# Vanishing, moving and immovable interfaces in fast reaction limits

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July 25, 2017

**Abstract**

We consider a type of singular limit problem called the *fast reaction limit*. The problem of the fast reaction limit involves studying the behaviour of solutions of reaction-diffusion systems when the reaction speeds are very fast. Fast reaction limits of two-component systems have been studied in recent decades. In most of these systems, the fast reaction terms of each component are represented by the same function. Fast reaction limits of systems with different fast reaction terms are still far from being well understood. In this paper, we focus on a reaction-diffusion system for which the reaction terms consist of monomial functions of various powers. The behaviour of interfaces arising in the fast reaction limit of this system is studied. Depending on the powers, three types of behaviour are observed: (i) the initial interface vanishes instantaneously, (ii) the interface propagates at a finite speed, and (iii) the interface does not move.

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# 1 Introduction

Singular limit analysis is an important method of reducing the freedom of systems and deriving essential dynamics. In recent decades, singular limits of reaction-diffusion systems have been studied intensively in cases where the effects of the reaction terms are very large compared with those of the other terms. This type of limit is called a *fast reaction limit*. Hilhorst et al. [13] considered a type of two-component reaction-diffusion system that originates from a chemical reaction, namely,

$$\begin{cases} u_t = \Delta u - kuv & \text{in } Q_T := \Omega \times (0, T], \\ v_t = -kuv & \text{in } Q_T, \end{cases} \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}$) with smooth boundary $\partial\Omega$, $T$ is a positive constant and $k$ is a positive parameter. It was shown that the limit of the solution $(u_k, v_k)$ of (1) as $k$ tends to infinity is represented by the solution of a certain free boundary problem; namely, the so-called one-phase Stefan problem. This result means that the supports of the limit functions $u$ and $v$ are separated by an interface that moves with a finite speed. Hilhorst et al. [14] and Eymard et al. [8] extended the previous study to systems of the following type:

$$\begin{cases} u_t = \Delta \phi(u) - kF(u, v) & \text{in } Q_T, \\ v_t = -kF(u, v) & \text{in } Q_T, \end{cases} \tag{2}$$

where $\phi$ is a nondecreasing smooth function and $F$ is smooth and nondecreasing in both arguments (see [14, 8] for further detail). A typical example of the reaction term $F(u, v)$ is given by $u^p v^q$, with constants $p, q \geq 1$. The limit problem of (2) can also be described by the one-phase Stefan problem, provided that $\phi(u) = u$ and $F(u, v) = u^p v^q$. Evans [7] investigated a system consisting of (1) with diffusion term $\Delta v$ in the equation for $v$. He demonstrated that the limit problem as $k$ tends to infinity is described by the two-phase Stefan problem with zero latent heat. Dancer et al. [6] and Crooks et al. [4] considered a Lotka-Volterra competition-diffusion system with large competition rates. It was shown that the habitats of two competing species are spatially segregated, and that the limit problem consists of the two-phase Stefan problem with zero latent heat. Murakawa and Ninomiya [20] investigated the limit problem of a three-component competition-diffusion system. For additional related work, we refer the reader to Bothe and Hilhorst [1], Bouillard et al. [3], and Hilhorst and Murakawa [12] for systems with reversible reaction terms; Murakawa [18] for approximations to degenerate parabolic problems; and Iida and Ninomiya [16], Iida et al. [15] and Murakawa [19] for approximations to nonlinear cross-diffusion systems. The two-component systems in the above references can be summarised in the following system:

$$\begin{cases} u_t = d_1 \Delta u + f(u) - kF(u,v) & \text{in } Q_T, \\ v_t = d_2 \Delta v + g(v) - kG(u,v) & \text{in } Q_T, \end{cases} \tag{3}$$

where $d_1, d_2$ are constants satisfying $d_1 > 0, d_2 \geq 0$, and $f, g, F, G$ are nonlinear functions. In [4, 6, 7, 8, 13, 14], each system satisfies $F \equiv G \not\equiv 0$. Under this restriction, it follows from (3) that

$$u_t - d_1 \Delta u - f(u) = v_t - d_2 \Delta v - g(v). \tag{4}$$

Because the equality in (4) is constantly satisfied for any $k > 0$, it plays a key role in proving the convergence of the fast reaction limit of (3). Similar situations are observed in cases where $F \equiv -G$ (see [1, 3, 12, 18]). We call $(F, G)$ a *balanced fast reaction pair* if there exists a constant $\ell$ such that $F \equiv \ell G$. Otherwise, we call it an *unbalanced fast reaction pair*. As stated above, there exist many results concerning the fast reaction limits of systems with balanced fast reaction pairs. However, systems with unbalanced fast reaction pairs have not yet been comprehensively studied. Therefore, we encounter the following natural question:

*Q. What happens for* (3) *with an unbalanced fast reaction pair as $k$ tends to infinity?*

In this paper, as a first step towards answering the above question we consider fast reaction limit of the following system, with a specific unbalanced fast reaction pair:

$$(\mathrm{P})_k \quad \begin{cases} u_t = \Delta u - k u^{m_1} v^{m_2} & \text{in } Q_T, \\ v_t = -k u^{m_3} v^{m_4} & \text{in } Q_T, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } S_T := \partial\Omega \times (0, T], \\ u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0 & \text{in } \Omega, \end{cases}$$

where $n$ is the outward normal unit vector to $\partial\Omega$, $m_i \geq 1$ $(i = 1, 2, 3, 4)$, $(m_1, m_2) \neq (m_3, m_4)$, and $u_0$ and $v_0$ are the initial data such that $u_0, v_0 \geq 0$,

$u_0 v_0 \equiv 0$, $u_0 \not\equiv 0$, and $v_0 \not\equiv 0$. Because $u_0 v_0 \equiv 0$, the regions occupied by $u_0$ and $v_0$ are separated from each other. In other words, an interface exists at the initial time. In this paper, we examine the behaviour of this initial interface. As a result, it turns out that the behaviour is completely different for various the powers $\mathbf{m} = (m_1, m_2, m_3, m_4)$.

As mentioned above, Hilhorst et al. [14] and Eymard et al. [8] have already studied balanced cases, i.e., where $(m_1, m_2) = (m_3, m_4) = (p, q)$. For unbalanced settings with $(m_1, m_2) \neq (m_3, m_4)$, only a few researchers have reported on the limit problem of $(\mathrm{P})_k$. Note that Noris et al. [21] considered a system of stationary Gross-Pitaevskii equations, derived from binary mixtures of Bose-Einstein condensates, which is closely related to the steady problem of (3) with $d_1 = d_2 > 0$, $f(u) = -u^3 + \lambda_1 u$, $g(v) = -v^3 + \lambda_2 v$ $(\lambda_1, \lambda_2 > 0)$, $F(u,v) = uv^2$ and $G(u,v) = u^2 v$. Conti et al. [5] and Hilhorst et al. [10] investigated multi-component competition-diffusion systems that correspond to unbalanced cases.

To clarify the relationship between the powers and the limit problem of $(\mathrm{P})_k$, we focus on the following four cases:

$$\begin{array}{lcl} \text{Case I} & : & \mathbf{m} = (m_1, 1, 1, 1), \\ \text{Case II} & : & \mathbf{m} = (1, m_2, 1, 1), \\ \text{Case III} & : & \mathbf{m} = (1, 1, m_3, 1), \\ \text{Case IV} & : & \mathbf{m} = (1, 1, 1, m_4). \end{array}$$

Our main results are as follows:

*Vanishing interface*: In Case I with $m_1 > 3$, $u_k$ converges to a solution of the heat equation defined in $Q_T$, and $v_k$ goes to zero as $k$ tends to infinity. This means that the initial interface vanishes instantaneously.

*Moving interfaces*: In Case II with $m_2 \geq 1$, $u_k$ converges to a solution of the one-phase Stefan problem with a latent heat $v_0^{m_2}/m_2$. Therefore, the interface arising in the limit problem moves with a finite speed. In Case IV with $1 \leq m_4 < 2$, similar results are obtained. The limit function is represented by a solution of the one-phase Stefan problem with a latent heat $v_0^{(2-m_4)}/(2-m_4)$.

