changing, there is no traveling wave.*

Proof. Let γ be positive. Theorem 4.4 ensures the existence and uniqueness of compact traveling wave $(\Gamma_0, \mathbf{c})$ of (1.2). Since γ and β are symmetric to

η_1, Proposition 4.1 means that Γ_0 is axisymmetric with respect to $\mathbf{e}(\eta_1)$ and $\mathbf{c}$ satisfies $\mathbf{c} = |\mathbf{c}|\,\mathbf{e}(\eta_1)$ or $\mathbf{c} = |\mathbf{c}|\,\mathbf{e}(\eta_1 + \pi)$. Similarly, Γ_0 is axisymmetric with respect to $\mathbf{e}(\eta_2)$ and $\mathbf{c}$ satisfies $\mathbf{c} = |\mathbf{c}|\,\mathbf{e}(\eta_2)$ or $\mathbf{c} = |\mathbf{c}|\,\mathbf{e}(\eta_2+\pi)$. By the assumption $|\eta_1 - \eta_2| \neq 0, \pi$, it follows that $\mathbf{c} = \mathbf{0}$.

Let us consider the sign-changing case. If there exists a compact traveling wave $(\Gamma_0, \mathbf{c})$ of (1.2), then $\mathbf{c} \neq \mathbf{0}$ as seen in Lemma 3.10. This implies that there is no traveling wave under the sign-changing condition. This completes the proof. □ □

4.2 Translation invariance

We here discuss an invariant property of compact traveling waves. In fact, we can construct easily another compact traveling wave if (1.2) has a compact traveling wave $(\Gamma_0, c\,\mathbf{e}(\eta))$.

Proposition 4.7. *Assume that* $(\Gamma_0^*, c^*\mathbf{e}(\eta^*))$ *is a compact traveling wave of* (1.2) *with* γ*. Then* (1.2) *with* $\widetilde{\gamma}(\theta) := \gamma(\theta) + \nu\beta(\theta)\cos(\theta - \eta^*)$ *for* $\nu \in \mathbb{R}$ *also has a compact traveling wave* $(\Gamma_0, \mathbf{c})$ *where*

$$(\Gamma_0, \mathbf{c}) = \begin{cases} (\Gamma_0^*, (c^* + \nu)\mathbf{e}(\eta^*)) & \text{if } c^* + \nu \geq 0, \\ (\Gamma_0^*, |c^* + \nu|\mathbf{e}(\eta^* + \pi)) & \text{if } c^* + \nu < 0. \end{cases}$$

Proof. By assumption, there is a solution (θ, x, y) satisfies (1.10) with $c = c^*$ and $\eta = \eta^*$. Especially, θ satisfies $\theta_s = \gamma(\theta) - c^*\beta(\theta)\cos(\theta - \eta^*)$. Let ν be a constant satisfying $c^* + \nu \geq 0$. Then we have

$$\widetilde{\gamma}(\theta) - (c^* + \nu)\beta(\theta)\cos(\theta - \eta^*) = \gamma(\theta) - c^*\beta(\theta)\cos(\theta - \eta^*) = \theta_s.$$

This means that $(\Gamma_0^*, (c^* + \nu)\mathbf{e}(\eta^*))$ is a compact traveling wave of (1.2) with $\widetilde{\gamma}$ instead of γ. As for the case $c^* + \nu < 0$, we obtain the same result; however we need to take $(c^* + \nu)\mathbf{e}(\eta) = |c^* + \nu|\mathbf{e}(\eta + \pi)$ in consideration since the scalar c of $c\,\mathbf{e}(\eta)$ must be positive by the definition of compact traveling wave. □ □

On account of the above lemma, we immediately obtain the following corollary.

Corollary 4.8. *Suppose that there exists a stationary compact traveling wave* $(\Gamma_0^*, \mathbf{0})$ *of* (1.2)*. Then* (1.2) *with* $\gamma(\theta)$ *replaced by* $\widetilde{\gamma}(\theta) := \gamma(\theta) + \nu\beta(\theta)\cos(\theta - \eta^*)$ *for any* $\eta^* \in [0, 2\pi)$ *and* $\nu \in \mathbb{R}$ *has a compact traveling wave* $(\Gamma_0^*, \nu\,\mathbf{e}(\eta^*))$.

