# A Reaction-Diffusion Approximation to a Cross-Diffusion System

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**Abstract**

In this paper it is discussed whether reaction and linear diffusion bring about a effect of nonlinear diffusion or not. It is proved that a cross–diffusion system for two competitive species is realized in a singular limit of a reaction–diffusion system with a small parameter under some assumptions.

## 1 Introduction

In this paper the following type of parabolic equations is called a *reaction–diffusion system*:

$$\boldsymbol{u}_t = \boldsymbol{D}\triangle\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}), \tag{1.1}$$

where

$$\boldsymbol{u} = \boldsymbol{u}(x,t) = {}^t(u_1(x,t),\cdots,u_M(x,t)), \quad \boldsymbol{f}(\boldsymbol{u}) = {}^t(f_1(\boldsymbol{u}),\cdots,f_M(\boldsymbol{u})),$$

$\boldsymbol{D}$ is a diagonal matrix whose elements are positive (or non-negative). In other words, a reaction–diffusion system consists of two parts: one is a kinetic term $\boldsymbol{f}(\boldsymbol{u})$; the other is a diffusion one $\boldsymbol{D}\triangle\boldsymbol{u}$. Many manuscripts reveals various dynamics of reaction–diffusion systems. Thus we meet the questions: "What sort of behavior can be exhibited by solutions to the reaction–diffusion system ?", or "How rich are the dynamics of the reaction–diffusion system ?" One of the ways to answer these questions is to "realize" the dynamics of parabolic systems which do not belong to reaction–diffusion systems in the reaction–diffusion systems. As remarkable researches from a related viewpont we refer to [15, 3], where Poláčik has proved that any finite dimensional vector field can be realized in the equation

$$u_t = \triangle u + f(u, \nabla u)$$

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in a bounded domain in $\mathbf{R}^N$ if $f$ is chosen appropriately. From the morphological point of view, Mimura et al. showed in [13] that reaction–diffusion systems can "realize" density–dependent diffusion models. They considered the colonies of some species of bacteria which exhibit the various spatial patterns. Though several density–dependent diffusion models for such spatial patterns had already been proposed (e.g., [6]), they obtained the similar spatial patterns even from a reaction–diffusion system by introducing "inactive state" of a bacterium explicitly. Their concept of modelling leads us to the concept of "reaction–diffusion approximation".

Our aim is to find a reaction–diffusion system which approximates a cross–diffusion system

$$\begin{cases} u^*_t = \triangle\alpha(u^*) + f(u^*, v^*), & x \in \Omega,\ t > 0, \\ v^*_t = \triangle\beta(u^*, v^*) + g(u^*, v^*), & x \in \Omega,\ t > 0 \end{cases} \tag{1.2}$$

under the Neumann boundary condition

$$\frac{\partial u^*}{\partial n} = 0,\ \frac{\partial v^*}{\partial n} = 0, \qquad x \in \partial\Omega,\ t > 0 \tag{1.3}$$

and the initial condition

$$u^*(x, 0) = u_0(x),\ v^*(x, 0) = v_0(x), \quad x \in \Omega. \tag{1.4}$$

For a typical example,

$$\begin{aligned} &\alpha(u) = (\alpha_0 + \alpha_1 u)u, \quad \beta(u, v) = (\beta_0 + \beta_1 u + \beta_2 v)v, \\ &f(u, v) = (f_0 - f_1 u - f_2 v)u, \quad g(u, v) = (g_0 - g_1 u - g_2 v)v, \end{aligned}$$

where $\alpha_0, \beta_0, f_j, g_j$ are positive constants and $\alpha_1, \beta_1, \beta_2$ are nonnegative ones. This example is one of the ecological models which Shigesada et al. [16] proposed in order to introduce the population pressure by interference between individuals into a Lotka–Volterra competition system. In this case $u^*$ and $v^*$ stand for population densities for two competing species. The species for $v^*$ has a tendency to move towards the lower distribution $u^*$ (also see [14]). Namely this system includes the "negative chemotactic effect". This effect induces the complex dynamics including the Hopf bifurcations and the segregation of a convex habitat between two similar species (see [5, 8, 10, 11, 12]). It is well-known in [7] that if $\Omega$ is convex there are no stable inhomogeneous equilibria in the competition–diffusion system, i.e., $\alpha_1 = \beta_1 = \beta_2 = 0$. It is shown in [10, 11] that the stable spatial segregation takes place if $\beta_1 > 0$, which is called *cross–diffusion induced instability.* In this paper we will show that the cross–diffusion system (1.2) is actually a singular limit of a reaction–diffusion system with a small parameter. Though reaction–diffusion systems do not seem to bring about the negative chemotactic effect, this fact might imply that reaction–diffusion systems include such a effect. This method also tells us the relationship between Turing's instability and the cross–diffusion induced instability, which is shown in [4].

Hereafter we assume that $\Omega$ is a bounded domain in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$, and $\alpha, \beta, f, g$ are smooth functions satisfying

$$\alpha \in C^4(\mathbf{R}),\ \beta \in C^4(\mathbf{R}^2),\ f \in C^2(\mathbf{R}^2),\ g \in C^2(\mathbf{R}^2), \tag{1.5}$$

$$\inf_{u>0} \alpha'(u) > 0, \qquad \inf_{u>0,v>0} \beta_v(u,v) > 0, \tag{1.6}$$

and

$$\begin{gathered} u_0 \in C^4(\overline{\Omega}), \quad v_0 \in C^4(\overline{\Omega}), \\ u_0(x) \geq 0, \quad v_0(x) \geq 0 \qquad \text{in } \overline{\Omega}. \end{gathered}$$

We can take constants $d_1, d_2, d_3$ and $d_4$ satisfying

$$\begin{cases} 0 < d_1 < \inf\limits_{u>0} \alpha'(u), & 0 < d_2 < \inf\limits_{u>0,v>0} \beta_v(u,v), \\ d_3 > 0, \quad d_3 \neq d_1, & d_4 > 0, \quad d_4 \neq d_2. \end{cases}$$

Set

$$a(u) := \alpha(u) - d_1 u, \quad b(u,v) := \beta(u,v) - d_2 v.$$

For a small positive parameter $\epsilon$, we consider an auxiliary semilinear parabolic system with fast reactions in $w$ and $z$:

$$\begin{cases} u_t = d_1 \triangle u + \triangle w + f(u,v), & x \in \Omega,\ t > 0, \\ v_t = d_2 \triangle v + \triangle z + g(u,v), & x \in \Omega,\ t > 0, \\ w_t = d_3 \triangle w + \dfrac{1}{\epsilon}(a(u) - w), & x \in \Omega,\ t > 0, \\ z_t = d_4 \triangle z + \dfrac{1}{\epsilon}(b(u,v) - z), & x \in \Omega,\ t > 0 \end{cases} \tag{1.7}$$

under the boundary condition

$$\frac{\partial u}{\partial n} = 0,\ \frac{\partial v}{\partial n} = 0,\ \frac{\partial w}{\partial n} = 0,\ \frac{\partial z}{\partial n} = 0, \qquad x \in \partial\Omega,\ t > 0 \tag{1.8}$$

and the initial condition

$$\begin{cases} u(x,0) = u_0(x),\ v(x,0) = v_0(x), \\ w(x,0) = a(u_0(x)),\ z(x,0) = b(u_0(x), v_0(x)), \quad x \in \Omega. \end{cases} \tag{1.9}$$