*Immovable interface*: In Case III with $m_3 > 1$, the initial interface does not move in the limit problem, and $u_k$ converges to a solution of a heat equation in the fixed domain $\Omega \setminus \mathrm{supp}(v_0)$.

In the 'vanishing interface' case, we assume that the initial data $u_0$ and $v_0$ are smooth. In the 'immovable interface' case, we suppose that $v_0$ is bounded below by a strictly positive constant on its support; that is, $v_0$ is discontinuous across the initial interface. These conditions are necessary in our proofs. However, we suspect that the behaviour of the interfaces would be the same as described above even if these conditions were absent. Case I with $1 < m_1 \leq 3$ and Case IV with $m_4 \geq 2$ remain open problems. We will discuss possible behaviours in these cases in the end of this section.

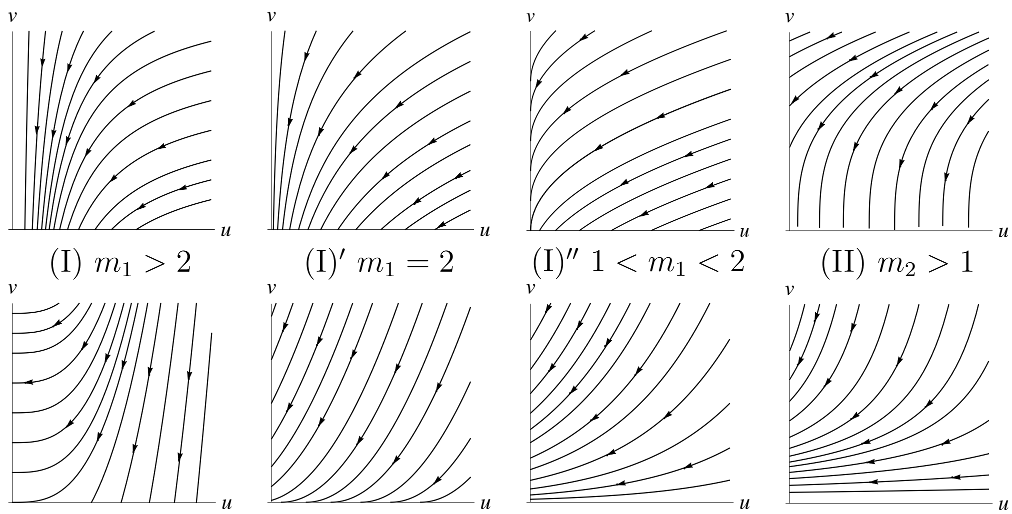
In the study of fast reaction limits of reaction-diffusion systems, it may be effective to consider certain sets called *reaction limit sets* (RLSs). For example, limit problems of many systems with balanced fast reaction pairs are characterised by the dynamics on RLSs corresponding to these systems (see [20]). In the following, we will explain how to understand our results for the system $(\mathrm{P})_k$ with unbalanced fast reaction pairs from the viewpoint of RLSs. The RLS $\mathcal{A}$ of $(\mathrm{P})_k$ is defined as a set of equilibria of the following fast reaction system:

$$\begin{cases} u_t = -ku^{m_1}v^{m_2}, \\ v_t = -ku^{m_3}v^{m_4}. \end{cases} \tag{5}$$

That is,

$$\mathcal{A} = \mathcal{A}_u \cup \mathcal{A}_v \cup \{(0,0)\},$$

where $\mathcal{A}_u := \{(u,0)\,|\,u > 0\}$ and $\mathcal{A}_v := \{(0,v)\,|\,v > 0\}$. Solutions of limit problems of $(\mathrm{P})_k$ as $k$ tends to infinity lie on $\mathcal{A}$ independently of $\mathbf{m} = (m_1, m_2, m_3, m_4)$. The limit solutions diffuse with a diffusion coefficient equal to one on $\mathcal{A}_u$, while they do not diffuse on $\mathcal{A}_v$. Moreover, the flux is discontinuous across the corner $(0,0)$ in $\mathcal{A}$. This discontinuity is one of the factors in producing the one-phase Stefan condition. In fact, when $m_3 = m_1$ and $m_4 = m_2$, the limit problem becomes the one-phase Stefan problem, and hence limit solutions of $(\mathrm{P})_k$ are characterised by $\mathcal{A}$. However, our results indicate that the behaviour of solutions to limit problems of $(\mathrm{P})_k$ vary depending on $\mathbf{m}$. In order to clarify the differnces, let us investigate the dynamics near $\mathcal{A}$. Each solution of the fast reaction system (5) moves




(I) $m_1 > 2$ (I)$'$ $m_1 = 2$ (I)$''$ $1 < m_1 < 2$ (II) $m_2 > 1$

(III) $m_3 > 1$ (IV) $1 < m_4 < 2$ (IV)$'$ $m_4 = 2$ (IV)$''$ $m_4 > 2$

Figure 1: Dynamics of (5) corresponding to each cases.

towards a point on $\mathcal{A}$ along one of the following orbits:

$$v = \begin{cases} v_0 + (u^{2-m_1} - u_0^{2-m_1})/(2-m_1) & (m_1 \neq 2) \\ v_0 + \log u - \log u_0 & (m_1 = 2) \end{cases} \qquad \text{in Case I,}$$

$$v = (m_2(u-u_0) + v_0^{m_2})^{1/m_2} \qquad \text{in Case II,}$$

$$v = v_0 + (u^{m_3} - u_0^{m_3})/m_3 \qquad \text{in Case III,}$$

$$v = \begin{cases} ((2-m_4)(u-u_0) + v_0^{2-m_4})^{1/(2-m_4)} & (m_4 \neq 2) \\ \exp(u - u_0 + \log v_0) & (m_4 = 2) \end{cases} \qquad \text{in Case IV.}$$

Thus, our problem is classified into eight groups, of which the dynamics are illustrated in Figure 1. Because the initial datum $(u_0, v_0)$ of $(\mathrm{P})_k$ satisfy $u_0 v_0 \equiv 0$, the initial datum belongs to $\mathcal{A}$. Hence, there is an initial interface separating the two regions $\{x \,|\, u_0(x) > 0, v_0(x) = 0\}$ and $\{x \,|\, v_0(x) > 0, u_0(x) = 0\}$. If $k$ is finite, then solutions $(u_k, v_k)$ of $(\mathrm{P})_k$ remain in the neighbourhood of the RLS $\mathcal{A}$ for any $t > 0$. When $u$ attempts to invade the region of $v$ as a result of diffusion, $u$ becomes positive temporarily near to the boundary of the region of $v$. It is important to consider the behaviour of $(u_k, v_k)$ at this moment. First, we consider Case I with $m_1 > 2$. In this case, the orbits of the fast reaction system (5) are illustrated in Figure 1 (I). If the first component of a data ponit belonging to

$\mathcal{A}_v$ becomes positive through a perturbation, then the solution immediately converges to $\mathcal{A}_u$ along the orbits for sufficiently large $k$. This observation indicates that $u$ can easily invade the region occupied by $v$. Thus, the initial region occupied by $v$ instantaneously vanishes, owing to the infinite speed of propagation of the diffusion. This suggests that the initial interface also vanishes in an instant for $m_1 > 2$, although a rigorous proof will only be given for the case where $m_1 > 3$. Next, we consider Case II. The solution orbits of the fast reaction system (5) are illustrated in Figure 1 (II). If the first component of a data point belonging to $\mathcal{A}_v$ becomes positive through a perturbation, then the solution moves into $\mathcal{A}_v$ along an orbit, and the value of $v$ decreases at this time. By repeating this process around the interface, $v$ gradually decreases, and the solution converges to $\mathcal{A}_u \cup (0, 0)$ over time. This means that the interface propagates with a finite speed, because the region occupied by $v$ is gradually invaded by $u$. Case IV with $1 < m_4 < 2$ can be also explained by using the same idea as Case II. Finally, we consider Case III. The dynamics of (5) are illustrated in Figure 1 (III). In this case, we note that the solution orbits are orthogonal to $\mathcal{A}_v$. In the limit state, any solution $(u, v)$ that is close to $\mathcal{A}_v$ converges to $\mathcal{A}_v$ without the value of $v$ changing, even though $u$ attempts to invade the region occupied by $v$. Therefore, the value of $v$ and its region do not change, which indicates that the interface is immovable.

