ENTIRE SOLUTIONS ORIGINATING FROM MONOTONE FRONTS TO THE ALLEN-CAHN EQUATION

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ABSTRACT. In this paper, we study entire solutions of the Allen-Cahn equation in one-dimensional Euclidean space. This equation is a scalar reaction-diffusion equation with a bistable nonlinearity. It is well-known that this equation admits three different types of traveling fronts connecting two of its three constant states. Under certain conditions on the wave speeds, the existence of entire solutions originating from three and four fronts is shown by constructing some suitable pairs of super-sub-solutions. Moreover, we show that there are no entire solutions originating from more than four fronts.

1. Introduction

In this paper, we consider the following reaction-diffusion equation

\[ u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, \ t \in \mathbb{R}, \]

where the function \( f(u) \in C^2(\mathbb{R}) \) satisfies

\[ f(0) = f(1) = 0, \quad f'(0), f'(1) < 0, \]

\[ f(a) = 0, \quad f'(a) > 0, \quad a \in (0, 1), \quad f(u) \neq 0 \text{ for } u \in (0, a) \cup (a, 1), \]

\[ \int_0^1 f(u) \, du > 0. \]

A typical example of \( f \) is \( f(u) = u(1-u)(u-a) \), where \( a \in (0, 1/2) \). This equation is often called the Allen-Cahn equation or the Nagumo equation. It is easy to see that the constant states \( u = 0 \) and \( u = 1 \) are stable and the constant state \( u = a \) is unstable for the kinetic equation (i.e., (1.1) without diffusion term), since \( f'(0) < 0, f'(1) < 0 \) and \( f'(a) > 0 \).

Due to the rich dynamics of this prototype equation (1.1), there have been a lot of research on the dynamical behaviors of (1.1). One of the main concerns on the dynamics of (1.1) is the existence of entire solutions. Here an entire solution means a classical solution defined for all \((x, t) \in \mathbb{R}^2\). One of typical examples of entire solutions is the traveling wave solution. A solution \( u \) of (1.1) is called a traveling wave solution, if \( u(x, t) = \Phi(x + vt) \) for some constant \( v \) (the wave speed) and some function \( \Phi \) (the wave profile). A traveling wave solution is called
a traveling front, if it connects two different constant states. In fact, (1.1) admits three different kinds of traveling fronts connecting states \( \{0, 1\} \), \( \{0, a\} \), \( \{a, 1\} \), respectively. The first one is the bistable connection and the latter two cases are the monostable connections. For the reader’s convenience, we recall in the followings the traveling front connecting states \( \{0, 1\} \), \( \{0, a\} \), \( \{a, 1\} \), resp.) with wave profile denoted by \( \psi_0 \), \( (\psi_1, \psi_2, \text{resp.}) \) and admissible wave speed denoted by \( v_0 \), \( (v_1, v_2, \text{resp.}) \).

By [1, 2], there exists a unique (up to translations) traveling front \( u(x, t) = \psi_0(x + v_0 t) \) of (1.1) connecting \( \{0, 1\} \) with the unique speed \( v_0 \). Note that, by setting \( z = x + v_0 t \), \( \psi_0 \) satisfies
\[
\psi_0''(z) - v_0 \psi_0'(z) + f(\psi_0(z)) = 0, \quad \psi_0'(z) > 0, \quad z \in \mathbb{R},
\]
\[
\psi_0(-\infty) = 0, \quad \psi_0(\infty) = 1.
\]
and the speed \( v_0 \) is given by
\[
v_0 = \frac{\int_0^1 f(\psi_0(t)) \, d\psi_0}{\int_{-\infty}^\infty (\psi_0'(z))^2 \, dz} > 0.
\]

By [1, 13], there exists a constant \( c_{1,\text{max}} \leq -2\sqrt{f'(a)} \) such that a traveling front \( u(x, t) = \psi_1(x + v_1 t) \) of (1.1) connecting \( \{0, a\} \) with speed \( v_1 \) exists for each \( v_1 \leq c_{1,\text{max}} \). Set \( z = x + v_1 t \). Then \( \psi_1(z) \) satisfies
\[
\psi_1''(z) - v_1 \psi_1'(z) + f(\psi_1(z)) = 0, \quad \psi_1'(z) > 0, \quad z \in \mathbb{R},
\]
\[
\psi_1(-\infty) = 0, \quad \psi_1(\infty) = a.
\]

Similarly, there exists a constant \( c_{2,\text{min}} \geq 2\sqrt{f'(a)} \) such that a traveling front \( u(x, t) = \psi_2(x + v_2 t) \) of (1.1) connecting \( \{a, 1\} \) with speed \( v_2 \) exists for each \( v_2 \geq c_{2,\text{min}} \). Set \( z = x + v_2 t \). Then \( \psi_2(z) \) satisfies
\[
\psi_2''(z) - v_2 \psi_2'(z) + f(\psi_2(z)) = 0, \quad \psi_2'(z) > 0, \quad z \in \mathbb{R},
\]
\[
\psi_2(-\infty) = a, \quad \psi_2(\infty) = 1.
\]

Note that \( v_0 > 0 \), \( v_1 < 0 < v_2 \), and \( 0 < \psi_0 < 1 \), \( 0 < \psi_1 < a \), \( a < \psi_2 < 1 \) in \( \mathbb{R} \).

In 1999, Hamel and Nadirashvili [11] constructed a new type of entire solutions originating from two fronts (at \( t = -\infty \)) for the Fisher-KPP equation (see also [12]). Since then, there have been many works devoted to the construction of entire solutions originating from two fronts for the scalar reaction-diffusion equations (see, e.g., [19, 8, 10, 3, 4, 14]). In particular, Yagisita [19] derived the existence of entire solutions which behave as two traveling fronts \( \psi_0(-x + ct) \) and \( \psi_0(x + ct) \) on the left \( x \)-axis and right \( x \)-axis as \( t \to -\infty \), respectively. Then, Fukao, Morita and Ninomiya [8] provided a simple proof for the results shown in [19] for the Allen-Cahn equation.

For the function \( f(u) \) satisfying (1.2), (1.3), according to the results shown in [11, 10], for any \( c_{11}, c_{12} \leq c_{1,\text{max}} \), there exists an entire solution of (1.1) which converges to \( \psi_1(x + c_{11} t) \) and \( \psi_1(-x + c_{12} t) \) on the left \( x \)-axis and right \( x \)-axis, respectively, as \( t \to -\infty \). Similarly, the existence of an entire solution of (1.1) which converges to \( \psi_2(-x + c_{21} t) \) and \( \psi_2(x + c_{22} t) \) on the
left $x$-axis and right $x$-axis, respectively, as $t \to -\infty$ can be shown for any $c_{21}, c_{22} \geq c_{2,\text{min}}.$

Later, in [14], Morita and Ninomiya proposed a unified method to construct all types of entire solutions including entire solutions with two merging fronts mentioned above.