Since we can rewrite (1.2) as

$$\begin{cases} u^*_t = d_1 \triangle u^* + \triangle a(u^*) + f(u^*, v^*), & x \in \Omega,\ t > 0, \\ v^*_t = d_2 \triangle v^* + \triangle b(u^*, v^*) + g(u^*, v^*), & x \in \Omega,\ t > 0, \end{cases} \tag{1.10}$$

we may expect that $(w,z)$ approximates to $(a(u),b(u,v))$ in (1.7) and that $(u,v)$ converges to the solution of (1.10) as $\epsilon$ tends to 0. Actually we will show later that the dynamics of (1.7) under (1.8) and (1.9) is close to that of (1.10) under (1.3) and (1.4) as $\epsilon \to +0$, if they are restricted to any bounded region. Notice that the system (1.7) is almost a reaction–diffusion system. Indeed, applying the linear transformation

$$\tilde{u} = u - \frac{w}{d_3 - d_1}, \quad \tilde{v} = v - \frac{z}{d_4 - d_2}, \quad \tilde{w} = w, \quad \tilde{z} = z,$$

i.e.,

$$u = \tilde{u} + \frac{\tilde{w}}{d_3 - d_1}, \quad v = \tilde{v} + \frac{\tilde{z}}{d_4 - d_2}, \quad w = \tilde{w}, \quad z = \tilde{z},$$

we obtain the following reaction–diffusion system

$$\begin{cases} \tilde{u}_t = d_1 \triangle \tilde{u} + f(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}, \tilde{v} + \dfrac{\tilde{z}}{d_4 - d_2}) \\ \qquad - \dfrac{1}{(d_3 - d_1)\epsilon}\Big(a(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}) - \tilde{w}\Big), & x \in \Omega,\ t > 0, \\ \tilde{v}_t = d_2 \triangle \tilde{v} + g(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}, \tilde{v} + \dfrac{\tilde{z}}{d_4 - d_2}) \\ \qquad - \dfrac{1}{(d_4 - d_2)\epsilon}\Big(b(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}, \tilde{v} + \dfrac{\tilde{z}}{d_4 - d_2}) - \tilde{z}\Big), & x \in \Omega,\ t > 0, \\ \tilde{w}_t = d_3 \triangle \tilde{w} + \dfrac{1}{\epsilon}\Big(a(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}) - \tilde{w}\Big), & x \in \Omega,\ t > 0, \\ \tilde{z}_t = d_4 \triangle \tilde{z} + \dfrac{1}{\epsilon}\Big(b(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}, \tilde{v} + \dfrac{\tilde{z}}{d_4 - d_2}) - \tilde{z}\Big), & x \in \Omega,\ t > 0. \end{cases} \tag{1.11}$$

It is not clear whether $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{z}$ in (1.11) can stand for some biological quantities. However, after we accomplished the present paper, we found another reaction–diffusion approximation to (1.2) under additional assumptions on $\alpha$ and $\beta$ (see [4]). In the later approximation the solutions of a reaction–diffusion system like (1.11) can stand for the population densities of some parts of the competing species which are described by the model of Shigesada et al. in [16]; besides, our later approximation gives us a better understanding of cross–diffusion in biological models.

We remark that the existence of local solutions of (1.7) follows from that of (1.11).

Theorem 1.1. *Assume* (1.5) *and* (1.6). *Fix positive numbers* $d_1, d_2, d_3, d_4$ *and functions* $a(r), b(r,s)$ *as above. For positive constants* $R_1$ *and* $R_2$, *there exist functions* $\tilde{a}(r), \tilde{b}(r,s), \tilde{f}(r,s)$ *and* $\tilde{g}(r,s)$ *such that*

$$\tilde{a}(r) = a(r), \quad \tilde{b}(r,s) = b(r,s), \quad \tilde{f}(r,s) = f(r,s), \quad \tilde{g}(r,s) = g(r,s) \\ \text{for any } (r,s) \in [0,R_1] \times [0,R_2] \tag{1.12}$$

*and that the solution* $(u, v, w, z) = (u(x,t), v(x,t), w(x,t),\ z(x,t))$ *of* (1.7)–(1.9) *with* $a,\ b,\ f,\ g$ *replaced by* $\tilde{a},\ \tilde{b},\ \tilde{f},\ \tilde{g}$ *respectively exists globally in time.*

*If the solution* $(u^*, v^*) = (u^*(x,t), v^*(x,t))$ *of* (1.2)–(1.4) *belongs to* $C^4(\overline{\Omega} \times [0,T]) \times C^4(\overline{\Omega} \times [0,T])$ *and*

$$0 \le u^*(x,t) \le R_1, \quad 0 \le v^*(x,t) \le R_2 \qquad in\ \overline{\Omega} \times [0,T]$$

*for some positive constant* $T$*, then the following inequalities hold*

$$\begin{cases} \|u - u^*\|_{C^0([0,T];L^2(\Omega))} & \le \ c_1\epsilon, \\ \|v - v^*\|_{C^0([0,T];L^2(\Omega))} & \le \ c_1\epsilon, \\ \|w - a(u^*)\|_{C^0([0,T];L^2(\Omega))} & \le \ c_1\epsilon, \\ \|z - b(u^*, v^*)\|_{C^0([0,T];L^2(\Omega))} & \le \ c_1\epsilon \end{cases} \tag{1.13}$$

*for* $\epsilon > 0$ *where* $c_1$ *is a positive constant independent of* $\epsilon$ *and* $(u, v, w, z)$*. Moreover, if* $N \le 4$*, then the following inequalities also hold:*

$$\begin{cases} \|\nabla(u - u^*)\|_{C^0([0,T];L^2(\Omega))} & \le \ c_2\epsilon^{3/4}, \\ \|\nabla(v - v^*)\|_{C^0([0,T];L^2(\Omega))} & \le \ c_2\epsilon^{3/4}, \\ \|\nabla(w - a(u^*))\|_{C^0([0,T];L^2(\Omega))} & \le \ c_2\epsilon^{3/4}, \\ \|\nabla(z - b(u^*, v^*))\|_{C^0([0,T];L^2(\Omega))} & \le \ c_2\epsilon^{3/4}, \\ \|\triangle(u - u^*)\|_{C^0([0,T];L^2(\Omega))} & \le \ c_3\epsilon^{1/4}, \\ \|\triangle(v - v^*)\|_{C^0([0,T];L^2(\Omega))} & \le \ c_3\epsilon^{1/4}, \\ \|\triangle(w - a(u^*))\|_{C^0([0,T];L^2(\Omega))} & \le \ c_3\epsilon^{1/4}, \\ \|\triangle(z - b(u^*, v^*))\|_{C^0([0,T];L^2(\Omega))} & \le \ c_3\epsilon^{1/4} \end{cases} \tag{1.14}$$

*for* $0 < \epsilon \le \epsilon_0$ *where* $c_2$*,* $c_3$ *and* $\epsilon_0$ *are positive constants independent of* $\epsilon$ *and* $(u, v, w, z)$*.*

As long as $(u(x,t), v(x,t))$ belongs to the region $[0, R_1] \times [0, R_2]$, $(u, v, w, z)$ in Theorem 1.1 is the solution of (1.7)–(1.9) without the replacement of $a$, $b$, $f$ and $g$. Thus this theorem implies that solutions to the cross–diffusion system (1.2) can be approximated by the linear combinations of solutions to the reaction–diffusion system (1.11) in any bounded region in the phase space.