As described above, we may be able to infer or understand behaviour of solutions of limit problems with the help of examining the dynamics of fast reaction systems. The above discussion illustrates that in the study of fast reaction limits of systems with unbalanced fast reaction pairs, we need to consider the dynamics of solutions not only on the RLS, but also in the neighbourhood of the RLS.

Now, we consider the remaining cases that are left as open problems. The dynamics for Case I with $m_1 = 2$ are illustrated in Figure 1 (I)$'$. When the first component of a data point belonging to $\mathcal{A}_v$ becomes positive, the solution tends to $\mathcal{A}_u$ imediately. This situation is the same as that in Case I with $m_1 > 2$. Therefore, we believe that the initial interface vanishes instantaneously. The situation in Case I with $1 < m_1 < 2$, the dynamics of which are drawn in Figure 1 (I)$''$, is rather different from the case where $m_1 \geq 2$. A data point in the neighbourhood of $\mathcal{A}_v$ moves into $\mathcal{A}_v$ along an orbit, and the value of $v$ decreases at this time. From this observation, it seems that the interface might propagate with a finite speed. Looking at the dynamics from a different point of view, the orbits are tangent to the

$v$-axis. In the limit state, the value of $v$ could become zero very fast. From this viewpoint, the interface might vanish instantaneously. We cannot rule out other possibilities that are beyond the scope of this paper. The solution orbits in Case IV with $m_4 \geq 2$ are illustrated in Figure 1 (IV)$'$ and (IV)$''$. When $u$ tries to invade the region occupied by $v$, the value of $v$ gradually decreases. However, the solution orbits are almost orthogonal to $\mathcal{A}_v$ in a neighbourhood of $(0,0)$, and no orbits can attain $(0,0)$. Thus, we suspect that $u$ cannot invade the region of $v$ if $v_0$ is continuous across the interface. Therefore, dealing with the parameter ranges in this situation presents a very delicate and difficult task, which we cannot treat in this paper. It is important to study the eight groups of $(\mathrm{P})_k$ from a theoretical point of view, in order to understand the mechanisms of the fast reaction limit.

This paper is organised as follows. In Section 2, we consider Case I. By solving the second equation of $(\mathrm{P})_k$, we rewrite $(\mathrm{P})_k$ as a single equation with a non-local term. We estimate the non-local term by introducing comparison functions, and demonstrate the positivity of the limit function $u$. Indeed, for $m_1 > 3$ we can show that $u$ satisfies the heat equation in the whole domain $Q_T$. In Section 3, we consider Case II with $m_2 > 1$ and Case IV with $1 < m_4 < 2$. In these cases, we can rewrite $(\mathrm{P})_k$ as (2) by applying appropriate transformations. By repeating the arguments applied in previous studies [8, 9, 14], we show that the limit function satisfies the one-phase Stefan problem where the free boundary moves with a finite speed. In Section 4, we consider Case III with $m_3 > 1$. First, we obtain a priori estimates of $u_k$ and $v_k$ in several functional spaces. By applying the Kolmogorov–M. Riesz–Fréchet theorem, we show that the limit functions satisfy a weak form of free boundary problem, but that the free boundary does not move.

## 2 Singular limit of $(\mathrm{P})_k$ in Case I

In this section, we consider the limit problem of $(\mathrm{P})_k$ in Case I with $m_1 > 1$, namely,

$$(\mathrm{P_I})_k \qquad \begin{cases} u_t = \Delta u - k u^m v & \text{in } Q_T, \\ v_t = -kuv & \text{in } Q_T, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } S_T, \\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 & \text{in } \Omega, \end{cases}$$

where $m > 1$. Throughout this paper, the following assumption is imposed on the initial data:

(H1) $(u_0, v_0) \in C(\overline{\Omega}) \times L^\infty(\Omega)$, $u_0 v_0 \equiv 0$, $u_0 \not\equiv 0$, $v_0 \not\equiv 0$ and there exist positive constants $M_u$, $M_v$ such that $0 \le u_0 \le M_u$, $0 \le v_0 \le M_v$ in $\Omega$.

Under the assumption (H1), there exists a unique solution $(u_k, v_k)$ of $(\mathrm{P})_k$ (see, e.g., [9]) such that

$$\begin{aligned}
&u_k \in C([0,T]; C(\overline{\Omega})) \cap C^1((0,T]; C(\overline{\Omega})) \cap C((0,T]; W^{2,p}(\Omega)),\\
&v_k \in C^1([0,T]; L^\infty(\Omega)),\\
&0 \le u_k \le M_u \text{ and } 0 \le v_k \le M_v \text{ in } Q_T
\end{aligned} \tag{6}$$

for any $p > 1$. In addition to (H1), in this section, we assume the following hypotheses:

(H2) $u_0 \in C^2(\overline{\Omega})$, $v_0 \in C^\alpha(\overline{\Omega})$ $(\alpha \in (0,1))$, $\dfrac{\partial u_0}{\partial n} = 0$ on $\partial\Omega$.

We now state the main result in this section.

**Theorem 1.** *Suppose that* (H1) *and* (H2) *hold. Let* $(u_k, v_k)$ *be the solution of* $(\mathrm{P_I})_k$ *with* $m > 3$. *Then there exist functions* $u$ *and* $v$ *such that*

$$u_k \to u \qquad \textit{in } C(\overline{Q}_T), \tag{7}$$

$$v_k \to v \equiv 0 \qquad \textit{in } C(\overline{\Omega} \times [\rho, T]) \tag{8}$$

*as* $k$ *tends to infinity, where* $\rho$ *is an arbitrary constant in* $(0,T)$. *Moreover, the limit function* $u$ *belongs to* $C^{2,1}(\overline{Q}_T)$ *and satisfies the following heat equation:*

$$\begin{cases}
u_t = \Delta u & \textit{in } Q_T,\\
\dfrac{\partial u}{\partial n} = 0 & \textit{on } S_T,\\
u(\cdot, 0) = u_0 & \textit{in } \Omega.
\end{cases} \tag{9}$$

This result shows that $u$ becomes positive everywhere in $\Omega$ immediately. Therefore, the initial interface vanishes instantaneously in the limit problem. Furthermore, $v_0$ has no effect on the limit problem.

## 2.1 Positivity of the solution of $(\mathrm{P_I})_k$

By solving the second equation $v_t = -kuv$, $(\mathrm{P_I})_k$ is rewritten as a single parabolic equation:

$$\begin{cases} u_t = \Delta u - kv_0 u^m e^{-k\int_0^t u\,d\tau} & \text{in } Q_T, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } S_T, \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{10}$$

The aim of this subsection is to prove that the solution $u_k$ of (10) becomes positive in a short time. To this end, we construct a comparison function which is independent of $k$. We consider the following problem including a positive constant $\delta$:

$$\begin{cases} u_t = \Delta u - \delta u & \text{in } Q_T, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } S_T, \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{11}$$

We denote by $\underline{u}_\delta(x,t;u_0)$ the solution of (11) with a positive constant $\delta$ and an initial datum $u_0$. It will be shown that $\underline{u}_\delta$ satisfies $u_k \geq \underline{u}_\delta$ for arbitrary $k$ if $\delta$ is sufficiently large. We prepare the following auxiliary lemma.