4.3 The inverse problem

Here we consider the inverse problem. We first fix any strictly convex Jordan curve Γ_0^* and any velocity $\mathbf{c}^*$. Also let β be any positive function of θ. Consider the following question:

Q. *Is there a function* γ *such that* (1.2) *possesses the compact traveling wave* $(\Gamma_0^*, \mathbf{c}^*)$ *?*

The answer is positive as follows.

Proposition 4.9. *Let* Γ_0^* *and* $\mathbf{c}^*$ *be an arbitrary strictly convex Jordan curve and vector, respectively. Then there exists* γ *such that* $(\Gamma_0^*, \mathbf{c})$ *is a compact traveling wave of* (1.2).

Proof. Since Γ_0^* is strictly convex, the curvature θ_s of Γ_0^* is positive,. Recall that θ stands for the angle on Γ_0^*. Thus θ is invertible, namely, there exists a function θ^{-1} satisfying $s = \theta^{-1}(\theta(s))$. We define γ by

$$\gamma(\theta) := \tau(\theta^{-1}(\theta)) + c^*\beta(\theta)\cos(\theta - \eta^*), \tag{4.6}$$

where $\tau(s) := \theta_s(s)$ and (c^*, η^*) satisfies $\mathbf{c}^* = c^*\,\mathbf{e}(\eta^*)$. Then γ satisfies (1.10) with given $\theta(s)$ and (c^*, η^*). Since the matching condition in (1.10) are determined by only $\theta(s)$, $(\Gamma_0^*, \mathbf{c}^*)$ is a compact traveling wave of (1.2) with γ defined by (4.6). □ □

Note that there exists a driving force γ such that (1.2) has no compact traveling wave as discussed in the first part of Subsection 3.2.

4.4 The Wulff shape

In this subsection, we consider the application of our theorems to a typical example (1.5). As introduced in Section 1, the anisotropic interface equation (1.5) is obtained by considering the interfacial energy and the difference between bulk energies. In (1.2), setting

$$\beta(\theta) = b(\theta)\left(f(\theta) + f''(\theta)\right)^{-1}, \quad \gamma(\theta) = A\left(f(\theta) + f''(\theta)\right)^{-1},$$

we see that (1.2) corresponds to (1.5). Recall that A is a positive constant. Let us consider that the interfacial energy $f(\theta)$ is *strictly stable* (see [21]), that is,

$$f(\theta) + f''(\theta) > 0.$$

Since γ is positive, we can apply Theorem 3.1 to (1.5). Thus there exists a unique compact traveling wave $(\Gamma_0, \mathbf{c})$. Using the integration by parts, we easily check that

$$\int_0^{2\pi} \gamma(\theta)^{-1} e^{i\theta} d\theta = A^{-1} \int_0^{2\pi} (f(\theta) + f''(\theta)) e^{i\theta} d\theta = 0,$$

which implies that $X(0, \eta) = Y(0, \eta) = 0$ for any $\eta \in [0, 2\pi)$. Therefore it follows from Lemma 2.11 that $\mathbf{c} = \mathbf{0}$. Due to the uniqueness of compact traveling waves, this stationary solution of (1.5) is a unique compact traveling wave with $\mathbf{c} = \mathbf{0}$. We can confirm the same result in [21].

As $A = 1$, in particular, the closed domain whose boundary is the stationary solution Γ_0 is often called *the Wulff shape* (also called *the Wulff region* or *the Wulff crystal*). Thus we know that the convex traveling wave of (1.5) (namely, stationary solution) is represented by the dilation of the Wulff shape.

4.5 Non-convex compact traveling waves

As mention in Remark 2.4, we here give an example of non-convex traveling waves. In main results of this paper, the driving force γ is a Lipschitz function depending only on θ. Under this assumption, every traveling wave of (1.2) is strictly convex by Theorem 2.2. In order to make a non-convex traveling wave of (1.2), we need to violate this restriction for γ.