Extending these works to multiple fronts, we shall study entire solutions $u$ originating from $k$ fronts $\{(c_j, \phi_j), j = 1, 2, \cdots, k\}$ ($k \geq 2$) satisfying the condition

$$ c_1 < c_2 < \cdots < c_k $$

such that

$$ \lim_{t \to -\infty} \left\{ \sum_{1 \leq j \leq k} \sup_{w_{j-1}(t) < x < w_j(t)} |u(x, t) - \phi_j(x + c_j t + \theta_j)| \right\} = 0 $$

for some constants $\theta_1, \cdots, \theta_k$, where $w_j(t) := -(c_j + c_{j+1})t/2$, $w_0(t) := -\infty$ and $w_k(t) := \infty$.

Note that the condition (1.6) is quite natural, since two adjacent waves must intersect at some negative time, if $c_j > c_{j+1}$ for some $j \in \{1, \cdots, k-1\}$. We do not consider the case when $c_j = c_{j+1}$ for some $j$. It is a delicate case which is left for open. Also, in this paper, the new terminology “originating” is used, since we mainly focus on the behavior at $t = -\infty$.

Entire solutions originating from two fronts may be merging to a single front or a constant state (which is called annihilating) as $t \to \infty$.

For an entire solution $u$ originating from $k$ fronts, there is the sequence $\{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\}$ satisfying $\phi_j(-\infty) = \alpha_j$, $\phi_j(\infty) = \omega_j$ for $j = 1, \cdots, k$. We call it the sequence of $u$. Due to the continuity of entire solutions, we have $\alpha_{j+1} = \omega_j$ for $j = 1, \cdots, k-1$.

We can easily check that the following sequences

$$ \{0, 1, 1, 0\}, \{0, 1, 1, a\}, \{a, 0, 0, a\}, \{a, 1, 1, a\}, \{a, 1, 1, 0\} $$

cannot be the sequences of entire solutions originating from two fronts. For example, for the case $\{0, 1, 1, 0\}$, the speeds of the corresponding traveling fronts satisfy $c_1 > 0 > c_2$. The condition (1.6) is violated. The other cases can be checked similarly. Therefore, entire solutions originating from two fronts consist of the following seven types:

$$ \{0, a, a, 0\}, \{0, a, a, 1\}, \{a, 0, 0, 1\}, \{1, 0, 0, a\}, \{1, 0, 0, 1\}, \{1, a, a, 0\}, \{1, a, a, 1\}. $$

The first, the fifth and the seventh cases were constructed in [19, 8, 10] and the others are done in [14].

Hence we encounter the natural question: are there entire solutions originating from three or more fronts for (1.1)? The main purpose of this work is to construct entire solutions originating from $k$ fronts for $k \geq 3$ for equation (1.1).

The following theorem is the first main result of this paper.

**Theorem 1.1.** Let $(v_0, \psi_0)$, $(v_1, \psi_1)$ and $(v_2, \psi_2)$ be traveling fronts described as above such that

$$ -v_0 < v_1 \leq c_{1,\text{max}}. $$
Then there exists an entire solution of (1.1) such that

\[
\text{(1.9)} \quad \lim_{t \to -\infty} \left\{ \sup_{x \leq w_1(t)} |u(x, t) - \psi_0(-x + v_0 t + \theta)| + \sup_{w_1(t) \leq x \leq w_2(t)} |u(x, t) - \psi_1(x + v_1 t + \theta)| + \sup_{x \geq w_2(t)} |u(x, t) - \psi_2(x + v_2 t + \theta)| \right\} = 0
\]

for some constant \( \theta \), where

\[
w_1(t) := \frac{-(v_0 + v_1)t}{2}, \quad w_2(t) := \frac{(v_1 + v_2)t}{2}.
\]

Moreover, it holds

\[
\text{(1.10)} \quad \lim_{t \to \infty} \sup_{x \in \mathbb{R}} |u(x, t) - 1| = 0.
\]

This theorem shows us a new type of entire solution originating from three fronts with sequence \( \{1, 0, 0, a, a, 1\} \). Moreover, three fronts of this entire solution are annihilated as \( t \to \infty \). Note that the condition (1.8) on the speeds can be realized when we take the constant \( a \) such that \( v_0 > 2\sqrt{f'(a)} \). For example, when \( f(u) = u(1-u)(u-a) \), we have \( v_0 = \sqrt{2}(1/2-a) > 2\sqrt{a(1-a)} = 2\sqrt{f'(a)} \) if we consider \( 0 < a < (3 - \sqrt{6})/6 \).

For notational convenience, in the sequel we shall use \( \tilde{\psi}_1 \) to denote another traveling front connecting \( \{0, a\} \) with speed \( \tilde{v}_1 \leq c_{1, \max} \). Similar to Theorem 1.1, we can construct entire solutions originating from three fronts with sequence \( \{1, 0, 0, a, 0, a\} \) as follows.

**Theorem 1.2.** Let \((v_0, \psi_0), (v_1, \psi_1)\) and \((\tilde{v}_1, \tilde{\psi}_1)\) be traveling fronts described as above such that (1.8) holds. Then there exists an entire solution of (1.1) such that

\[
\text{(1.11)} \quad \lim_{t \to -\infty} \left\{ \sup_{x \leq w_1(t)} |u(x, t) - \psi_0(-x + v_0 t + \theta_1)| + \sup_{w_1(t) \leq x \leq w_2(t)} |u(x, t) - \psi_1(x + v_1 t + \theta_1)| + \sup_{x \geq w_2(t)} |u(x, t) - \tilde{\psi}_1(-x - \tilde{v}_1 t - \theta_2)| \right\} = 0
\]

for some constants \( \theta_1 \) and \( \theta_2 \), where

\[
w_1(t) := \frac{-(v_0 + v_1)t}{2}, \quad w_2(t) := \frac{(v_1 - \tilde{v}_1)t}{2}.
\]

Moreover, it holds

\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |u(x, t) - \psi_0(-x + v_0 t + \theta)| = 0
\]

for some constant \( \theta \).

Notice that three fronts of the entire solution constructed in Theorem 1.2 are merging to a single front as \( t \to \infty \).

Next, we have the following existence theorem for entire solutions originating from four fronts such that these four fronts are annihilated as \( t \to \infty \).
**Theorem 1.3.** Let \((v_0, \psi_0)\) and \((v_1, \psi_1)\) be traveling fronts described as in Theorem 1.1 such that (1.8) holds. Then there exists a symmetric (with respect to \(x = 0\)) entire solution of (1.1) such that

\[
\lim_{t \to \infty} \left\{ \sup_{x \leq w_1(t)} |u(x, t) - \psi_0(-x + v_0 t + \theta_1)| + \sup_{w_1(t) \leq x \leq 0} |u(x, t) - \psi_1(x + v_1 t + \theta_2)| + \sup_{0 \leq x \leq -w_1(t)} |u(x, t) - \psi_1(-x + v_1 t + \theta_2)| + \sup_{x \geq -w_1(t)} |u(x, t) - \psi_0(x + v_0 t + \theta_1)| \right\} = 0
\]

for some constants \(\theta_1\) and \(\theta_2\), where

\[
w_1(t) := \frac{(-v_0 + v_1)t}{2}.
\]

Moreover, the asymptotic behavior (1.10) holds.