**Lemma 2.** *There exists a unique classical solution $u_k \in C^{2,1}(\overline{Q}_T)$ (resp. $\underline{u}_\delta \in C^{2,1}(\overline{Q}_T)$) of (10) (resp. (11)). Moreover, it holds that*

$$0 < \underline{u}_\delta \leq M_u, \quad 0 < u_k \leq M_u \quad \textit{in } \overline{\Omega} \times (0,T].$$

*Proof.* By (H2) and a classical theory, (10) and (11) have unique solutions $u_k$ and $\underline{u}_\delta$ in $C^{2,1}(\overline{Q}_T)$, respectively (see Chapter 4 in [17]). According to the weak maximum principle and the Hopf lemma, we immediately have $0 \leq \underline{u}_\delta, u_k \leq M_u$. In addition, it follows from the strong maximum principle and the Hopf lemma that $\underline{u}_\delta, u_k$ are positive. □

**Lemma 3.** *Assume that $m > 3$. Then there exists a positive constant $t^* = t^*(\delta)$ such that*

$$\{(m-1)\underline{u}_\delta^{m-3}\,(\underline{u}_\delta)_t - 1\}\underline{u}_\delta \leq 0 \quad \textit{in } \operatorname{supp}(v_0) \times [0,t^*]. \tag{12}$$

*Proof.* It follows from $m > 3$ that $(m-1)\underline{u}_\delta(x,0)^{m-3}\left(\underline{u}_\delta\right)_t(x,0) = 0$ for any $x \in \operatorname{supp}(v_0)$. Here we used the property that $u_0 v_0 = 0$ in $\Omega$. Since $\underline{u}_\delta, \left(\underline{u}_\delta\right)_t \in C(\overline{Q}_T)$, the inequality (12) is ensured in a short time. This implies the existence of $t^*$. $\square$

**Lemma 4.** *Let $\delta > M_v/e$ and $t^* = t^*(\delta)$ be a positive constant defined in Lemma 3. Then*

$$u_k \geq \underline{u}_\delta \quad \text{in } \overline{Q}_{t^*} \tag{13}$$

*for any $k > 0$.*

*Proof.* For the sake of simplicity, we use $\underline{u}$ instead of $\underline{u}_\delta$ in this proof. Set $W := u_k - \underline{u} + \varepsilon$ for any constant $\varepsilon \in (0, \varepsilon_k)$, where $\varepsilon_k$ is a positive constant dependent on $k$ and is specified later. Lemma 2 ensures that $W \in C^{2,1}(\overline{Q}_T)$. According to (10) and (11), the function $W$ satisfies the problem

$$\begin{cases} W_t = \Delta W - k v_0 u_k^m e^{-k\int_0^t u_k d\tau} + \delta \underline{u} & \text{in } Q_T, \\ \dfrac{\partial W}{\partial n} = 0 & \text{on } S_T, \\ W(\cdot, 0) \equiv \varepsilon & \text{in } \Omega. \end{cases} \tag{14}$$

We show $W > 0$ in $Q_{t^*}$ by a contradiction argument. Suppose that $(x_0, t_0)$ is a point in $Q_{t^*}$ satisfying

$$\begin{cases} W(x_0, t_0) = 0, \\ W(x,t) > 0 \quad \text{for} \quad (x,t) \in \Omega \times [0, t_0). \end{cases} \tag{15}$$

At this minimum point $(x_0, t_0)$, we have $W_t(x_0, t_0) \leq 0$ and $\Delta W(x_0, t_0) \geq 0$. We denote by $I_0$ the reaction term in (14) at the minimum point, namely,

$$I_0 := -k v_0(x_0) u_k(x_0, t_0)^m e^{-k\int_0^{t_0} u_k(x_0,\tau)\, d\tau} + \delta \underline{u}(x_0, t_0).$$

First, we consider the case where $x_0 \in \Omega \setminus \operatorname{supp}(v_0)$. Then it immediately follows that $I_0 = \delta \underline{u}(x_0, t_0) > 0$. Thus, we get

$$0 \geq W_t(x_0, t_0) = \Delta W(x_0, t_0) + I_0 > 0. \tag{16}$$

This contradicts the assumption (15); namely the point $(x_0, t_0)$ satisfying (15) cannot exist in $(\Omega \setminus \operatorname{supp}(v_0)) \times (0, t^*]$ for $\varepsilon > 0$.

Next, we consider the case where $x_0 \in \operatorname{supp}(v_0) \cap \Omega$. We divide $I_0$ into three parts $I_1$, $I_2$ and $I_3$ as follows:

$$I_0 = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &:= -kv_0(x_0)\Big(u_k(x_0,t_0)^m - \underline{u}(x_0,t_0)^m\Big)e^{-k\int_0^{t_0} u_k(x_0,\tau)\,d\tau},\\
I_2 &:= \underline{u}(x_0,t_0)\left(\delta - kv_0(x_0)\underline{u}(x_0,t_0)^{m-1}e^{-k\int_0^{t_0}\underline{u}(x_0,\tau)\,d\tau}\right),\\
I_3 &:= kv_0(x_0)\underline{u}(x_0,t_0)^m\left(e^{-k\int_0^{t_0}\underline{u}(x_0,\tau)\,d\tau} - e^{-k\int_0^{t_0}u_k(x_0,\tau)\,d\tau}\right).
\end{aligned}$$

By $\underline{u}(x_0,t_0) = u_k(x_0,t_0)+\varepsilon$, we obtain

$$I_1 \geq kv_0 e^{-k\int_0^{t_0} u_k\,d\tau} m{u_k}^{m-1}\varepsilon \geq 0. \tag{17}$$

Note that $I_2$ is represented by

$$I_2 = \underline{u}\left(\delta - v_0 z_k e^{-z_k} e^{z_k - k\int_0^{t_0}\underline{u}\,d\tau}\right),$$

where $z_k(x_0,t_0) := k\underline{u}(x_0,t_0)^{m-1}$. Since $(x_0,t_0) \in \operatorname{supp}(v_0)\times[0,t^*]$ and

$$z_k - k\int_0^{t_0}\underline{u}\,d\tau = k\int_0^{t_0}\left\{(m-1)\underline{u}^{m-3}\underline{u}_t - 1\right\}\underline{u}\,d\tau,$$

it follows from Lemma 3 that

$$e^{z_k - k\int_0^{t_0}\underline{u}\,d\tau} \leq 1.$$

Thus, we obtain

$$I_2 \geq \underline{u}\left(\delta - v_0 z_k e^{-z_k} e^{z_k - k\int_0^{t_0}\underline{u}\,d\tau}\right) \geq \underline{u}\left(\delta - \frac{v_0}{e}\right) > 0 \tag{18}$$

by virtue of the assumption $\delta > M_v/e$. Moreover, since $\underline{u} - u_k \leq \varepsilon$ on $\overline{\Omega}\times[0,t_0]$, we see that

$$I_3 \geq kv_0\underline{u}^m e^{-k\int_0^{t_0}\underline{u}\,d\tau}\left(1 - e^{k\varepsilon t_0}\right) \geq -kv_0\underline{u}^m(e^{k\varepsilon t_0}-1). \tag{19}$$

By (17), (18) and (19), it can be concluded that

$$I_0 \geq \underline{u}\left(\delta - \frac{v_0}{e} - kv_0\underline{u}^{m-1}(e^{k\varepsilon t_0}-1)\right).$$

Now, we specify $\varepsilon_k$. We assume that $\varepsilon_k$ satisfies

$$\delta - \frac{M_v}{e} > kM_v M_u^{m-1}(e^{k\varepsilon_k t_*} - 1).$$

Then $I_0 > 0$ for any $\varepsilon \in (0, \varepsilon_k]$, and we have (16) again. This contradicts the assumption (15). Thus, if $\varepsilon \in (0, \varepsilon_k]$, then there exists no point $(x_0, t_0)$ satisfying (15) in $(\mathrm{supp}\,(v_0) \cap \Omega) \times (0, t^*]$. Therefore, we know that $W = u_k - \underline{u} + \varepsilon > 0$ in $Q_{t^*}$ when $\varepsilon \in (0, \varepsilon_k]$. Letting $\varepsilon$ go to zero, we get the estimate $u_k \geq \underline{u}$. By the continuity of $\underline{u}$ and $u_k$, we obtain the desired estimate (13) in $\overline{\Omega} \times [0, t^*]$. $\square$

## 2.2 Proof of Theorem 1

We first show that $u_k$ satisfies (7). Let us denote by $\bar{u}$ the solution of the heat equation (9). It follows from the comparison principle that $u_k \leq \overline{u}$ in $\overline{Q}_T$. To ensure that $u_k$ converges to $\overline{u}$, we construct a family $\{U_k\}$ satisfying

$$U_k \leq u_k \quad \text{in } \overline{Q}_T \tag{20}$$

for sufficiently large $k$, and

$$U_k \to \overline{u} \quad \text{in } C(\overline{Q}_T) \tag{21}$$

as $k$ tends to infinity.