The proof of this theorem will be given in §2. We will construct $\tilde{a}$, $\tilde{b}$, $\tilde{f}$ and $\tilde{g}$ by suitably truncating $a$, $b$, $f$ and $g$ respectively around the bounded region $[0, R_1] \times [0, R_2]$. The constants $c_1$, $c_2$ and $c_3$ depend on $R_1, R_2, u^*, v^*$, and thus on $T$.

See [1] for the existence, uniqueness, and regularity of a local solution of (1.2)-(1.4), where $(u_0, v_0) \in W^1_p(\Omega)^2$ for $p > N$. In particular, if $\alpha, \beta, f$ and $g$ are sufficiently smooth, then the local solution instantly becomes sufficiently smooth up to the boundary. See also [2] and the references therein. Thus the assumptions for $u^*$ and $v^*$ are not so restricted. As for its global existence, see, e.g., [9, 17, 18].

## 2 Modification of equations

To prove Theorem 1.1, we will construct the functions $\tilde{a}$, $\tilde{b}$, $\tilde{f}$, and $\tilde{g}$ in this section. First we introduce the following stronger assumption for $a$, $b$, $f$ and $g$ than (1.5) and (1.6):

**(A)** There exist positive constants $k_i$ $(i = 1, 2, 3, 4)$ satisfying

$$\begin{cases} a'(r) \geq k_1, \\ b_v(r,s) \geq k_2, \\ \displaystyle\sum_{j=1}^{3} \left| \frac{d^j}{du^j} a(r) \right| + \sum_{1 \leq j+l \leq 3,\ j,l \geq 0} \left| \frac{\partial^{j+l}}{\partial u^j \partial v^l} b(r,s) \right| \\ \qquad + \left| \displaystyle\int_0^s b_{uu}(r,\sigma) d\sigma \right| + \left| \displaystyle\int_0^s b_{uuu}(r,\sigma) d\sigma \right| \leq k_3, \\ |f_u(r,s)| + |f_v(r,s)| + |f_{uu}(r,s)| + |f_{uv}(r,s)| + |f_{vv}(r,s)| \leq k_4, \\ |g_u(r,s)| + |g_v(r,s)| + |g_{uu}(r,s)| + |g_{uv}(r,s)| + |g_{vv}(r,s)| \leq k_5 \end{cases} \tag{2.1}$$

for any $r, s \in \mathbf{R}$.

THEOREM 2.1. *Let* $(u^*, v^*) = (u^*(x,t), v^*(x,t))$ *(resp.* $(u, v, w, z) = (u(x,t), v(x,t), w(x,t), z(x,t))$*) be the solution of* (1.2) – (1.4) (resp. (1.7) – (1.9)) *in* $t \in [0, T]$. *Assume* **(A)** *and*

$$\begin{aligned} &\|u_t^*\|_{L^\infty(\Omega)} + \|v_t^*\|_{L^\infty(\Omega)} + \|\nabla u^*\|_{L^\infty(\Omega)} + \|\nabla v^*\|_{L^\infty(\Omega)} \\ &\quad + \|d_3 \triangle a(u^*) - a'(u^*) u_t^*\|_{H^1(\Omega)} \\ &\quad + \|d_4 \triangle b(u^*, v^*) - b_u(u^*, v^*) u_t^* - b_v(u^*, v^*) v_t^*\|_{H^1(\Omega)} \leq M_1 \end{aligned} \tag{2.2}$$

*for* $0 \leq t \leq T$. *Then* (1.13) *holds.*

The proof will be given in the next section.

For positive numbers $\delta$ and $R$ we can easily choose a $C^\infty$-function $\chi(x; \delta, R)$ as follows:

$$\chi(x; \delta, R) = \begin{cases} 1 & \text{for } x \in [0, R], \\ 0 & \text{for } x \in (-\infty, -2\delta] \cup\ [R + 2\delta, \infty), \end{cases}$$

and

$$0 \leq \chi(x; \delta, R) \leq 1, \qquad \sup_{-\infty < x < \infty} |\chi'(x; \delta, R)| \leq \frac{1}{\delta}.$$

LEMMA 2.2. *Assume* (1.5) *and* (1.6). *Let* $R_1$ *and* $R_2$ *be positive numbers, and set*

$$m_1 := \min_{u \in [0, R_1]} a'(u), \qquad m_2 := \min_{(u,v) \in [0,R_1] \times [0,R_2]} b_v(u, v).$$

*If $\delta_1$ and $\delta_2$ are positive but so small, then* **(A)** *holds true for the following functions $\tilde{a}$, $\tilde{b}$, $\tilde{f}$, $\tilde{g}$ and some positive constants $k_1, \cdots, k_5$:*

$$\begin{aligned}
\tilde{a}(u) &:= m_1 u + a(0) + \int_0^u \chi_1(s)\,(a'(s) - m_1)\,ds, \\
\tilde{b}(u,v) &:= m_2 v + \chi_2(v)\left(\chi_1(u) b(u,0) + \int_0^v \chi_3(u,s)\,(b_v(u,s) - m_2)\,ds\right), \\
\tilde{f}(u,v) &:= \chi_3(u,v) f(u,v), \\
\tilde{g}(u,v) &:= \chi_3(u,v) g(u,v),
\end{aligned}$$

*where*

$$\begin{aligned}
\chi_1(u) &:= \chi(u; \delta_1, R_1), \\
\chi_2(v) &:= \chi(v; \frac{1}{\delta_2}, R_2), \\
\chi_3(u,v) &:= \chi(u; \delta_1, R_1)\chi(v; \delta_1, R_2).
\end{aligned}$$

We can easily check (1.12). Since the support of $\int_0^v \chi_3(u,s)\,(b_v(u,s) - m_2)\,ds$ is not compact, we cannot obtain the boundedness of $\int_0^v \int_0^\sigma \chi_3(u,s)\,(b_v(u,s) - m_2)\,dsd\sigma$ and its derivatives. Therefore it is necessary to multiply $\int_0^v \chi_3(u,s)\,(b_v(u,s) - m_2)\,ds$ by $\chi_2(v)$ in the definition of $\tilde{b}$.

*Proof.* We show (2.1) only for $\tilde{b}$. If $\delta_1$ is so small, then

$$\chi_3(u,v)\,(b_v(u,v) - m_2) \geq -\frac{m_2}{4} \quad \text{for } (u,v) \in \mathbf{R}^2.$$

We can choose $\delta_2$ so small that

$$\chi_2'(v)\left(\chi_1(u) b(u,0) + \int_0^v \chi_3(u,s)\,(b_v(u,s) - m_2)\,ds\right) \geq -\frac{m_2}{4} \quad \text{for } (u,v) \in \mathbf{R}^2.$$

Differentiating $\tilde{b}$ in $v$, we have

$$\begin{aligned}
\tilde{b}_v(u,v) &= m_2 + \chi_2'(v)\left(\chi_1(u) b(u,0) + \int_0^v \chi_3(u,s)\,(b_v(u,s) - m_2)\,ds\right) \\
&\quad + \chi_2(v)\chi_3(u,v)\,(b_v(u,v) - m_2) \\
&\geq m_2 - \frac{m_2}{4} - \frac{m_2}{4} \\
&= \frac{m_2}{2}.
\end{aligned}$$