Finally, we have the following nonexistence theorem for entire solutions originating from \(k\) fronts for \(k \geq 5\).

**Theorem 1.4.** Under the condition (1.6), there are no entire solutions originating from \(k\) fronts if \(k \geq 5\).

Since the comparison principle is available for (1.1), it is well-known that an entire solution exists if we can find a suitable pair of super-sub-solutions (see, e.g., [8, 10, 14]). Therefore, the main task of finding entire solutions originating from multiple fronts is to construct some suitable super-sub-solutions with the desired properties. One of the main ideas in [14] is to find an auxiliary rational function with certain properties in order to construct a suitable pair of super-sub-solutions. The form of this auxiliary function depends on the equilibrium states which are connected by those two traveling fronts under consideration. Although the method of finding an auxiliary function can be applied to the construction of entire solutions originating from three fronts, finding this useful rational function is by no means trivial.

Due to the increase of the number of fronts, we were unable to construct a suitable auxiliary function for an entire solution originating from four fronts. Instead, the super-sub-solutions constructed for deriving entire solutions originating from three fronts are used effectively to construct an entire solution originating from four fronts.

Besides the works on the scalar equations, there are many works on the entire solutions originating from two fronts for systems of two reaction-diffusion equations. We refer the reader to, for examples, [15, 9, 17, 18, 16, 20]. For the discrete version of (1.1), the same results as in Theorem 1.1 and Theorem 1.2 have been shown in [5].

The rest of this paper is organized as follows. First, a proof of Theorem 1.4 is given in §2. In §3, we first give an auxiliary function linking three traveling fronts of (1.1). Then we provide some useful properties of this auxiliary function and derive the key estimates (see Lemma 3.4) for the later construction of super-sub-solutions. In §4, we use this auxiliary function to construct a pair of super-sub-solutions and give a proof of Theorem 1.1 on the existence of entire solutions originating from three fronts with sequence \(\{1, 0, 0, a, a, 1\}\). For the sequence \(\{1, 0, 0, a, a, 0\}\), since the proof of Theorem 1.2 is similar to that of Theorem 1.1, we only point out the main differences in §5. Finally, in §6, we give a proof of Theorem 1.3.
2. Proof of Theorem 1.4

First we introduce a sequence \( \{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\} \) which is called a terminated sequence if \( \{\alpha_{k-1}, \omega_{k-1}, \alpha_k, \omega_k\} \) is one of the cases in (1.7). This means that, for a terminated sequence \( \{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\} \), there is no entire solution originating from \( k+1 \) fronts with any sequence \( \{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k, \alpha_{k+1}, \omega_{k+1}\} \) such that \( \alpha_{k+1} = \omega_k \) for some \( \omega_{k+1} \in \{0, a, 1\} \setminus \{\omega_k\} \). Note that \( \{0, 1\} \) and \( \{a, 1\} \) are terminated sequences. For example, if a sequence starts from \( a \), then the possible sequences are \( \{a, 0\} \) or \( \{a, 1\} \). But, the latter is already terminated. In fact, the possible terminated sequences starting from \( a \) are the following two:

\[
\{a, 1\}, \{a, 0, 0, 1\},
\]

because two waves must intersect at some negative time for the case \( \{a, 0, 0, a\} \). Thus the longest terminated sequence starting from \( a \) is \( \{a, 0, 0, 1\} \). Using this argument, we can prove Theorem 1.4.

Proof of Theorem 1.4. Let \( u \) be an entire solution of (1.1) originating from \( k \) fronts and let \( \{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\} \) be the sequence of \( u \). As stated above, the longest terminated sequence starting from \( a \) is \( \{a, 0, 0, 1\} \). This means that there are no entire solutions originating from \( k \) fronts for \( k \geq 5 \) if \( \alpha_1 = a \).

For the case where the sequence starts from \( 0 \), \( \{0, 1\} \) is terminated, while \( \{0, a\} \) is not terminated. By combing \( \{0, a\} \) and the longest terminated sequence starting from \( a \), we can conclude that the longest terminated sequence starting from \( 0 \) is \( \{0, a, a, 0, 0, 1\} \).

Similarly, let us consider the case where the sequence starts from \( 1 \). If it starts from \( \{1, a\} \), then the longest terminated one is \( \{1, a, a, 0, 0, 1\} \). If it starts from \( \{1, 0\} \), then combing \( \{0, a, 0, 0, 1\} \), we can obtain the longest terminated sequence \( \{1, 0, 0, a, a, 0, 0, 1\} \). Thus the longest terminated sequence starting from \( 1 \) is \( \{1, 0, 0, a, a, 0, 0, 1\} \). Therefore, there are no sequences for entire solutions originating from \( k \) fronts for \( k \geq 5 \).

Remark 1. Taking the symmetry into account, we can check that the only possible sequences with \( k = 3 \) are \( \{1, 0, 0, a, a, 1\} \) and \( \{1, 0, 0, a, a, 0\} \). Moreover, the only possible sequence with \( k = 4 \) is \( \{1, 0, 0, a, a, 0, 0, 1\} \).

3. Some function linking three-front dynamics

Set \( c_1 := -v_0 \), \( c_2 := v_1 \leq c_{1_{\text{max}}} \) and \( c_3 := v_2 \geq c_{2_{\text{min}}} \). Let \( \phi_i = \phi_i(x + c_i t), i = 1, 2, 3 \), be traveling fronts of (1.1) that satisfy

\[
\begin{align*}
\phi_i''(s) - c_i \phi_i'(s) + f(\phi_i(s)) &= 0, \quad s \in \mathbb{R}, \\
\phi_i(-\infty) &= \alpha_i, \quad \phi_i(\infty) = \omega_i,
\end{align*}
\]

where \( (\alpha_1, \omega_1, \alpha_2, \omega_2, \alpha_3, \omega_3) = (1, 0, 0, a, a, 1) \). Here the prime denotes the derivative with respect to \( s \). Note that \( \phi_1(z) = \psi_0(-z) \) and \( \phi_i = \psi_{i-1}, i = 2, 3 \). In this and next sections, we assume

\[
\phi_1(0) = \frac{a}{2}, \quad \phi_2(0) = \frac{a}{2}, \quad \phi_3(0) = \frac{1+a}{2}.
\]
By the nondegenerate condition on $f$, for $p \leq 0$, there are positive constants $\beta_i, \gamma_i, \ i = 1, 2, 3$, and $K > 0$ such that

$$\begin{align*}
|\phi'_i(x + p)| &\leq K\exp(\beta_i(x + p)), \ x \leq -p, \\
|\phi'_i(x + p)| &\leq K\exp(-\gamma_i(x + p)), \ x \geq -p.
\end{align*}$$

(3.3)

In addition, there is a constant $\tau > 0$ such that

$$\begin{align*}
\frac{|\phi_1(x - p) - 1|}{|\phi_1(x - p)|} &\leq \tau, \ x \leq p, \\
\frac{|\phi_1(x - p) - 0|}{|\phi_2(x + p) - a|} &\leq \tau, \ x \geq p, \\
\frac{|\phi_2'(x + p)|}{|\phi_2'(x + p) - a|} &\leq \tau, \ x \geq -p, \\
\frac{|\phi_3'(x + p)|}{|\phi_3'(x + p) - 1|} &\leq \tau, \ x \geq -p.
\end{align*}$$

(3.4)

The key auxiliary function we found for linking three fronts is as follows.