Here we present a numerical result of non-convex traveling waves of (1.2) with the driving force γ which depends not only on θ but also y. To this end, we first make a non-convex compact traveling wave $(\Gamma_0, \mathbf{c})$ with $\mathbf{c} = \mathbf{0}$. Let γ be a function of y with $\gamma(y) = \gamma(-y)$ and (θ, x, y) be a solution of

$$\begin{cases} \theta_s = \gamma(y) & \text{in } (0, \ell), \\ x_s = -\sin\theta & \text{in } (0, \ell), \\ y_s = \cos\theta & \text{in } (0, \ell), \\ \theta(0) = 0, \ \theta(\ell) = \pi/2, & \\ x(0) = y(0) = 0, & \end{cases} \tag{4.7}$$

for a constant $\ell > 0$. By the symmetry of γ, we can extend the solution

(x, y, θ) to the interval $[0, 4\ell]$ by

$$x_s = -\sin\Theta,\ \ y_s = \cos\Theta,\ \ \Theta := \begin{cases} \theta(s) & \text{in } [0, \ell) \\ \pi - \theta(s-\ell) & \text{in } [\ell, 2\ell) \\ \pi + \theta(s-2\ell) & \text{in } [2\ell, 3\ell) \\ 2\pi - \theta(s-3\ell) & \text{in } [3\ell, 4\ell) \end{cases},$$

and we obtain a Jordan curve $\Gamma_0 = \{(x(s), y(s)) \mid s \in [0, L),\ L = 4\ell\}$. In order to get a non-convex traveling wave Γ_0, $\gamma(y(s))$ must be at least negative at some $s \in (0, \ell)$. Let y_* be defined by

$$y_* = \int_0^\ell \cos\theta\, ds.$$

We suppose for instance that γ has two points y_1, y_2 such that $0 < y_1 < y_2 < y_*$, $\gamma(y_1) = \gamma(y_2) = 0$ and

$$\gamma = \begin{cases} > 0 & \text{in } (0, y_1) \cup (y_2, y_*), \\ < 0 & \text{in } (y_1, y_2). \end{cases} \tag{4.8}$$

By (4.7), we have the following relationship:

$$\int_0^{y(s)} \gamma(y)\, dy = \int_0^{\theta(s)} \cos\theta\, d\theta.$$

This equality gives

$$\gamma(0) \geq 0, \quad \int_0^{y_1} \gamma(y) dy < 1 \left(= \int_0^{y_*} \gamma(y) dy \right), \tag{4.9}$$

which is due to the fact that $\cos\theta$ is non-negative in $[0, \pi/2]$. Then we can give an example

$$\gamma(y) := g_0(y^2 - 1)(y^2 - 4),$$

with some positive constant g_0, which satisfies the above condition (4.8) with $y_1 = 1$ and $y_2 = 2$. We take g_0 so small that the second condition of (4.9) holds. Since

$$\int_0^2 \gamma(y) dy = \frac{16 g_0}{15} > 0,$$

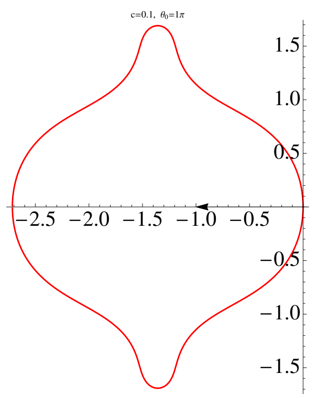


Figure 4.1: The non-convex compact traveling wave given by (4.10) with $g_0 = 1/3$.

we have

$$\int_0^y \gamma(y)dy > 0, \qquad \text{for } y > 0.$$

Moreover, it is easily seen that there is a positive constant y_3 such that

$$\int_0^{y_3} \gamma(y)dy = 1.$$

From this, Γ_0 is a non-convex traveling wave with $\mathbf{c} = \mathbf{0}$. Repeating the same argument as in subsection 4.2, we also see the existence of a non-convex compact traveling wave with the velocity $\mathbf{c} = c\,\mathbf{e}(0)(\neq \mathbf{0})$ when γ is replaced by $\gamma(y) + c\cos\theta$ and $\beta \equiv 1$. Let us define γ and β by

$$\gamma(\theta, y) := g_0(y^2 - 1)(y^2 - 4) + \frac{1}{2}\cos\theta, \quad \beta(\theta) := 1. \tag{4.10}$$

Then, as seen in Figure 4.1, there numerically exists a compact traveling wave $(\Gamma_0, \mathbf{c})$ such that Γ_0 is non-convex and $\mathbf{c} = \mathbf{e}(0)/2$.

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