The other conditions of (2.1) can be checked. □

*Proof of Theorem 1.1.* The inequalities (1.13) are a direct consequence of Lemma 2.2 and Theorem 2.1. Notice that the global existence of $(u, v, w, z)$ is guaranteed by the fact: the grow-up rates of the nonlinear terms in (1.11) are less than or equal to some affine functions of $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})$ after the replacement of $(a, b, f, g)$ with $(\tilde{a}, \tilde{b}, \tilde{f}, \tilde{g})$. Since the fourth derivatives of $\tilde{a}$ and $\tilde{b}$ in Lemma 2.2 are bounded, the latter part of Theorem 1.1 is deduced from Lemma 2.2 and the following theorem which will be proved in the next section. $\square$

THEOREM 2.3. *Assume* **(A)**, (2.2), $N \leq 4$ *and*

$$\begin{aligned}
\|\triangle u^*\|_{L^\infty(\Omega)} + \|\triangle v^*\|_{L^\infty(\Omega)} + \left\|\frac{\partial}{\partial n}\triangle a(u^*)\right\|_{L^2(\partial\Omega)} + \left\|\frac{\partial}{\partial n}\triangle b(u^*, v^*)\right\|_{L^2(\partial\Omega)} \\
+ \|d_3\triangle a(u^*) - a'(u^*)u_t^*\|_{H^2(\Omega)} \\
+ \|d_4\triangle b(u^*, v^*) - b_u(u^*, v^*)u_t^* - b_v(u^*, v^*)v_t^*\|_{H^2(\Omega)} \leq M_2
\end{aligned} \tag{2.3}$$

*for* $0 \leq t \leq T$. *Then* (1.14) *holds.*

# 3 Proof

Let $\|\cdot\|$ be a $L^2$-norm and $(\cdot, \cdot)$ an inner product in $L^2(\Omega)$. Let $(u^*, v^*) = (u^*(x,t), v^*(x,t))$ and $(u, v, w, z) = (u(x,t), v(x,t), w(x,t), w(x,t))$ be as in Theorem 2.1. Hereafter for the simplicity of notation, the positive constants independent of $\epsilon$ and $(u, v, w, z)$ (namely, depending only on $d_1, \cdots, d_4, k_1, \cdots, k_5, M_1, M_2, T, \Omega, N$, and $\epsilon_0$) is denoted by $c_i$ $(i = 1, 2, \cdots)$.

Set

$$\begin{aligned}
&U := u - u^*, \quad V := v - v^*, \quad W := w - a(u^*), \quad Z := z - b(u^*, v^*), \\
&w^* := a(u^*), \quad z^* := b(u^*, v^*),
\end{aligned}$$

which satisfy

$$\begin{cases}
U_t = d_1\triangle U + \triangle W + f(u^* + U, v^* + V) - f(u^*, v^*), \\
V_t = d_2\triangle V + \triangle Z + g(u^* + U, v^* + V) - g(u^*, v^*), \\
W_t = d_3\triangle W + \dfrac{1}{\epsilon}(a(u^* + U) - a(u^*) - W) + d_3\triangle w^* - w_t^*, \\
Z_t = d_4\triangle Z + \dfrac{1}{\epsilon}(b(u^* + U, v^* + V) - b(u^*, v^*) - Z) + d_4\triangle z^* - z_t^*.
\end{cases} \tag{3.1}$$

Define

$$\begin{aligned}
A(u) &:= \int_0^u a(s)ds, \\
B(u,v) &:= \int_0^v b(u,s)ds, \\
E_1(t) &:= \int_\Omega (A(u^*+U) - A(u^*) - A'(u^*)U)\, dx, \\
E_2(t) &:= \int_\Omega (B(u^*+U, v^*+V) - B(u^*,v^*) - B_u(u^*,v^*)U - B_v(u^*,v^*)V)\, dx.
\end{aligned}$$

*Proof of Theorem 2.1.* Differentiating $E_1$ in $t$, we have

$$\begin{aligned}
\frac{dE_1}{dt} &= \int_\Omega \{A'(u^*+U)(u_t^*+U_t) - A'(u^*)u_t^* - A''(u^*)u_t^*U - A'(u^*)U_t\}\, dx, \\
&= \int_\Omega \Big\{(a(u^*+U) - a(u^*))U_t + \Big(a(u^*+U) - a(u^*) - a'(u^*)U\Big)u_t^*\Big\}\, dx, \\
&= \int_\Omega \Bigg\{\Big(a(u^*+U) - a(u^*))(d_1\triangle U + \triangle W + f(u^*+U, v^*+V) - f(u^*,v^*)\Big) \\
&\quad + \int_0^1\int_0^1 a''(u^*+\theta_1\theta_2 U)\theta_1 U^2 u_t^* d\theta_1 d\theta_2\Bigg\} dx. \qquad (3.2)
\end{aligned}$$

The first term of the right hand side of (3.2) is estimated as follows:

$$\begin{aligned}
&\int_\Omega (a(u^*+U) - a(u^*))(d_1\triangle U + \triangle W + f(u^*+U, v^*+V) - f(u^*,v^*))dx \\
&\le -\int_\Omega \Big(a'(u^*+U)(\nabla u^* + \nabla U) - a'(u^*)\nabla u^*\Big)\cdot d_1\nabla U dx \\
&\quad + \int_\Omega (a(u^*+U) - a(u^*))\triangle W dx + k_3k_4\|U\|(\|U\| + \|V\|), \\
&\le \int_\Omega (a(u^*+U) - a(u^*))\triangle W dx - \int_\Omega d_1 a'(u^*+U)|\nabla U|^2 dx \\
&\quad + d_1k_3M_1\|U\|\ \|\nabla U\| + k_3k_4\|U\|(\|U\| + \|V\|) \\
&\le (a(u^*+U) - a(u^*), \triangle W) - \frac{d_1k_1}{2}\|\nabla U\|^2 + c_4(\|U\|^2 + \|V\|^2), \qquad (3.3)
\end{aligned}$$

where $c_4 := 3k_3k_4/2 + d_1k_3^2M_1^2/(2k_1)$. Substituting (3.3) into (3.2), we get

$$\frac{dE_1}{dt} \le (a(u^*+U) - a(u^*), \triangle W) - \frac{d_1k_1}{2}\|\nabla U\|^2 + c_5(\|U\|^2 + \|V\|^2), \qquad (3.4)$$

where $c_5 := c_4 + k_3M_1/2$. Taking an inner product between the third equation of (3.1) and $-\triangle W$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla W\|^2 &= -d_3\|\triangle W\|^2 - \frac{1}{\epsilon}(a(u^*+U)-a(u^*),\triangle W)\\ &\quad +(\nabla(d_3\triangle w^* - w_t^*),\nabla W) - \frac{1}{\epsilon}\|\nabla W\|^2\\ &\le -d_3\|\triangle W\|^2 - \frac{1}{\epsilon}(a(u^*+U)-a(u^*),\triangle W)\\ &\quad -\frac{1}{2\epsilon}\|\nabla W\|^2 + c_6\epsilon, \qquad (3.5)\end{aligned}$$

where $c_6 := M_1^2/2$. The above two inequalities (3.4) and (3.5) immediately imply

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|\nabla W\|^2 + \frac{1}{\epsilon}E_1\right) &\le -d_3\|\triangle W\|^2 - \frac{1}{2\epsilon}\|\nabla W\|^2 - \frac{d_1k_1}{2\epsilon}\|\nabla U\|^2\\ &\quad +\frac{c_5}{\epsilon}(\|U\|^2+\|V\|^2) + c_6\epsilon. \qquad (3.6)\end{aligned}$$