**Lemma 3.1.** Set

$$Q(y, z, w) = z + (1 - z)(1 - y)z(w - a) + y(a - z)(1 - w)$$

(3.5)

$$\left(1 - y \right) z \left(1 - a \right) + (a - z)(1 - w).$$

Then the following three statements hold:

(i) $Q$ can be rewritten as

$$Q(y, z, w) = \begin{cases} 
\frac{(1 - a)(w - y)}{(1 - y)z(1 - a) + (a - z)(1 - w)}, \\
w + (a - z)(1 - w) \frac{y - w}{(1 - y)z(1 - a) + (a - z)(1 - w)}. 
\end{cases}$$

(3.6)

(ii) There exist functions $Q_i$, $i = 1, 2, 3$, such that

$$\begin{align*}
Q_y(y, z, w) &= (a - z)(1 - w)Q_1(y, z, w), \\
Q_z(y, z, w) &= (1 - y)(1 - w)Q_2(y, z, w), \\
Q_w(y, z, w) &= (1 - y)zQ_3(y, z, w).
\end{align*}$$

(iii) There exist functions $R_j$, $j = 1, \cdots, 16$, such that

$$\begin{align*}
Q_{yy}(y, z, w) &= zR_1(y, z, w) = (a - z)R_2(y, z, w) = (1 - w)R_3(y, z, w), \\
Q_{zz}(y, z, w) &= (1 - y)R_4(y, z, w) = (1 - w)R_5(y, z, w) \\
&\quad = yR_6(y, z, w) + (w - a)R_7(y, z, w), \\
Q_{ww}(y, z, w) &= (1 - y)R_8(y, z, w) = zR_9(y, z, w) = (a - z)R_{10}(y, z, w), \\
Q_{yz}(y, z, w) &= (1 - w)R_{11}(y, z, w), \quad Q_{zw}(y, z, w) = (1 - y)R_{12}(y, z, w), \\
Q_{yw}(y, z, w) &= (1 - y)R_{13}(y, z, w) = zR_{14}(y, z, w) \\
&\quad = (a - z)R_{15}(y, z, w) = (1 - w)R_{16}(y, z, w).
\end{align*}$$

*Proof.* Obviously, the function $Q(y, z, w)$ defined by (3.5) allows the expression as (3.6).
By a simple calculation, we can derive
\[ Q_y(y, z, w) = \frac{a(1 - z)(a - z)(1 - w)}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^2}, \]
\[ Q_z(y, z, w) = \frac{(1 - a)a(1 - y)(1 - w)(w - y)}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^2}, \]
\[ Q_w(y, z, w) = \frac{a(1 - a)(1 - y)^2 z(1 - z)}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^2}. \]

Hence, the conclusion (ii) holds.

For the statement (iii), we compute the second derivative of function $Q$ and obtain that
\[ Q_{yy}(y, z, w) = \frac{2(1 - a)az(a - z)(1 - z)(1 - w)}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3}, \]
\[ Q_{zz}(y, z, w) = -\frac{2(1 - a)a(1 - y)(1 - w)(w - y)[w - a - y(1 - a)]}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3}, \]
\[ Q_{ww}(y, z, w) = \frac{2(1 - a)a(1 - y)^2 z(a - z)(1 - z)}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3}, \]
\[ Q_{yz}(y, z, w) = -\frac{(1 - a)a(1 - w)^2[(y - w)z + a(1 - 2y - z + w + yz)]}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3}, \]
\[ Q_{yw}(y, z, w) = -\frac{2(1 - a)a(1 - y)z(a - z)(1 - z)(1 - w)}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3}, \]
\[ Q_{zw}(y, z, w) = -\frac{(1 - a)a(1 - y)^2[(w - y)z + a(-1 + w + z - 2wz + yz)]}{[(1 - y)z(1 - a) + (a - z)(1 - w)]^3}. \]

Thus, we get the conclusion (iii) and the lemma is proved.

With this auxiliary function $Q$, we can construct a suitable pair of super-sub-solutions. For this, we put $u(x, t) = U(\xi, t)$ with $\xi := x + \overline{c}t$ and $\overline{c} = (c_1 + c_2)/2 = (-v_0 + v_1)/2$. Then (1.1) becomes
\[ (3.7) \quad U_t = U_{\xi\xi} - \overline{c}U_\xi + f(U), \quad \xi \in \mathbb{R}. \]

We can easily check that (3.7) has traveling wave solutions
\[ U = \phi_1'(\xi - s_1t), \quad \phi_2(\xi + s_1t), \quad \phi_3(\xi + s_2t), \]
where $s_1 := (c_2 - c_1)/2 = (v_1 + v_0)/2 > 0$, by (1.8), and
\[ s_2 := c_3 - \overline{c} = (2c_3 - c_1 - c_2)/2 = (2v_2 + v_0 - v_1)/2 > (v_0 - v_1)/2 > s_1. \]