In order to accomplish the purpose, we confirm that the reaction term of (10) converges to zero as $k$ tends to infinity. If $x$ does not belong to $\mathrm{supp}\,(v_0)$, then we see at once that

$$kv_0(x)u_k(x,t)^m e^{-k\int_0^t u_k(x,\tau)\,d\tau} = 0$$

for $t \in [0, T]$ and $k > 0$. Suppose that $x$ belongs to $\mathrm{supp}\,(v_0)$. Using the inequality $s^2 e^{-s} \leq 4/e^2$ for $s \geq 0$, we have

$$kv_0(x)u_k(x,t)^m e^{-k\int_0^t u_k(x,\tau)\,d\tau} \leq kv_0(x)u_k(x,t)^m \frac{4}{\left(ke\int_0^t u_k(x,\tau)\,d\tau\right)^2}. \tag{22}$$

Define

$$\gamma(t) := \int_0^t \min_{x\in\overline{\Omega}} \underline{u}_\delta \, d\tau,$$

where $\underline{u}_\delta$ is the solution of (11) with $\delta > M_v/e$. It follows from Lemma 4 and (22) that

$$kv_0 u_k^m e^{-k\int_0^t u_k\,d\tau} \le \frac{4M_v M_u^{m-1}}{ke^2\gamma(t)^2} u_k \tag{23}$$

for $t \in (0, t^*]$. Let $k^*$ be a positive constant satisfying

$$\gamma(t^*) = (k^*)^{-1/4}.$$

By $\gamma(0) = 0$ and the continuity of $\gamma(t)$, there exists a time $t_k \in (0, t^*)$ such that

$$\gamma(t_k) = k^{-1/4}$$

for any $k > k^*$. We note here that $t_k$ converges to zero as $k$ tends to infinity, because $\min_{x\in\overline{\Omega}} \underline{u}_\delta$ is positive for $t > 0$. Hereafter we assume that $k > k^*$. Hence, the inequality (23) and the monotonicity of $\gamma$ ensure

$$kv_0 u_k^m e^{-k\int_0^t u_k\,d\tau} \le \frac{4M_v M_u^{m-1}}{k^{1/2}e^2} u_k \tag{24}$$

for $t \in [t_k, t^*]$. From the inequality (22) and Lemma 4, we have

$$kv_0(x)u_k(x,t)^m e^{-k\int_0^t u_k(x,\tau)\,d\tau} \le \frac{4M_v M_u^{m-1}}{k\left(e\int_0^{t^*} \underline{u}_\delta(x,\tau;u_0)\,d\tau\right)^2} u_k \le \frac{4M_v M_u^{m-1}}{ke^2\gamma(t^*)^2} u_k \le \frac{4M_v M_u^{m-1}}{k^{1/2}e^2} u_k$$

for $t \in [t^*, T]$. Therefore, we conclude that (24) holds in $\Omega \times [t_k, T]$.

Now, we construct a family $\{U_k\}$ with properties (20) and (21). Let $\delta_1$ be a positive constant larger than $M_v/e$. It follows from Lemma 4 that

$$u_k(x,t) \ge \underline{u}_{\delta_1}(x,t;u_0) \tag{25}$$

for any $(x,t) \in \overline{Q}_{t^*}$. Set $\delta_2 = 4M_v M_u^{m-1}/(k^{1/2}e^2)$. By the inequality (24), $\underline{u}_{\delta_2}(x,t;\,\underline{u}_{\delta_1}(\cdot,t_k;u_0))$ satisfies

$$u_k(x,t) \ge \underline{u}_{\delta_2}(x,t-t_k;\,\underline{u}_{\delta_1}(\cdot,t_k;u_0)) \tag{26}$$

for $(x,t) \in \overline{\Omega} \times [t_k, T]$. We recall that $t_k < t^*$ under the condition $k > k^*$. We define $U_k$ as follows:

$$U_k(x,t) := \begin{cases} \underline{u}_{\delta_1}(x,t;u_0) & (0 \le t \le t_k), \\ \underline{u}_{\delta_2}(x,t-t_k;\,\underline{u}_{\delta_1}(\cdot,t_k;u_0)) & (t_k < t \le T). \end{cases} \tag{27}$$

Combining (25) with (26), we have

$$u_k \geq U_k \quad \text{in} \quad \overline{Q}_T.$$

Taking $u_k \leq \overline{u}$ into account, we get the desired estimate $U_k \leq u_k \leq \overline{u}$ in $\overline{Q}_T$. Notice that $W = \overline{u} - U_k$ satisfies $W \geq 0$ in $Q_T$ and

$$\begin{cases} W_t = \Delta W + \delta_1 \underline{u}_{\delta_1} & \text{in } Q_{t_k}, \\ \dfrac{\partial W}{\partial n} = 0 & \text{on } S_{t_k}. \end{cases}$$

Let $\overline{W}$ be the solution of $\overline{W}_t = \delta_1 M_u$ in $[0, t_k]$ and $\overline{W}(0) = \max_{\overline{\Omega}} W(\cdot, 0) = 0$. By the weak maximum principle, we know that $W(\cdot, t) \leq \overline{W}(\cdot, t) = \max_{\overline{\Omega}} W(\cdot, 0) + \delta_1 M_u t = \delta_1 M_u t$ in $\Omega$ for each time $t \in [0, t_k]$. Repeating this argument on $[t_k, T]$, we can deduce that $W(\cdot, t) \leq \max_{\overline{\Omega}} W(\cdot, t_k) + \delta_2 M_u (t - t_k)$ in $\Omega$ for any $t \in [t_k, T]$. Thus, we obtain

$$\begin{aligned} \max_{\overline{Q}_T} |\overline{u} - U_k| &\leq \max_{(x,t) \in \overline{\Omega} \times [0, t_k]} \left| \overline{u}(x,t) - \underline{u}_{\delta_1}(x,t) \right| + \max_{(x,t) \in \overline{\Omega} \times [t_k, T]} \left| \overline{u}(x,t) - \underline{u}_{\delta_2}(x, t - t_k) \right| \\ &\leq (2 t_k \delta_1 + T \delta_2) M_u. \end{aligned} \tag{28}$$

Since the right-hand side of (28) converges to zero as $k$ tends to infinity, we see that

$$u_k \to \overline{u} \quad \text{in} \quad C(\overline{Q}_T)$$

as $k$ tends to infinity.

Finally we show that $v_k$ satisfies (8). It follows from (25) that

$$0 \leq v_k = v_0 e^{-k \int_0^t u_k \, d\tau} \leq v_0 e^{-k \int_0^{\min\{t^*, t\}} u_k \, d\tau} \leq v_0 e^{-k \int_0^{\min\{t^*, t\}} \underline{u}_{\delta_1} \, d\tau}. \tag{29}$$

Thus, the right-hand side converges to zero because $\underline{u}_{\delta_1}$ is positive for any $t > 0$ and independent of $k$. Therefore, we complete the proof of Theorem 1. □

**Remark 1.** *In this section, we treated Case I with $m_1 > 3$. Considering the dynamics explained in Section 1, we expect that we obtain the same results even if $2 < m_1 \leq 3$. However, we can not apply the proof of Theorem 1 to the case with $2 < m_1 \leq 3$ because our proof relies on the construction of $U_k$ in* (27).