Next we consider the derivative of $E_2$:

$$\begin{aligned}\frac{dE_2}{dt} &= \int_\Omega \Big\{B_u(u^*+U,v^*+V)(u_t^*+U_t) + B_v(u^*+U,v^*+V)(v_t^*+V_t)\\ &\quad -B_u(u^*,v^*)u_t^* - B_v(u^*,v^*)v_t^* - B_{uu}(u^*,v^*)u_t^*U - B_{uv}(u^*,v^*)v_t^*U\\ &\quad -B_u(u^*,v^*)U_t - B_{uv}(u^*,v^*)u_t^*V - B_{vv}(u^*,v^*)v_t^*V - B_v(u^*,v^*)V_t\Big\}dx\\ &= \int_\Omega \Bigg\{\Big(B_u(u^*+U,v^*+V) - B_u(u^*,v^*) - B_{uu}(u^*,v^*)U - B_{uv}(u^*,v^*)V\Big)u_t^*\\ &\quad +\Big(B_v(u^*+U,v^*+V) - B_v(u^*,v^*) - B_{uv}(u^*,v^*)U - B_{vv}(u^*,v^*)V\Big)v_t^*\\ &\quad +\Big(B_u(u^*+U,v^*+V) - B_u(u^*,v^*)\Big)U_t + (b(u^*+U,v^*+V) - b(u^*,v^*))V_t\Bigg\}dx\\ &\le \frac{k_3M_1}{2}(\|U\|^2+\|V\|^2) + (b(u^*+U,v^*+V)-b(u^*,v^*),\triangle Z) - d_2k_2\|\nabla V\|^2\\ &\quad +k_3\{M_1(\|U\|+\|V\|) + \|\nabla U\| + \|\nabla V\|\}(d_1\|\nabla U\| + \|\nabla W\|)\\ &\quad +d_2k_3\{M_1(\|U\|+\|V\|) + \|\nabla U\|\}\|\nabla V\| + 2k_3(k_4+k_5)(\|U\|^2+\|V\|^2)\\ &\le (b(u^*+U,v^*+V)-b(u^*,v^*),\triangle Z) - \frac{d_2k_2}{2}\|\nabla V\|^2\\ &\quad +c_7(\|U\|^2+\|V\|^2+\|\nabla U\|^2+\|\nabla W\|^2), \qquad (3.7)\end{aligned}$$

with some positive constant $c_7$. Similarly we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla Z\|^2 &= -d_4\|\triangle Z\|^2 - \frac{1}{\epsilon}(b(u^*+U, v^*+V) - b(u^*, v^*), \triangle Z)\\ &\quad -\frac{1}{\epsilon}\|\nabla Z\|^2 + (\nabla(d_4\triangle z^* - z_t^*), \nabla Z)\\ &\leq -d_4\|\triangle Z\|^2 - \frac{1}{\epsilon}(b(u^*+U, v^*+V) - b(u^*, v^*), \triangle Z)\\ &\quad -\frac{1}{2\epsilon}\|\nabla Z\|^2 + c_8\epsilon, \qquad (3.8)\end{aligned}$$

where $c_8 := M_1^2/2$. Combining two inequalities (3.7) and (3.8), we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|\nabla Z\|^2 + \frac{1}{\epsilon}E_2\right) &\leq -d_4\|\triangle Z\|^2 - \frac{1}{2\epsilon}\|\nabla Z\|^2 - \frac{d_2k_2}{2\epsilon}\|\nabla V\|^2\\ &\quad +\frac{c_7}{\epsilon}(\|U\|^2 + \|V\|^2 + \|\nabla U\|^2 + \|\nabla W\|^2) + c_8\epsilon. \qquad (3.9)\end{aligned}$$

Combine (3.6) and (3.9). If $\gamma \geq \max\{4c_7, 4c_7/(d_1k_1)\}$, then we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\nabla Z\|^2 + \frac{1}{\epsilon}E_2 + \frac{\gamma}{2}\|\nabla W\|^2 + \frac{\gamma}{\epsilon}E_1\right)\\ &\leq -\gamma d_3\|\triangle W\|^2 - d_4\|\triangle Z\|^2 - \frac{\gamma}{4\epsilon}\|\nabla W\|^2 - \frac{1}{2\epsilon}\|\nabla Z\|^2 - \frac{\gamma d_1k_1}{4\epsilon}\|\nabla U\|^2\\ &\quad -\frac{d_2k_2}{2\epsilon}\|\nabla V\|^2 + \frac{\gamma c_5 + c_7}{\epsilon}(\|U\|^2 + \|V\|^2) + (\gamma c_6 + c_8)\epsilon. \qquad (3.10)\end{aligned}$$

The assumption **(A)** implies

$$\begin{aligned}E_1 &\geq \frac{k_1}{2}\|U\|^2,\\ E_2 &\geq -\frac{k_3}{2}\|U\|^2 - k_3\|U\|\|V\| + \frac{k_2}{2}\|V\|^2 \geq -\left(\frac{k_3}{2} + \frac{k_3^2}{k_2}\right)\|U\|^2 + \frac{k_2}{4}\|V\|^2.\end{aligned}$$

Taking $\gamma$ so large as

$$\gamma \geq \max\left\{\frac{4k_3}{k_1}\left(\frac{1}{2} + \frac{k_3}{k_2}\right), 4c_7, \frac{4c_7}{d_1k_1}\right\},$$

we have

$$\gamma E_1 + E_2 \geq c_9(\|U\|^2 + \|V\|^2),$$

where

$$c_9 := \min\left\{\frac{k_2}{4}, \frac{k_1\gamma}{4}\right\}.$$

Thus, (3.10) and the above inequality mean

$$\begin{aligned}
&\frac{d}{dt}\left\{\left(\frac{1}{2}\|\nabla Z\|^2+\frac{1}{\epsilon}E_2+\frac{\gamma}{2}\|\nabla W\|^2+\frac{\gamma}{\epsilon}E_1\right)e^{-c_{10}t}\right\}\\
&\quad\le\ -\Bigg(\gamma d_3\|\triangle W\|^2+d_4\|\triangle Z\|^2+\frac{\gamma}{4\epsilon}\|\nabla W\|^2+\frac{1}{2\epsilon}\|\nabla Z\|^2+\frac{\gamma d_1k_1}{4\epsilon}\|\nabla U\|^2\\
&\qquad\quad+\frac{d_2k_2}{2\epsilon}\|\nabla V\|^2\Bigg)e^{-c_{10}t}+(\gamma c_6+c_8)\epsilon e^{-c_{10}t}
\end{aligned}$$

for $c_{10}:=(\gamma c_5+c_7)/c_9$. Finally, we obtain the first and second inequalities of (1.13).

Lemma 3.1. *Let $\lambda$ and $\xi$ be constants. If a $C^1$–function $X(t)$ satisfies*

$$X'(t)\le\lambda(\xi-X)$$

*for $0<t\le T$ and $X(0)\le\xi$, then $X(t)\le\xi$ for $0\le t\le T$.*

This lemma can be easily checked. So, the proof is omitted.