Now we consider
\[ U(\xi, t) = Q(\phi_1, \phi_2, \phi_3), \quad \phi_1 = \phi_1(\xi - q_1(t)), \quad \phi_2 = \phi_2(\xi + q_2(t)), \quad \phi_3 = \phi_3(\xi + q_3(t)), \]
where $q_i(t) < 0$, $i = 1, 2, 3$, and $-q_2(t) < -q_3(t)$. Set
\[ \mathcal{T}[U] := U_t - U_{\xi\xi} + \overline{c}U_\xi - f(U). \]
Then
\[ T[Q(\phi_1, \phi_2, \phi_3)] = -Q_y \phi'_1(q'_1 - s_1) + Q_z \phi'_2(q'_2 - s_1) + Q_w \phi'_3(q'_3 - s_2) - G(\phi_1, \phi_2, \phi_3) - H(\phi_1, \phi_2, \phi_3) \]
where
\[
G(\phi_1, \phi_2, \phi_3) := Q_{yy}\{\phi'_1\}^2 + Q_{zz}\{\phi'_2\}^2 + Q_{ww}\{\phi'_3\}^2 + 2[Q_{yz}\phi'_1\phi'_2 + Q_{yz}\phi'_1\phi'_3 + Q_{zw}\phi'_2\phi'_3],
\]
\[
H(\phi_1, \phi_2, \phi_3) := f(Q) - Q_y f(\phi_1) - Q_z f(\phi_2) - Q_w f(\phi_3).
\]
From (3.6) and Lemma 3.1 we see that
\[
H(1, z, w) = f(Q(1, z, w)) - Q_y(1, z, w) f(1) - Q_z(1, z, w) f(z) - Q_w(1, z, w) f(w) = 0,
\]
\[
H(y, 0, w) = f(Q(y, 0, w)) - Q_y(y, 0, w) f(y) - Q_z(y, 0, w) f(0) - Q_w(y, 0, w) f(w) = 0,
\]
\[
H(y, a, w) = f(Q(y, a, w)) - Q_y(y, a, w) f(y) - Q_z(y, a, w) f(a) - Q_w(y, a, w) f(w) = 0,
\]
\[
H(y, z, 1) = f(Q(y, z, 1)) - Q_y(y, z, 1) f(y) - Q_z(y, z, 1) f(z) - Q_w(y, z, 1) f(1) = 0,
\]
which implies that there is a smooth function $H_1$ satisfying
\[
H(y, z, w) = (1 - y)(a - z)(1 - w)H_1(y, z, w).
\]
Since $Q(0, z, a) = z$ and $Q_z(0, z, a) = 1$, we have
\[
H(0, z, a) = f(Q(0, z, a)) - Q_z(0, z, a) f(z) = 0,
\]
which implies $H_1(0, z, a) = 0$. Applying the mean value theorem to $H_1$ yields
\[
H_1(y, z, w) = \int_0^1 H_{1y}(\theta y, z, \theta w + (1 - \theta)a) d\theta \cdot y + \int_0^1 H_{1w}(\theta y, z, \theta w + (1 - \theta)a) d\theta \cdot (w - a).
\]
Thus we obtain
\[
(3.8) \quad \begin{cases} 
H(y, z, w) = (1 - y)(a - z)[y H_{11}(y, z, w) + (w - a)H_{12}(y, z, w)], \\
H(y, z, w) = (1 - w)(a - z)[y H_{21}(y, z, w) + (w - a)H_{22}(y, z, w)]
\end{cases}
\]
for some functions $H_{ij}$, $i, j = 1, 2$.

**Lemma 3.2.** For $q_1, q_2, q_3 \leq -\delta < 0$, there exist positive constants $\epsilon_1, \epsilon_2$ and $\epsilon_3$ such that
\[
Q_y(\phi_1(q - q_1), \phi_2(q + q_2), \phi_3(q + q_3)) \geq \epsilon_1 \text{ for } q \leq -q_2,
\]
\[
Q_z(\phi_1(q - q_1), \phi_2(q + q_2), \phi_3(q + q_3)) \geq \epsilon_2 \text{ for } q_1 \leq \xi \leq -q_3,
\]
\[
Q_w(\phi_1(q - q_1), \phi_2(q + q_2), \phi_3(q + q_3)) \geq \epsilon_3 \text{ for } \xi \geq -q_2.
\]
Proof. Recall (3.2). Then
\[
\frac{a}{2} \leq \phi_1(\xi - q_1) \leq 1, \quad 0 \leq \phi_2(\xi + q_2) \leq \frac{a}{2}, \quad a \leq \phi_3(\xi + q_3) \leq \frac{1+a}{2} \text{ for } \xi \leq q_1,
\]
\[
0 \leq \phi_1(\xi - q_1) \leq \frac{a}{2}, \quad 0 \leq \phi_2(\xi + q_2) \leq \frac{a}{2}, \quad a \leq \phi_3(\xi + q_3) \leq \frac{1+a}{2} \text{ for } q_1 \leq \xi \leq -q_2,
\]
\[
0 \leq \phi_1(\xi - q_1) \leq \frac{a}{2}, \quad \frac{a}{2} \leq \phi_2(\xi + q_2) \leq a, \quad a \leq \phi_3(\xi + q_3) \leq \frac{1+a}{2} \text{ for } -q_2 \leq \xi \leq -q_3,
\]
\[
0 \leq \phi_1(\xi - q_1) \leq \frac{a}{2}, \quad \frac{a}{2} \leq \phi_2(\xi + q_2) \leq a, \quad \frac{1+a}{2} \leq \phi_3(\xi + q_3) \leq 1 \text{ for } \xi \geq -q_3,
\]
when \(q_1, q_2, q_3 \leq -\delta\). Then we have
\[
(3.9) \quad \frac{a(1-a)}{4} \leq (1-a)(1-\phi_1)\phi_2 + (a-\phi_2)(1-\phi_3) \leq \frac{3a(1-a)}{2}
\]
for \(\xi \in \mathbb{R}, q_1, q_2, q_3 \leq -\delta\).

By Lemma 3.1, for \(q_1, q_2, q_3 < -\delta\), we derive that
\[
Q_y(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) = \frac{a(1-\phi_2)(a-\phi_2)(1-\phi_3)^2}{[(1-\phi_1)\phi_2(1-a) + (a-\phi_2)(1-\phi_3)]^2} \geq \frac{2-a}{3a(1-a)/2^2} = \frac{2-a}{18}
\]
for \(\xi \leq -q_2\), and
\[
Q_z(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) = \frac{(1-a)a(1-\phi_1)(1-\phi_3)(\phi_3 - \phi_1)}{[(1-\phi_1)\phi_2(1-a) + (a-\phi_2)(1-\phi_3)]^2} \geq \frac{2-a}{3a(1-a)/2^2} = \frac{2-a}{18}
\]
for \(q_1 \leq \xi \leq -q_3\), and
\[
Q_w(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) = \frac{a(1-a)(1-\phi_1)^2\phi_2(1-\phi_2)}{[(1-\phi_1)\phi_2(1-a) + (a-\phi_2)(1-\phi_3)]^2} \geq \frac{(2-a)^2}{3a(1-a)/2^2} = \frac{(2-a)^2}{18}
\]
for \(\xi \geq -q_2\). Therefore, the lemma follows. \(\square\)

From Lemma 3.1, it is easy to check that there exists a positive constant \(C\) such that
\[
|R_j(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))| \leq C,
\]
\[
|H_{mn}(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))| \leq C,
\]
for $\xi \in \mathbb{R}$, $q_1, q_2, q_3 < -\delta$, $j = 1, \cdots, 16$, and $m, n = 1, 2$. Now, we define a function $F(\phi_1, \phi_2, \phi_3)$ as follows

$$F(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))$$

$$:= -Q_y(\phi_1, \phi_2, \phi_3)\phi_1'(\xi - q_1) + Q_z(\phi_1, \phi_2, \phi_3)\phi_2'(\xi + q_2) + Q_w(\phi_1, \phi_2, \phi_3)\phi_3'(\xi + q_3).$$

Then the function $F$ is bounded above for $\xi \in \mathbb{R}$, $q_1, q_2, q_3 < -\delta$, since $Q_y, Q_z, Q_w, -\phi_1', \phi_2'$ and $\phi_3'$ are bounded above for $\xi \in \mathbb{R}$, $q_1, q_2, q_3 < -\delta$.