# 3 Singular limits of $(\mathrm{P})_k$ in Cases II and IV

In this section, we treat Problems $(\mathrm{P})_k$ in Case II with $m_2 \geq 1$, say $(\mathrm{P}_{\mathrm{II}})_k$, and Case IV with $1 \leq m_4 < 2$, say $(\mathrm{P}_{\mathrm{IV}})_k$. We can easily derive the limit problems of $(\mathrm{P}_{\mathrm{II}})_k$ and $(\mathrm{P}_{\mathrm{IV}})_k$ as $k$ tends to infinity by change of variables. Let $(u_k, v_k)$ be the solution of $(\mathrm{P}_{\mathrm{II}})_k$ (resp. $(\mathrm{P}_{\mathrm{IV}})_k$) and put $w_k := v_k^{m_2}$ (resp. $w_k := v_k^{2-m_4}$). Then $(u_k, w_k)$ solves the following problem with balanced fast reaction pair:

$$\begin{cases} u_t = \Delta u - kuw^\ell & \text{in } Q_T, \\ w_t = -\mu kuw^\ell & \text{in } Q_T, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } S_T, \\ u(\cdot,0) = u_0, \quad w(\cdot,0) = w_0 & \text{in } \Omega. \end{cases} \tag{30}$$

Here, $(\ell, \mu) = (1, m_2)$ in Case II and $(\ell, \mu) = (1/(2-m_4), 2-m_4)$ in Case IV, and $w_0 := v_0^\mu$. As mentioned in Section 1, the fast reaction limit of Problem (30) is well-known. Therefore, we can immediately establish the limit problems of $(\mathrm{P}_{\mathrm{II}})_k$ and $(\mathrm{P}_{\mathrm{IV}})_k$.

Let us summarise the results for (30).

**Theorem 5.** *Assume that* (H1) *holds, where we replace $v_0$ with $w_0$. Let $(u_k, w_k)$ be the solution of* (30) *with $\ell \geq 1$. Then there exists a function $z$ such that*

$$\begin{aligned} & z \in L^\infty(Q_T), \quad z_+ \in L^2(0,T;H^1(\Omega)), \\ & -M_v \leq z \leq \mu M_u \quad \text{a.e. in } Q_T, \\ & u_k \to z_+/\mu \quad \text{strongly in } L^2(Q_T) \text{ and weakly in } L^2(0,T;H^1(\Omega)), \\ & w_k \to z_- \quad \text{strongly in } L^2(Q_T) \end{aligned}$$

*as $k$ tends to infinity. Here $s_\pm := \max\{0, \pm s\}$ for $s \in \mathbb{R}$. Moreover, $z$ satisfy*

$$-\int_\Omega (\mu u_0 - w_0)\varphi(0)dx + \iint_{Q_T} \{-z\varphi_t + \nabla z_+ \cdot \nabla\varphi\}\, dxdt = 0 \tag{31}$$

*for all functions $\varphi \in H^1(Q_T)$ such that $\varphi(\cdot, T) = 0$.*

The relation (31) is known as a weak formulation of the classical Stefan problem, that is, the limit functions of $u_{k_n}$ and $w_{k_n}$ are represented by the weak solution of the Stefan problem. The Stefan problem is well-known as a free boundary problem. We rewrite Problem (31) as an explicit free boundary problem. To this end, we introduce some notation using

$$u := z_+/\mu \quad \text{and} \quad w := z_-, \tag{32}$$

where $z_+$ and $z_-$ are the functions defined in Theorem 5, as follows:

$$\begin{aligned}
&\Omega^u(t) := \{x \in \Omega \mid u(x,t) > 0\}, \quad \Omega^w(t) := \{x \in \Omega \mid w(x,t) > 0\},\\
&Q_T^u := \bigcup_{0<t\le T} \Omega^u(t) \times \{t\}, \quad Q_T^w := \bigcup_{0<t\le T} \Omega^w(t) \times \{t\},\\
&\Gamma(t) := \Omega \setminus (\Omega^u(t) \cup \Omega^w(t)), \quad \Gamma := \bigcup_{0\le t\le T} \Gamma(t) \times \{t\}.
\end{aligned}$$

The definitions of $u$ and $w$ immediately imply $Q_T^u \cap Q_T^w = \emptyset$. We remark that

$$\begin{aligned}
&u = 0 \quad \text{in } Q_T^w, \quad w = 0 \quad \text{in } Q_T^u,\\
&u_k \to u \quad \text{strongly in } L^2(Q_T) \text{ and weakly in } L^2(0,T;H^1(\Omega)),\\
&w_k \to w \quad \text{strongly in } L^2(Q_T)
\end{aligned}$$

by Theorem 5.

**Theorem 6.** *Assume that* (H1) *is satisfied. Let $u$ and $w$ be the functions defined in* (32). *Suppose that $\Gamma(t)$ is a smooth closed orientable hypersurface satisfying $\Gamma(t) \cap \partial\Omega = \emptyset$ for all $t \in [0,T]$ and that $\Gamma(t)$ moves smoothly. Also suppose that the functions $u$ and $w$ are smooth in $\overline{Q_T^u}$ and $\overline{Q_T^w}$, respectively. Then $w(x,t) = w_0(x)$ holds true for $(x,t) \in \overline{Q_T^w}$. Moreover, the function $u$ in $\overline{Q_T^u}$ satisfies the following free boundary problem:*

$$\begin{cases}
u_t = \Delta u & \text{in } Q_T^u,\\
\dfrac{w_0}{\mu} V = -\dfrac{\partial u}{\partial n_\Gamma} & \text{on } \Gamma,\\
u = 0 & \text{on } \Gamma,\\
\dfrac{\partial u}{\partial n} = 0 & \text{on } S_T,\\
u(\cdot,0) = u_0 & \text{in } \Omega^u(0),
\end{cases} \tag{33}$$

*where $n_\Gamma$ is the unit normal vector on $\Gamma(t)$ oriented from $\Omega^u(t)$ to $\Omega^w(t)$, and $V$ is the normal speed of the free boundary $\Gamma(t)$.*

We note that the value of $w_0$ in the second equation of (33) should be taken as the limit from $\Omega^w(t)$ and $\partial u/\partial n_\Gamma(x,t) = \lim_{h\to+0}(u(x,t) - u(x - hn_\Gamma, t))/h$ for $x \in \Gamma(t)$ and $t \in [0,T]$. We omit the proofs of Theorems 5 and 6 because these are proved in a manner analogous to the proofs in [8, 9, 14].

Problem (33) corresponds to the classical formulation of the one-phase Stefan problem which is a typical model of ice-water phase transition problems. Here the latent heat coefficient coincides with $w_0|_{\Gamma(t)}/\mu$.

Summarising the above discussion, we conclude that the limit function $u$ of the solutions $u_k$ to $(\mathrm{P}_{\mathrm{II}})_k$ (resp. $(\mathrm{P}_{\mathrm{IV}})_k$) solves the one-phase Stefan problem with a latent heat $v_0^{m_2}|_{\Gamma(t)}/m_2$ (resp. $v_0^{2-m_4}|_{\Gamma(t)}/(2-m_4)$). Thus, the free boundary moves with finite speed as in (33) in Case II with $m_2 \geq 1$ and Case IV with $1 \leq m_4 < 2$.

# 4 Singular limit of $(\mathrm{P})_k$ in Case III

This section is devoted to Problem $(\mathrm{P})_k$ in Case III, namely,

$$(\mathrm{P}_{\mathrm{III}})_k \qquad \begin{cases} u_t = \Delta u - kuv & \text{in } Q_T, \\ v_t = -ku^m v & \text{in } Q_T, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } S_T, \\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 & \text{in } \Omega, \end{cases}$$

where $m > 1$. We show that a free boundary appears in the limit problem as $k$ tends to infinity, however, the free boundary does not move. In particular, when we treat the free boundary in the strong form, we impose an additional hypothesis on $v_0$ as follows:

(H3) there exists a positive constant $m_v$ such that $v_0 \geq m_v$ in $\mathrm{supp}\,(v_0)$.