We will show the inequality for $Z=z-b(u^*,v^*)$ in (1.13). By (3.1), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|Z\|^2\ &\le\ -d_4\|\nabla Z\|^2+\frac{k_3}{\epsilon}(\|U\|+\|V\|)\|Z\|-\frac{1}{\epsilon}\|Z\|^2+M_1\|Z\|\\
&\le\ -d_4\|\nabla Z\|^2+\frac{k_3^2}{\epsilon}(\|U\|^2+\|V\|^2)+M_1^2\epsilon-\frac{1}{4\epsilon}\|Z\|^2,\\
&\le\ (2k_3^2c_1^2+M_1^2)\epsilon-\frac{1}{4\epsilon}\|Z\|^2.
\end{aligned}$$

By Lemma 3.1,

$$\|Z\|^2\le 4(2k_3^2c_1^2+M_1^2)\epsilon^2.$$

The inequality for $W=w-a(u^*)$ in (1.13) can be proved similarly. This completes the proof of Theorem 2.1. □

Remark 3.2. It is difficult to estimate the terms $\triangle W$ and $\triangle Z$ in the first and second equations of (3.1). To overcome this difficulty, we have introduced the functionals $E_1(t)$ and $E_2(t)$ instead of $\|U\|^2$ and $\|V\|^2$. For example, we have chosen $E_1(t)$ in order that $(a(u^*+U)-a(u^*),\triangle W)$ in (3.4) cancels out that of (3.5).

We prepare the following lemma for the proof of Theorem 2.3.

Lemma 3.3. *Let $\lambda(t;\epsilon),\rho(t;\epsilon)$ be non-negative continuous functions in $t$ and satisfy*

$$\int_0^T\lambda(t;\epsilon)dt\le\overline{\lambda},\quad\int_0^T\rho(t;\epsilon)dt\le\overline{\rho}(\epsilon)$$

*where $\overline{\lambda}$ is independent of $\epsilon$. Assume a non-negative $C^1$–function $X(t;\epsilon)$ and a non-negative continuous function $Y(t;\epsilon)$ satisfy*

$$X_t \le -Y + \lambda(t;\epsilon)X + \rho(t;\epsilon)$$

*for $0 < t \le T$. Then,*

$$X(t;\epsilon) \le \{X(0;\epsilon) + \overline{\rho}(\epsilon)\}e^{\overline{\lambda}}, \quad \int_0^t Y(s;\epsilon)ds \le \{X(0;\epsilon) + \overline{\rho}(\epsilon)\}e^{\overline{\lambda}} \tag{3.11}$$

*for $0 \le t \le T$.*

*Proof.* Since

$$\frac{d}{dt}\left(Xe^{-\int_0^t \lambda(\tau;\epsilon)d\tau}\right) \le (-Y + \rho)e^{-\int_0^t \lambda(\tau;\epsilon)d\tau},$$

we have

$$X(t;\epsilon) + \int_0^t Y(s;\epsilon)e^{\int_s^t \lambda(\tau;\epsilon)d\tau}ds \le X(0;\epsilon)e^{\int_0^t \lambda(\tau;\epsilon)d\tau} + \int_0^t \rho(s;\epsilon)e^{\int_s^t \lambda(\tau;\epsilon)d\tau}ds.$$

We can easily check (3.11). □

*Proof of Theorem 2.3.* Owing to Lemma 3.3, it follows from the first and second inequalities of (1.13) and (3.10) that

$$\int_0^T (\|\triangle W\|^2 + \|\triangle Z\|^2)dt \le c_{11}\epsilon.$$

By (3.1), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla U\|^2 &\le -d_1\|\triangle U\|^2 + \|\triangle W\|\ \|\triangle U\| + k_4(\|U\| + \|V\|)\|\triangle U\| \\ &\le -\frac{d_1}{2}\|\triangle U\|^2 + \frac{1}{d_1}\left\{\|\triangle W\|^2 + k_4^2(\|U\| + \|V\|)^2\right\}.\end{aligned}$$

Lemma 3.3 shows us that these inequalities and (1.13) imply

$$\int_0^T \|\triangle U\|^2 dt \le c_{12}\epsilon. \tag{3.12}$$

It is similarly seen that

$$\int_0^T \|\triangle V\|^2 dt \le c_{12}\epsilon. \tag{3.13}$$

Multiplying the first equation of (3.1) by $-a'(u^*)\triangle U$ and integrating over $\Omega$ yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla U|^2 a'(u^*)dx - \frac{1}{2}\int_\Omega |\nabla U|^2 a''(u^*)u_t^* dx + \int_\Omega U_t a''(u^*)\nabla U\cdot\nabla u^* dx\\ &\le -d_1\int_\Omega |\triangle U|^2 a'(u^*)dx - (\triangle W, a'(u^*)\triangle U) + k_3k_4(\|U\|+\|V\|)\|\triangle U\|.\end{aligned}$$

Since

$$\|U_t\| \le d_1\|\triangle U\| + \|\triangle W\| + k_4(\|U\|+\|V\|),$$

the above inequality implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla U|^2 a'(u^*)dx &\le -\frac{d_1k_1}{2}\|\triangle U\|^2 - (\triangle W, a'(u^*)\triangle U) + \frac{1}{4}\|\triangle W\|^2\\ &\quad + c_{13}(\|U\|^2+\|V\|^2+\|\nabla U\|^2). \qquad (3.14)\end{aligned}$$

Similarly, operating $\triangle$ to the third equation of (3.1), multiplying it by $\triangle W$, and integrating over $\Omega$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\triangle W\|^2 &= -d_3\|\nabla\triangle W\|^2 + d_3\int_{\partial\Omega}\triangle W\frac{\partial}{\partial n}\triangle W dS + (\triangle(d_3\triangle w^* - w_t^*), \triangle W)\\ &\quad + \frac{1}{\epsilon}(\triangle(a(u^*+U)-a(u^*)), \triangle W) - \frac{1}{\epsilon}\|\triangle W\|^2. \qquad (3.15)\end{aligned}$$

LEMMA 3.4. *Assume (2.3). Then, there exists a positive constant $c_{14}$ independent of $\epsilon$ and $(u,v,w,z)$ such that*

$$d_3\left|\int_{\partial\Omega}\triangle W\frac{\partial}{\partial n}\triangle W dS\right| \le \frac{d_3}{4}\|\nabla\triangle W\|^2 + \frac{1}{4\epsilon}\|\triangle W\|^2 + c_{14}(\epsilon^{1/2}+\epsilon), \qquad (3.16)$$

$$d_4\left|\int_{\partial\Omega}\triangle Z\frac{\partial}{\partial n}\triangle Z dS\right| \le \frac{d_4}{4}\|\nabla\triangle Z\|^2 + \frac{1}{4\epsilon}\|\triangle Z\|^2 + c_{14}(\epsilon^{1/2}+\epsilon). \qquad (3.17)$$

*Proof.* The equations (1.7) and (1.8) imply

$$\frac{\partial}{\partial n}\triangle w = 0 \quad \text{on } \partial\Omega,$$

and hence

$$\frac{\partial}{\partial n}\triangle W = -\frac{\partial}{\partial n}\triangle a(u^*) \quad \text{on } \partial\Omega.$$

Then,

$$\begin{aligned}
\left|\int_{\partial\Omega} \triangle W \frac{\partial}{\partial n} \triangle W dS\right| &\leq \left\| \frac{\partial}{\partial n} \triangle W \right\|_{L^2(\partial\Omega)} \|\triangle W\|_{L^2(\partial\Omega)} \\
&\leq M_2 \|\triangle W\|_{L^2(\partial\Omega)} \\
&\leq c_{15} \|\triangle W\|_{H^{1/2}(\Omega)} \\
&\leq c_{16} \left( \|\nabla \triangle W\|^{1/2} \|\triangle W\|^{1/2} + \|\triangle W\| \right),
\end{aligned}$$

which is reduced to (3.16). Similarly, (3.17) can be checked. $\square$

Set

$$I_1 := \triangle(a(u^* + U) - a(u^*)) - a'(u^*)\triangle U.$$

It follows from the chain rule that

$$\begin{aligned}
I_1 &= a''(u^* + U)(|\nabla u^* + \nabla U|^2 - |\nabla u^*|^2) + (a''(u^* + U) - a''(u^*))|\nabla u^*|^2 \\
&\quad + (a'(u^* + U) - a'(u^*))\triangle u^* + (a'(u^* + U) - a'(u^*))\triangle U.
\end{aligned}$$