The next lemma shows that the function $F$ has a positive lower bound for $\xi \in \mathbb{R}$ and $q_1, q_2, q_3 < -\delta$, if $\delta$ is sufficiently large.

**Lemma 3.3.** There exists a sufficiently large constant $\delta$ such that

$$F(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) > 0 \text{ for } \xi \in \mathbb{R}, q_1, q_2, q_3 \leq -\delta.$$

Moreover, $F(\phi_1, \phi_2, \phi_3) = F(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))$ satisfies

\begin{align}
F(\phi_1, \phi_2, \phi_3) &\geq \frac{1}{2}Q_y|\phi_1'|(\xi - q_1)| \quad \text{for } \xi \leq q_1, \\
F(\phi_1, \phi_2, \phi_3) &\geq \frac{1}{2}\left[Q_y|\phi_1'(\xi - q_1)| + Q_z|\phi_2'(\xi + q_2)|\right] \quad \text{for } q_1 \leq \xi \leq -q_2, \\
F(\phi_1, \phi_2, \phi_3) &\geq \frac{1}{2}\left[Q_z|\phi_2'(\xi + q_2)| + Q_w|\phi_3'(\xi + q_3)|\right] \quad \text{for } -q_2 \leq \xi \leq -q_3, \\
F(\phi_1, \phi_2, \phi_3) &\geq \frac{1}{2}Q_w|\phi_3'(\xi + q_3)| \quad \text{for } \xi \geq -q_3,
\end{align}

when $q_1, q_2, q_3 \leq -\delta$.

**Proof.** Since $\phi_1(-\infty) = 1$, $\phi_3(-\infty) = a$, $\phi_1(\xi - q_1)$ is decreasing and $\phi_3(\xi + q_3)$ is increasing for $\xi \in \mathbb{R}$, there exists a $q_0 < q_1$ such that $\phi_1(q_0 - q_1) = \phi_3(q_0 + q_3)$, $\phi_1(\xi - q_1) > \phi_3(\xi + q_3)$ for $\xi < q_0$ and $\phi_1(\xi - q_1) < \phi_3(\xi + q_3)$ for $q_0 < \xi \leq q_1$. For $q_0 \leq \xi \leq q_1$, we have $Q_z(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \geq 0$ and

$$F(\phi_1, \phi_2, \phi_3) = -Q_y\phi_1' + Q_z\phi_2' + Q_w\phi_3' \geq -Q_y\phi_1' \geq \frac{1}{2}Q_y|\phi_1'|$$

by $Q_w, \phi_2', \phi_3' \geq 0$. If $\xi < q_0$, we know that $Q_z(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) < 0$. From (3.3)-(3.4), we have $|\phi_1'| \geq (1 - \phi_1)/\tau$ and $|\phi_2'| \leq Ke^{\beta_2q_2}$. Then we compute that

\begin{align}
F(\phi_1, \phi_2, \phi_3) &- \frac{1}{2}Q_y|\phi_1'| \geq \frac{1}{2}Q_y|\phi_1'| + Q_z\phi_2' + Q_w\phi_3' \geq \frac{1}{2}Q_y|\phi_1'| + Q_z\phi_2' \\
&\geq \frac{a(1 - \phi_2)(a - \phi_2)(1 - \phi_3)^2(1 - \phi_1)}{2\tau[(1 - \phi_1)\phi_2(1 - a) + (a - \phi_2)(1 - \phi_3)]^2} \\
&\quad + \frac{(1 - a)a(1 - \phi_3)(1 - \phi_1)(\phi_3 - \phi_1)}{[(1 - \phi_1)\phi_2(1 - a) + (a - \phi_2)(1 - \phi_3)]^2}Ke^{\beta_2q_2} \\
&\geq \frac{a(1 - \phi_1)(1 - \phi_3)}{2\tau[3a(1 - a)/2]^2} \left[ \left(1 - \frac{a}{2}\right) \left( \frac{a}{2} - \frac{1}{2} \right) + 2\tau(1 - a)(a - 1)Ke^{-\beta_2}\right] \\
&\geq 0
\end{align}

for $\delta$ sufficiently large. Therefore, (3.10) holds for $\xi \leq q_1$ and $q_1, q_2, q_3 \leq -\delta$.

For $\xi \geq q_1$, since $Q_y, Q_z, Q_w \geq 0$, $\phi_1' < 0$ and $\phi_2', \phi_3' > 0$, we get the conclusion. \(\square\)
With Lemma 3.3, we now state and prove the following key lemma on the estimates to be used later in verifying super-sub-solutions.

**Lemma 3.4.** There is a positive constant $M$ such that

$$
(3.14) \quad \left| \frac{H(\phi_1, \phi_2, \phi_3) + G(\phi_1, \phi_2, \phi_3)}{F(\phi_1, \phi_2, \phi_3)} \right| \leq \begin{cases} 
M(|\phi'_2| + |\phi'_3|) & \text{for } \xi \leq 0, \\
M(|\phi'_1| + |\phi'_3|) & \text{for } 0 \leq \xi \leq -\frac{q_3 + q_2}{2}, \\
M(|\phi'_1| + |\phi'_2|) & \text{for } \xi \geq -\frac{q_3 + q_2}{2},
\end{cases}
$$

for $q_1, q_2, q_3 < -\delta$ with $\delta \gg 1$.

**Proof.** For the simplicity of notation, we denote the functions $H_{ij}(\phi_1, \phi_2, \phi_3)$ ($i, j = 1, 2$) by $H_{ij}$. Similarly we also omit $(\phi_1, \phi_2, \phi_3)$ for $H(\phi_1, \phi_2, \phi_3)$, $G(\phi_1, \phi_2, \phi_3)$, $Q_y(\phi_1, \phi_2, \phi_3)$ and so on.