Our convergence result for Case III is as follows:

**Theorem 7.** *Assume that* (H1) *holds. Let* $(u_k, v_k)$ *be the solution of* $(\mathrm{P}_{\mathrm{III}})_k$. *Then there exist subsequences* $\{u_{k_n}\}$ *and* $\{v_{k_n}\}$ *of* $\{u_k\}$ *and* $\{v_k\}$, *respectively, and* $u, v, \eta$ *such that*

$$u, u^{\frac{m}{2}} \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)),\ v \in L^\infty(Q_T),\ \eta \in H^{-1}(Q_T), \tag{34}$$

$$0 \leq u \leq M_u, \quad 0 \leq v \leq M_v, \quad uv = 0 \quad \text{a.e. in } Q_T, \tag{35}$$

$$\eta \geq 0 \quad \text{in } H^{-1}(Q_T), \tag{36}$$

$$u_{k_n} \to u,\ u_{k_n}^{\frac{m}{2}} \to u^{\frac{m}{2}} \quad \textit{strongly in } L^p(Q_T) \textit{ weakly in } L^2(0,T;H^1(\Omega)), \tag{37}$$

$$v_{k_n} \rightharpoonup v \quad \textit{weakly in } L^p(Q_T), \tag{38}$$

$$\left|\nabla u_{k_n}^{\frac{m}{2}}\right|^2 \overset{*}{\rightharpoonup} \eta \quad \textit{weakly} * \textit{ in } H^{-1}(Q_T) \tag{39}$$

*for any* $p \geq 1$ *as* $k_n$ *tends to infinity. Moreover,* $u$*,* $v$ *and* $\eta$ *satisfy*

$$\iint_{Q_T} \left\{ -\left(\frac{1}{m}u^m - v\right)\varphi_t + \frac{2}{m}u^{\frac{m}{2}}\nabla u^{\frac{m}{2}} \cdot \nabla\varphi \right\} dxdt + \frac{4(m-1)}{m^2}\ {}_{H^{-1}(Q_T)}\langle \eta, \varphi \rangle_{H_0^1(Q_T)} = 0 \tag{40}$$

*for all* $\varphi \in H_0^1(Q_T)$*.*

We will prove $\eta = \left|\nabla u^{\frac{m}{2}}\right|^2$ under the additional condition (H3). In order to give an explicit equation of motion for the free boundary, we define $\Omega^u(t)$, $\Omega^v(t)$, $Q_T^u$, $Q_T^v$, $\Gamma(t)$ and $\Gamma$ using $u$ and $v$ defined in Theorem 7 similarly to Section 3.

**Theorem 8.** *Assume* (H1) *and* (H3)*. Let* $u, v$ *and* $\eta$ *be the functions satisfying* (34)–(40)*. Suppose that* $\Gamma(t)$ *is a smooth, closed and orientable hypersurface satisfying* $\Gamma(t) \cap \partial\Omega = \emptyset$ *for all* $t \in [0,T]$ *and that* $\Gamma(t)$ *smoothly moves with a normal speed* $V$ *from* $\Omega^u(t)$ *to* $\Omega^v(t)$*. Also suppose that* $u$ *(resp.* $v$*) is smooth in* $\overline{Q_T^u}$ *(resp.* $\overline{Q_T^v}$*) and that* $\eta \in L^1_{\rm loc}(Q_T)$*. Then the following relations hold.*

$$V \equiv 0 \textit{ on } \Gamma, \textit{ that is, } \Omega^u(t) \equiv \Omega^u(0),\ \Omega^v(t) \equiv \Omega^v(0),\ \Gamma(t) \equiv \Gamma(0),$$

$$\begin{array}{ll} u_t = \Delta u & \textit{in } \Omega^u(0) \times (0,T], \\ u = 0 & \textit{on } \Gamma(0) \times (0,T], \\ v = v_0, \quad \eta = \left|\nabla u^{\frac{m}{2}}\right|^2 & \textit{in } Q_T. \end{array}$$

This result indicates that the initial interface $\Gamma(0)$ does not move. Thus, $u$ can not invade the region occupied by $v$, and hence $u$ solves the heat equation in the fixed domain $\Omega^u(0)$ with the homogeneous Dirichlet boundary condition on $\Gamma(0)$.

## 4.1 A priori estimates for the solution of $(\mathrm{P}_{\mathrm{III}})_k$

We establish a priori estimates for the solution $(u_k, v_k)$ of $(\mathrm{P}_{\mathrm{III}})_k$.

**Lemma 9.** *There exist positive constants $C_1, C_2, C_3$ and $C_4$ independent of $k$ such that*

$$\|u_k v_k\|_{L^1(Q_T)} \leq \frac{C_1}{k}, \tag{41}$$

$$\|u_k\|_{L^2(0,T;H^1(\Omega))} \leq C_2, \tag{42}$$

$$\left\|u_k^{\frac{m}{2}}\right\|_{L^2(0,T;H^1(\Omega))} \leq C_3, \tag{43}$$

$$\left\|\left|\nabla u_k^{\frac{m}{2}}\right|^2\right\|_{H^{-1}(Q_T)} \leq C_4.$$

*Proof.* Integrating the equation for $u_k$ over $Q_T$ and using (6) yield the estimate (41). Multiplying the equation for $u_k$ by $u_k$ (resp. $u_k^{m-1}$) and integrating by parts, we obtain (42) (resp. (43)). It follows from the equations for $u_k$ and $v_k$ that

$$\left(\frac{1}{m}u_k^m - v_k\right)_t - u_k^{m-1}\Delta u_k = 0.$$

Multiplying it by a test function $\varphi \in H_0^1(Q_T)$ and integrating by parts, we have

$$\iint_{Q_T} \left\{-\left(\frac{1}{m}u_k^m - v_k\right)\varphi_t + \frac{4(m-1)}{m^2}\left|\nabla u_k^{\frac{m}{2}}\right|^2 \varphi + \frac{2}{m}u_k^{\frac{m}{2}}\nabla u_k^{\frac{m}{2}} \cdot \nabla\varphi\right\} dxdt = 0. \tag{44}$$

By virtue of (6), (43) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\iint_{Q_T}\left|\nabla u_k^{\frac{m}{2}}\right|^2 \varphi dxdt\right| &\leq \frac{m^2}{4(m-1)}\left\|\frac{1}{m}u_k^m - v_k\right\|_{L^2(Q_T)}\|\varphi_t\|_{L^2(Q_T)} + \frac{mM_u^{\frac{m}{2}}}{2(m-1)}\left\|\nabla u_k^{\frac{m}{2}}\right\|_{L^2(Q_T)}\|\nabla\varphi\|_{L^2(Q_T)} \\ &\leq C_4\|\varphi\|_{H_0^1(Q_T)}\end{aligned}$$

for all $\varphi \in H_0^1(Q_T)$, which completes the proof. $\square$

We now turn to the spatio-temporal-shift estimates.

**Lemma 10.** *There exist positive constants $C_5$ and $C_6$ independent of $k$ such that*

$$\int_0^T\int_{\Omega_\xi} |u_k(x+\xi,t) - u_k(x,t)|^p \, dxdt \leq C_5|\xi|^2, \tag{45}$$

$$\int_0^{T-\tau}\int_{\Omega} |u_k(x,t+\tau) - u_k(x,t)|^p \, dxdt \leq C_6\tau \tag{46}$$

*for all* $p \geq 2$, $\xi \in \mathbb{R}^N$ *and* $\tau \in (0,T)$. *Here,* $\Omega_\xi := \{x \in \Omega \mid x + r\xi \in \Omega \ \text{ for } \ 0 \leq r \leq 1\}$.

*Proof.* In a similar fashion to the proof in [11], the following estimate holds:

$$\int_0^T \int_{\Omega_\xi} |u_k(x+\xi,t) - u_k(x,t)|^2 \, dxdt \leq {C_2}^2 |\xi|^2$$

for all $\xi \in \mathbb{R}^N$. Since $u_k$ is uniformly bounded in $L^\infty(Q_T)$ with respect to $k$, we obtain (45). Similarly, (46) holds. Thus, the proof is analogous to that in [11]. $\square$

## 4.2 Proof of Theorem 7

In view of (6), Lemmas 9, 10, and the Kolmogorov–M. Riesz–Fréchet theorem [2, Theorem IV.25 and Corollary IV.26], there exist subsequences $\{u_{k_n}\}$, $\{v_{k_n}\}$ of $\{u_k\}$, $\{v_k\}$ and $u, v, \eta$ such that (34)–(39) are satisfied. The equation (44) holds for an arbitrary function $\varphi \in H_0^1(Q_T) \cap L^\infty(0,T; W^{1,\infty}(\Omega))$. Passing to the limit in $k$ along the subsequences, we obtain (40) for all $\varphi \in H_0^1(Q_T) \cap L^\infty(0,T;\, W^{1,\infty}(\Omega))$. Since $H_0^1(Q_T) \cap L^\infty(0,T;\, W^{1,\infty}(\Omega))$ is dense in $H_0^1(Q_T)$, we complete the proof. $\square$

## 4.3 Proof of Theorem 8

*Step 1* (*Equation on* $\Gamma$).