Then, we have

$$\begin{aligned}
|(I_1, \triangle W)| &\leq c_{17}(\|U\| + \|\nabla U\|)\|\triangle W\| + c_{17}\|\nabla U\|^2_{L^4(\Omega)}\|\triangle W\| \\
&\quad + c_{17}\|U\|_{L^4(\Omega)}\|\triangle U\| \ \|\triangle W\|_{L^4(\Omega)}. \qquad (3.18)
\end{aligned}$$

The assumption $N \leq 4$ ensures the inclusion $H^1(\Omega) \subset L^4(\Omega)$ and the existence of a positive constant $c_{18}$ such that

$$\|U\|_{L^4(\Omega)} \leq c_{18}(\|\nabla U\| + \|U\|), \quad \|\nabla U\|_{L^4(\Omega)} \leq c_{18}(\|\triangle U\| + \|U\|).$$

Here we also used an elliptic estimate for $U$ under the boundary conditions (1.3) and (1.8). There exists a positive constant $c_{19}$ such that

$$\begin{aligned}
c_{17}\|\nabla U\|^2_{L^4(\Omega)}\|\triangle W\| &\leq 2c_{17}c_{18}^2(\|\triangle U\|^2 + \|U\|^2)\|\triangle W\| \\
&\leq \frac{d_1 k_1}{4}\|\triangle U\|^2 + \frac{1}{8}\|\triangle W\|^2 + c_{19}\|U\|^4 \\
&\quad + c_{19}\|\triangle U\|^2\|\triangle W\|^2, \\
c_{17}\|U\|_{L^4(\Omega)}\|\triangle U\| \ \|\triangle W\|_{L^4(\Omega)} &\leq c_{17}c_{18}^2(\|\nabla U\| + \|U\|)\|\triangle U\|(\|\nabla \triangle W\| + \|\triangle W\|) \\
&\leq \frac{d_3\epsilon}{4}\|\nabla \triangle W\|^2 + \frac{c_{19}}{\epsilon}(\|U\|^2 + \|\nabla U\|^2)\|\triangle U\|^2 \\
&\quad + c_{19}(\|U\|^2 + \|\nabla U\|^2) + c_{19}\|\triangle U\|^2\|\triangle W\|^2.
\end{aligned}$$

Using the above inequalities, we can estimate $(I_1, \triangle W)$ as follows:

$$\begin{aligned}
|(I_1, \triangle W)| &\leq \frac{d_3\epsilon}{4}\|\nabla \triangle W\|^2 + \frac{1}{4}\|\triangle W\|^2 + \frac{d_1 k_1}{4}\|\triangle U\|^2 + (4c_{17}^2 + c_{19})(\|U\|^2 + \|\nabla U\|^2) \\
&\quad + \frac{c_{19}}{\epsilon}\|\triangle U\|^2 \left( \|U\|^2 + \|\nabla U\|^2 + 2\epsilon\|\triangle W\|^2 \right) + c_{19}\|U\|^4. \qquad (3.19)
\end{aligned}$$

Combining (3.15), (3.16), and (3.19) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\triangle W\|^2 &\leq -\frac{d_3}{2}\|\nabla\triangle W\|^2 - \frac{1}{2\epsilon}\|\triangle W\|^2 + c_{14}(\epsilon^{1/2}+\epsilon) + M_2\|\triangle W\| \\ &\quad + \frac{d_1 k_1}{4\epsilon}\|\triangle U\|^2 + \frac{1}{\epsilon}(a'(u^*)\triangle U, \triangle W) + \frac{4c_{17}^2 + c_{19}}{\epsilon}(\|U\|^2 + \|\nabla U\|^2) \\ &\quad + \frac{c_{19}}{\epsilon^2}\|\triangle U\|^2\left(\|U\|^2 + \|\nabla U\|^2 + 2\epsilon\|\triangle W\|^2\right) + \frac{c_{19}}{\epsilon}\|U\|^4. \end{aligned} \tag{3.20}$$

The inequalities (3.14) and (3.20) imply that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\epsilon}\int_\Omega |\nabla U|^2 a'(u^*)dx + \|\triangle W\|^2\right) \\ &\leq -\frac{d_1 k_1}{4\epsilon}\|\triangle U\|^2 - \frac{d_3}{2}\|\nabla\triangle W\|^2 - \frac{1}{8\epsilon}\|\triangle W\|^2 + c_{14}(\epsilon^{1/2}+\epsilon) + 2M_2^2\epsilon \\ &\quad + \frac{c_{13} + 4c_{17}^2 + c_{19}}{\epsilon}(\|U\|^2 + \|\nabla U\|^2 + \|V\|^2) \\ &\quad + \frac{c_{19}}{\epsilon^2}\|\triangle U\|^2\left(\|U\|^2 + \|\nabla U\|^2 + 2\epsilon\|\triangle W\|^2\right) + \frac{c_{19}}{\epsilon}\|U\|^4. \end{aligned} \tag{3.21}$$

Recall that

$$\frac{1}{\epsilon}\int_\Omega |\nabla U|^2 a'(u^*)dx + \|\triangle W\|^2 \geq \frac{k_1}{\epsilon}\|\nabla U\|^2 + \|\triangle W\|^2.$$

Fix an arbitrary positive number $\epsilon_0$. Taking (1.13) into account, we can derive from (3.21)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\epsilon}\int_\Omega |\nabla U|^2 a'(u^*)dx + \|\triangle W\|^2\right) \\ &\leq -\frac{d_1 k_1}{4\epsilon}\|\triangle U\|^2 - \frac{d_3}{2}\|\nabla\triangle W\|^2 - \frac{1}{8\epsilon}\|\triangle W\|^2 + c_{20}\epsilon^{1/2} + c_{20}\|\triangle U\|^2 \\ &\quad + c_{20}\left(1 + \frac{1}{\epsilon}\|\triangle U\|^2\right)\left(\frac{k_1}{\epsilon}\|\nabla U\|^2 + \|\triangle W\|^2\right), \end{aligned} \tag{3.22}$$

if $0 < \epsilon \leq \epsilon_0$. Since (3.12) guarantees that the assumptions of Lemma 3.3 hold true with

$$\rho(t;\epsilon) = 2c_{20}(\epsilon^{1/2} + \|\triangle U\|^2), \quad \lambda(t;\epsilon) = 2c_{20}\left(1 + \frac{1}{\epsilon}\|\triangle U\|^2\right),$$

we can apply Lemma 3.3 to (3.22) and obtain

$$\|\nabla U\|^2 \leq c_{21}\epsilon^{3/2}, \quad \int_0^T \|\triangle U\|^2 dt \leq c_{21}\epsilon^{3/2}, \quad \int_0^T \|\nabla\triangle W\|^2 dt \leq c_{21}\epsilon^{1/2} \tag{3.23}$$

for $0 < \epsilon \le \epsilon_0$.