First, we estimate $|H/F|$. For $\xi \leq q_1$, by (3.8), Lemma 3.2, (3.4) and (3.10), we have

$$
(3.15) \quad \left| \frac{H}{F} \right| \leq \left| \frac{2(1 - \phi_1)\phi_2[\phi_1H_{11} + (\phi_3 - a)H_{12}]}{Q_y|\phi'_1|} \right|
\leq \frac{2\tau}{\epsilon_1} |C|\phi_2| + C|\phi_3 - a| \leq \frac{2C\tau^2}{\epsilon_1} (|\phi'_2| + |\phi'_3|).
$$

For $q_1 \leq \xi \leq 0$, by (3.8), Lemma 3.2, (3.4) and (3.11), we have

$$
(3.16) \quad \left| \frac{H}{F} \right| \leq \left| \frac{2(1 - \phi_1)\phi_2[\phi_1H_{11} + (\phi_3 - a)H_{12}]}{Q_y|\phi'_1| + Q_z|\phi'_2|} \right|
\leq \frac{2|1 - \phi_1||\phi_2||\phi_1||H_{11}| + 2|1 - \phi_1||\phi_2||\phi_3 - a||H_{12}|}{Q_y|\phi'_1|}
\leq \frac{2C\tau^2|\phi'_2| + 2C\tau^2|\phi'_3|}{\epsilon_1}.
$$

From (3.15)-(3.16), we obtain that

$$
(3.17) \quad \left| \frac{H}{F} \right| \leq M_1(|\phi'_2(\xi + q_2)| + |\phi'_3(\xi + q_3)|) \quad \text{for } \xi \leq 0.
$$

For $0 \leq \xi \leq -q_2$, by (3.8), Lemma 3.2, (3.4) and (3.11), we have

$$
(3.18) \quad \left| \frac{H}{F} \right| \leq \left| \frac{2(1 - \phi_1)\phi_2[\phi_1H_{11} + (\phi_3 - a)H_{12}]}{Q_z|\phi'_2|} \right|
\leq \frac{2|1 - \phi_1||\phi_2||\phi_1||H_{11}| + 2|1 - \phi_1||\phi_2||\phi_3 - a||H_{12}|}{Q_z|\phi'_2|}
\leq \frac{2C\tau^2|\phi'_1| + 2C\tau^2|\phi'_3|}{\epsilon_2}.
$$
For $-q_2 \leq \xi \leq (-q_3 - q_2)/2$, by (3.8), Lemma 3.2, (3.4) and (3.12), we have

\begin{equation}
\left| \frac{H}{F} \right| \leq \left| \frac{2(1 - \phi_3)(a - \phi_2)[\phi_1 H_{21} + (\phi_3 - a)H_{22}]}{Q_z |\phi_2'|} \right|
\leq \frac{2|1 - \phi_3||a - \phi_2||\phi_1||H_{21}| + 2|1 - \phi_3||a - \phi_2||\phi_3 - a||H_{22}|}{Q_z |\phi_2'|}
\leq \frac{2C \tau^2 |\phi_1'|}{\epsilon_2} + \frac{2C \tau^2 |\phi_3'|}{\epsilon_2}.
\end{equation}

From (3.18)-(3.19), we obtain that

\begin{equation}
\left| \frac{H}{F} \right| \leq M_2(|\phi_1'(\xi - q_1)| + |\phi_3'(\xi + q_3)|) \quad \text{for} \quad 0 \leq \xi \leq -\frac{q_3 + q_2}{2}.
\end{equation}

For $(-q_3 - q_2)/2 \leq \xi \leq -q_3$, by (3.8), Lemma 3.2, (3.4) and (3.12), we have

\begin{equation}
\left| \frac{H}{F} \right| \leq \left| \frac{2(1 - \phi_3)(a - \phi_2)[\phi_1 H_{21} + (\phi_3 - a)H_{22}]}{Q_z |\phi_2'| + Q_w |\phi_3'|} \right|
\leq \frac{2|1 - \phi_3||a - \phi_2||\phi_1||H_{21}| + 2|1 - \phi_3||a - \phi_2||\phi_3 - a||H_{22}|}{Q_w |\phi_3'|}
\leq \frac{2C \tau^2 |\phi_1'|}{\epsilon_2} + \frac{2C \tau^2 |\phi_3'|}{\epsilon_3}.
\end{equation}

For $\xi \geq -q_3$, by (3.8), Lemma 3.2, (3.4) and (3.13), we have

\begin{equation}
\left| \frac{H}{F} \right| \leq \left| \frac{2(1 - \phi_3)(a - \phi_2)[\phi_1 H_{21} + (\phi_3 - a)H_{22}]}{Q_w |\phi_3'|} \right|
\leq \frac{2\tau}{\epsilon_3} \left[ C|\phi_1'| + C|a - \phi_2| \right] \leq \frac{2C \tau^2}{\epsilon_3} (|\phi_1'| + |\phi_2'|).
\end{equation}

From (3.21)-(3.22), we obtain that

\begin{equation}
\left| \frac{H}{F} \right| \leq M_3(|\phi_1'(\xi - q_1)| + |\phi_2'(\xi + q_2)|) \quad \text{for} \quad \xi \geq -\frac{q_3 + q_2}{2}.
\end{equation}

Next, we estimate $|G/F|$. For $\xi \leq q_1$, by Lemma 3.1(ii), Lemma 3.2, (3.4) and (3.10), we have

\begin{equation}
\left| \frac{G}{F} \right| \leq \frac{2|Q_{yy}(\phi_1')^2 + Q_{zz}(\phi_2')^2 + Q_{ww}(\phi_3')^2 + 2Q_{yz}(\phi_1'\phi_2' + 2Q_{yw}\phi_1'\phi_3' + 2Q_{zw}\phi_2'\phi_3'|}{Q_y |\phi_1'|}
\leq 2 \left[ \frac{|\phi_2||R_1||\phi_1'|}{\epsilon_1} + \frac{|1 - \phi_1||R_4||\phi_2'|^2 + |1 - \phi_1||R_8||\phi_3'|^2}{\epsilon_1 |\phi_1'|} \right]
\leq \frac{2C \tau K}{\epsilon_1} |\phi_2'| + \frac{C \tau K}{\epsilon_1} |\phi_3'| + 2 \left( \frac{C}{\epsilon_1} |\phi_2'| + \frac{C}{\epsilon_1} |\phi_3'| + \frac{C \tau}{\epsilon_1} |\phi_2'| |\phi_3'| \right)
\leq M_4 (|\phi_2'| + |\phi_3'|).
\end{equation}
For $q_1 \leq \xi \leq 0$, by Lemma 3.1(ii), Lemma 3.2, (3.4) and (3.11), we have
\[
\left| \frac{G}{F} \right| \leq 2 \left( \left| \frac{Q_{yy}(\phi'_1)^2 + Q_{zz}(\phi'_2)^2 + Q_{ww}(\phi'_3)^2 + 2Q_{yz}\phi'_1\phi'_2 + 2Q_{yz}\phi'_1\phi'_3 + 2Q_{zw}\phi'_2\phi'_3}{Q_y|\phi'_1| + Q_z|\phi'_2|} \right) \\
+ 4 \left( \left| \frac{Q_{yz}|\phi'_2|}{\epsilon_1} + \left| \frac{Q_{yw}|\phi'_2|}{\epsilon_2} \right| + \left| \frac{Q_{zw}|\phi'_3|}{\epsilon_2} \right| \right) \right) \\
\leq 2 \left[ \frac{C\tau K}{\epsilon_1} |\phi'_1| + \frac{C\tau K}{\epsilon_2} |\phi'_2| + \frac{C}{\epsilon_3} |\phi'_3| + 2 \left( \frac{C}{\epsilon_1} |\phi'_1| + \frac{C}{\epsilon_2} |\phi'_2| + \frac{C}{\epsilon_3} |\phi'_3| \right) \right] \\
\leq M_5(|\phi'_2| + |\phi'_3|).
\]