We deduce from (40) and the definitions of $Q_T^u$ and $Q_T^v$ that

$$\begin{aligned} -\iint_{Q_T^u} \frac{1}{m} u^m \varphi_t \, dxdt + \frac{2}{m} \iint_{Q_T^u} u^{\frac{m}{2}} \nabla u^{\frac{m}{2}} \cdot \nabla \varphi \, dxdt + \iint_{Q_T^v} v \varphi_t \, dxdt \\ + \frac{4(m-1)}{m^2} \left( \iint_{Q_T^u} \eta \varphi \, dxdt + \iint_{Q_T^v} \eta \varphi \, dxdt \right) = 0 \end{aligned} \tag{47}$$

for all $\varphi \in C_0^\infty(Q_T)$. The fact that $u \in L^2(0,T;\, H^1(\Omega))$ and $u = 0$ in $Q_T^v$ leads us to

$$u = 0 \quad \text{on } \Gamma(t).$$

Since $u$ and $v$ are smooth in $\overline{Q_T^u}$ and $\overline{Q_T^v}$, respectively, we obtain

$$
\begin{aligned}
&-\frac{1}{m}\iint_{Q_T^u} u^m \varphi_t \, dxdt = \iint_{Q_T^u} u^{m-1} u_t \varphi \, dxdt, \\
&\iint_{Q_T^v} v\varphi_t \, dxdt = -\iint_{Q_T^v} v_t \varphi \, dxdt + \int_0^T \int_{\Gamma(t)} \tilde{v} V \varphi \, dxdt, \\
&\frac{2}{m}\iint_{Q_T^u} u^{\frac{m}{2}} \nabla u^{\frac{m}{2}} \cdot \nabla\varphi \, dxdt = -\iint_{Q_T^u} \left( u^{m-1}\Delta u + \frac{4(m-1)}{m^2} \left|\nabla u^{\frac{m}{2}}\right|^2 \right) \varphi \, dxdt,
\end{aligned}
$$

where $\tilde{v} = \tilde{v}(\cdot, t)$ denotes the boundary value of $v$ on $\partial\Omega^v(t)$ for $t \in [0, T]$. Therefore, we have

$$
\begin{aligned}
&\iint_{Q_T^u} \left\{ u^{m-1}(u_t - \Delta u) - \frac{4(m-1)}{m^2} \left( \left|\nabla u^{\frac{m}{2}}\right|^2 - \eta \right) \right\} \varphi \, dxdt \\
&\quad + \iint_{Q_T^v} \left\{ -v_t + \frac{4(m-1)}{m^2} \eta \right\} \varphi \, dxdt + \int_0^T \int_{\Gamma(t)} \tilde{v} V \varphi \, dxdt = 0
\end{aligned}
\tag{48}
$$

for all $\varphi \in C_0^\infty(Q_T)$. This implies

$$
\tilde{v} V = 0 \quad \text{on } \Gamma. \tag{49}
$$

*Step 2* (*The interface does not move*).

It follows from the equation for $v_k$ in $(\mathrm{P}_{\mathrm{III}})_k$ that

$$
\iint_{Q_T^v} v_k \varphi_t \, dxdt = k \iint_{Q_T^v} u_k^m v_k \varphi \, dxdt \geq 0, \quad \iint_{Q_T^v} \left|\nabla u_k^{\frac{m}{2}}\right|^2 \varphi \, dxdt \geq 0
$$

for all $k > 0$ and $\varphi \in C_{0+}^\infty(Q_T^v) := \{\varphi \in C_0^\infty(Q_T^v) \mid \varphi \geq 0\}$. Passing to the limit in $k$ along the subsequences, we have

$$
\iint_{Q_T^v} v\varphi_t \, dxdt \geq 0, \quad \iint_{Q_T^v} \eta\varphi \, dxdt \geq 0. \tag{50}
$$

On the other hand, (47) leads to

$$
\iint_{Q_T^v} v\varphi_t \, dxdt + \frac{4(m-1)}{m^2} \iint_{Q_T^v} \eta\varphi \, dxdt = 0 \tag{51}
$$

for any $\varphi \in C_{0+}^\infty(Q_T^v)$. Combining (50) and (51), we obtain

$$
\iint_{Q_T^v} \eta\varphi \, dxdt = 0 \tag{52}
$$

for all $\varphi \in C^\infty_{0+}(Q^v_T)$. In particular, $\eta = 0$ holds in $Q^v_T$. Therefore, (48) implies $v_t = 0$ in $Q^v_T$. Hence, $v \equiv v_0$ in $\overline{Q^v_T}$ since $\Omega^v(t) \subset \Omega^v(0)$ $(t > 0)$ and we assumed that $v$ is smooth in $\overline{Q^v_T}$. Especially, (H3) implies that $\tilde{v} \geq m_v > 0$ on $\Gamma$. Thereby, in view of (49), we have

$$V = 0 \quad \text{on } \Gamma.$$

This means that

$$\Omega^u(t) = \Omega^u(0), \quad \Omega^v(t) = \Omega^v(0), \quad \Gamma(t) = \Gamma(0) \quad \text{for } t \in (0, T].$$

*Step 3* (*Equation for u*).

We deduce from the equation for $u_k$ that

$$\int_0^T \int_{\Omega^u(0)} (u_{kt} - \Delta u_k)\varphi \, dxdt = -k \int_0^T \int_{\Omega^u(0)} u_k v_k \varphi \, dxdt$$

for all $\varphi \in C_0^\infty(\Omega^u(0) \times (0, T))$. Since the right hand side is zero because $\operatorname{supp}(v_k) \subset \operatorname{supp}(v_0)$, we have

$$\int_0^T \int_{\Omega^u(0)} (-u_k \varphi_t + \nabla u_k \cdot \nabla \varphi) \, dxdt = 0.$$

Tending to the limit in $k$ along the subsequence and integrating by parts yield

$$\int_0^T \int_{\Omega^u(0)} (u_t - \Delta u)\varphi \, dxdt = 0.$$

Thus, $u$ satisfies the heat equation, namely,

$$u_t = \Delta u \quad \text{in } Q^u_T = \Omega^u(0) \times (0, T].$$

It follows from (48) and (52) that

$$\iint_{Q_T} \left( \eta - \left| \nabla u^{\frac{m}{2}} \right|^2 \right) \varphi \, dxdt = 0$$

for all $\varphi \in C_0^\infty(Q_T)$. Hence, we have $\eta = \left| \nabla u^{\frac{m}{2}} \right|^2$. Therefore, the proof is complete. $\square$

**Remark 2.** *In this paper, we investigated four basic cases. We can extend our results to slightly more general cases by appropriate transformations. For instance, let us consider* $(\mathrm{P})_k$ *with* $\mathbf{m} = (m_1, m_2, 1, 1)$*, where* $m_1 > 1$ *and* $m_2 > 1$*. By a similar argument to that in Section 3, the problem can be transformed into one of the type treated in Case I. Hence, a similar conclusion to Theorem 1 is ensured for* $(\mathrm{P})_k$ *with* $\mathbf{m} = (m_1, m_2, 1, 1)$*,* $m_1 > 3,\ m_2 > 1$*. Similarly, we can verify that* $(\mathrm{P})_k$ *with* $\mathbf{m} = (1, m_2, m_3, 1)$ *(*$m_2, m_3 > 1$*) is transformed into a type of Case III. Hence, similar results to Theorems 7 and 8 hold for* $(\mathrm{P})_k$ *with* $\mathbf{m} = (1, m_2, m_3, 1)$*.*


# Acknowledgements

This work was partially supported by JSPS KAKENHI Grant nos. 26287024, 26287025, 26400205, 15H03635 and 15K04963, and JST CREST, Research Area: Modeling Methods Allied with Modern Mathematics (Research Supervisor: Takashi Tsuboi), Project: Theory on Mathematical Modeling for Spatio-Temporal Patterns Arising in Biology (Research Director: Shin-Ichiro Ei). The authors would like to thank the referees for their carefully reading of our manuscript and their valuable comments.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