Next we will show the inequality for $\|\nabla V\|^2$. Multiplying the second equation of (3.1) by $-b_v(u^*, v^*)\triangle V$ and integrating over $\Omega$ yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla V|^2 b_v(u^*, v^*)dx \quad &\le \quad -\frac{d_2 k_2}{2}\|\triangle V\|^2 - (\triangle Z, b_v(u^*, v^*)\triangle V) + \frac{1}{4}\|\triangle Z\|^2 \\ &\quad +c_{22}(\|U\|^2 + \|V\|^2 + \|\nabla V\|^2). \end{aligned} \tag{3.24}$$

Let us operate $\triangle$ to the last equation of (3.1), multiply it by $\triangle Z$, and integrate over $\Omega$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\triangle Z\|^2 \quad &= \quad -d_4\|\nabla\triangle Z\|^2 + d_4\int_{\partial\Omega}\triangle Z\frac{\partial}{\partial n}\triangle Z dS + (\triangle(d_4\triangle z^* - z_t^*), \triangle Z) \\ &\quad +\frac{1}{\epsilon}(\triangle(b(u^*+U, v^*+V) - b(u^*, v^*)), \triangle Z) - \frac{1}{\epsilon}\|\triangle Z\|^2. \end{aligned} \tag{3.25}$$

Setting

$$I_2 := \triangle(b(u^*+U, v^*+V) - b(u^*, v^*)) - b_v(u^*, v^*)\triangle V,$$

we can see, in the similar manner to the argument to obtain (3.18), that

$$\begin{aligned}|(I_2, \triangle Z)| \quad &\le \quad c_{23}(\|U\| + \|\nabla U\| + \|V\| + \|\nabla V\|)\|\triangle Z\| \\ &\quad +c_{23}\left(\|\nabla U\|^2_{L^4(\Omega)} + \|\nabla V\|^2_{L^4(\Omega)}\right)\|\triangle Z\| + c_{23}\|\triangle U\|\ \|\triangle Z\| \\ &\quad +c_{23}\left(\|U\|_{L^4(\Omega)} + \|V\|_{L^4(\Omega)}\right)\|\triangle V\|\ \|\triangle Z\|_{L^4(\Omega)}. \end{aligned}$$

As in deriving (3.19), the above inequality is reduced to

$$\begin{aligned}|(I_2, \triangle Z)| \quad &\le \quad \frac{d_4\epsilon}{4}\|\nabla\triangle Z\|^2 + \frac{1}{4}\|\triangle Z\|^2 + \frac{d_2 k_2}{4}\|\triangle V\|^2 + c_{24}\|\triangle U\|^2 \\ &\quad +c_{24}\left(\|U\|^2 + \|\nabla U\|^2 + \|V\|^2 + \|\nabla V\|^2\right) \\ &\quad +\frac{c_{24}}{\epsilon}\|\triangle V\|^2\left(\|U\|^2 + \|\nabla U\|^2 + \|V\|^2 + \|\nabla V\|^2\right) \\ &\quad +c_{24}\left(\|\triangle U\|^2 + \|\triangle V\|^2\right)\|\triangle Z\|^2 + c_{24}(\|U\|^4 + \|V\|^4). \end{aligned} \tag{3.26}$$

Combining (3.24), (3.25), (3.26) and (3.17), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\epsilon}\int_\Omega |\nabla V|^2 b_v(u^*, v^*)dx + \|\triangle Z\|^2\right) \\ &\le \quad -\frac{d_2 k_2}{4\epsilon}\|\triangle V\|^2 - \frac{d_4}{2}\|\nabla\triangle Z\|^2 - \frac{1}{8\epsilon}\|\triangle Z\|^2 + c_{14}(\epsilon^{1/2} + \epsilon) + 2M_2^2\epsilon \\ &\quad +\frac{c_{24}}{\epsilon}\|\triangle U\|^2 + \frac{c_{22}+c_{24}}{\epsilon}\left(\|U\|^2 + \|\nabla U\|^2 + \|V\|^2 + \|\nabla V\|^2\right) \\ &\quad +\frac{c_{24}}{\epsilon^2}\|\triangle V\|^2\left(\|U\|^2 + \|\nabla U\|^2 + \|V\|^2 + \|\nabla V\|^2\right) \\ &\quad +\frac{c_{24}}{\epsilon}\left(\|\triangle U\|^2 + \|\triangle V\|^2\right)\|\triangle Z\|^2 + \frac{c_{24}}{\epsilon}(\|U\|^4 + \|V\|^4). \end{aligned} \tag{3.27}$$

Hence, we can obtain

$$\|\nabla V\|^2 \le c_{25}\epsilon^{3/2}, \quad \int_0^T \|\triangle V\|^2 dt \le c_{25}\epsilon^{3/2}, \quad \int_0^T \|\nabla\triangle Z\|^2 dt \le c_{25}\epsilon^{1/2} \tag{3.28}$$

for $0 < \epsilon \le \epsilon_0$, using (1.13), (3.13), (3.23) and Lemma 3.3.

Due to (3.5),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla W\|^2 &\le \frac{k_3}{\epsilon}\left(M_1\|U\| + \|\nabla U\|\right)\|\nabla W\| - \frac{1}{2\epsilon}\|\nabla W\|^2 + c_6\epsilon \\ &\le \frac{1}{4\epsilon}\left(c_{26}\epsilon^{3/2} - \|\nabla W\|^2\right),\end{aligned}$$

where $0 < \epsilon \le \epsilon_0$. Lemma 3.1 and the above inequality imply the third inequality of (1.14). The fourth inequality of (1.14) can be also seen similarly.

Hereafter we will prove the remaining four inequalities of (1.14) for $0 < \epsilon \le \epsilon_0$. Since

$$\frac{\partial}{\partial n}(d_1\triangle U + \triangle W) = 0 \quad \text{on } \partial\Omega$$

by (3.1), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\triangle U\|^2 &\le -d_1\|\nabla\triangle U\|^2 + \|\nabla\triangle W\|\ \|\nabla\triangle U\| \\ &\quad +\|\nabla(f(u^*+U, v^*+V) - f(u^*, v^*))\|\ \|\nabla\triangle U\| \\ &\le -\frac{d_1}{2}\|\nabla\triangle U\|^2 + \frac{1}{d_1}\|\nabla\triangle W\|^2 + c_{27}\epsilon^{3/2}.\end{aligned}$$

Integrating the above in $t \in [0, T]$ and using (3.23) yield the fifth inequality of (1.14). Similarly we can show the sixth inequality of (1.14).

Consider the estimates of $\|\triangle W\|$ and $\|\triangle Z\|$. By (3.20), (3.25), (3.17) and (3.26), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\triangle W\|^2 &\le -\frac{d_3}{2}\|\nabla\triangle W\|^2 + c_{28}\epsilon^{1/2} + \frac{c_{28}}{\epsilon}\|\triangle U\|^2 + \frac{c_{19}}{\epsilon}\|\triangle U\|^2\|\triangle W\|^2, \\ \frac{1}{2}\frac{d}{dt}\|\triangle Z\|^2 &\le -\frac{d_4}{2}\|\nabla\triangle Z\|^2 + c_{29}\epsilon^{1/2} + \frac{c_{29}}{\epsilon}(\|\triangle U\|^2 + \|\triangle V\|^2) \\ &\quad +\frac{c_{29}}{\epsilon}(\|\triangle U\|^2 + \|\triangle V\|^2)\|\triangle Z\|^2.\end{aligned}$$

The last two inequalities of (1.14) follow from the above inequalities, (3.23), (3.28) and Lemma 3.3. $\square$

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