Then we obtain that
\[
(3.24) \quad \left| \frac{G}{F} \right| \leq M_6(|\phi'_2(\xi + q_2)| + |\phi'_3(\xi + q_3)|) \quad \text{for } \xi \leq 0.
\]

For $0 \leq \xi \leq -q_2$, by (3.8), Lemma 3.2, (3.4) and (3.11), we have
\[
\left| \frac{G}{F} \right| \leq 2 \left( \left| \frac{Q_{yy}(\phi'_1)^2 + Q_{zz}(\phi'_2)^2 + Q_{ww}(\phi'_3)^2 + 2Q_{yz}\phi'_1\phi'_2 + 2Q_{yz}\phi'_1\phi'_3 + 2Q_{zw}\phi'_2\phi'_3}{Q_y|\phi'_1| + Q_z|\phi'_2|} \right) \\
+ 4 \left( \left| \frac{Q_{yz}|\phi'_2|}{\epsilon_1} + \left| \frac{Q_{yw}|\phi'_2|}{\epsilon_2} \right| + \left| \frac{Q_{zw}|\phi'_3|}{\epsilon_2} \right| \right) \right) \\
\leq 2 \left[ \frac{C}{\epsilon_1} |\phi'_1| + \frac{C\tau K}{\epsilon_2} (|\phi'_1| + |\phi'_3|) + \frac{C\tau K}{\epsilon_2} |\phi'_3| + 2 \left( \frac{C}{\epsilon_1} |\phi'_1| + \frac{C}{\epsilon_2} |\phi'_2| + \frac{C}{\epsilon_3} |\phi'_3| \right) \right] \\
\leq M_7(|\phi'_1| + |\phi'_3|).
\]

For $-q_2 \leq \xi \leq (-q_3 - q_2)/2$, by Lemma 3.1(ii), Lemma 3.2, (3.4) and (3.12), we have
\[
\left| \frac{G}{F} \right| \leq 2 \left( \left| \frac{Q_{yy}(\phi'_1)^2 + Q_{zz}(\phi'_2)^2 + Q_{ww}(\phi'_3)^2 + 2Q_{yz}\phi'_1\phi'_2 + 2Q_{yz}\phi'_1\phi'_3 + 2Q_{zw}\phi'_2\phi'_3}{Q_y|\phi'_1| + Q_z|\phi'_2|} \right) \\
+ 4 \left( \left| \frac{Q_{yz}|\phi'_2|}{\epsilon_2} + \left| \frac{Q_{yw}|\phi'_2|}{\epsilon_3} \right| + \left| \frac{Q_{zw}|\phi'_3|}{\epsilon_2} \right| \right) \right) \\
\leq 2 \left[ \frac{C\tau K}{\epsilon_2} |\phi'_1| + \frac{C\tau K}{\epsilon_3} (|\phi'_1| + |\phi'_3|) + \frac{C}{\epsilon_3} |\phi'_3| + 2 \left( \frac{C}{\epsilon_2} |\phi'_1| + \frac{C}{\epsilon_3} |\phi'_2| + \frac{C}{\epsilon_2} |\phi'_3| \right) \right] \\
\leq M_8(|\phi'_1| + |\phi'_3|).
\]

Then we obtain that
\[
(3.25) \quad \left| \frac{G}{F} \right| \leq M_9(|\phi'_1(\xi - q_1)| + |\phi'_3(\xi + q_3)|) \quad \text{for } 0 \leq \xi \leq -\frac{q_3 + q_2}{2}.
\]
For \((-q_3 - q_2)/2 \leq \xi \leq -q_3\), by Lemma 3.1(ii), Lemma 3.2, (3.4) and (3.12), we have

\[
|G| \leq 2|Q_{yy}(\phi'_1)^2 + Q_{zz}(\phi'_2)^2 + Q_{ww}(\phi'_3)^2 + 2Q_{yz}\phi'_1\phi'_2 + 2Q_{yw}\phi'_1\phi'_3 + 2Q_{zw}\phi'_2\phi'_3|/Q_{zz}\phi'_2 + Q_w\phi'_3|
\]

\[
\leq 2 \left( \frac{|a - \phi_2||R_2||\phi'_1|}{\epsilon_2} + \frac{|Q_{zz}||\phi'_2|}{\epsilon_2} + \frac{|a - \phi_2||R_{10}||\phi'_3|}{\epsilon_3} \right)
+ 4 \left( \frac{|Q_{yz}||\phi'_1|}{\epsilon_2} + \frac{|Q_{yw}||\phi'_1|}{\epsilon_3} + \frac{|Q_{zw}||\phi'_2|}{\epsilon_3} \right)
\]

\[
\leq 2 \left[ \frac{C\tau K}{\epsilon_3} |\phi'_1| + \frac{C\tau K}{\epsilon_3} |\phi'_2| + \frac{C\tau K}{\epsilon_3} |\phi'_3| + 2 \left( \frac{C\tau}{\epsilon_2} |\phi'_1| + \frac{C}{\epsilon_3} |\phi'_1| + \frac{C}{\epsilon_3} |\phi'_2| \right) \right]
\]

\[
\leq M_{11}(|\phi'_1| + |\phi'_2|).
\]

Then we obtain that

\[
(3.26) \quad \left| \frac{G}{F} \right| \leq M_{12}(|\phi'_1(\xi - q_1)| + |\phi'_2(\xi + q_2)|) \quad \text{for} \quad \xi \geq -\frac{q_3 + q_2}{2}.
\]

The lemma is proved by combining (3.17), (3.20), (3.23), (3.24), (3.25) and (3.26). \(\Box\)

Therefore, from (3.3) and (3.14), we have

\[
|G(\phi_1, \phi_2, \phi_3) + H(\phi_1, \phi_2, \phi_3)|
\]

\[
\leq F(\phi_1, \phi_2, \phi_3)KM\{e^{\beta_2(\xi + q_2)} + e^{\beta_3(\xi + q_3)}\}
\]

\[
\leq F(\phi_1, \phi_2, \phi_3)KM\{e^{\beta_2q_2} + e^{\beta_3q_3}\}
\]

for \(\xi \leq 0\);

\[
|G(\phi_1, \phi_2, \phi_3) + H(\phi_1, \phi_2, \phi_3)|
\]

\[
\leq F(\phi_1, \phi_2, \phi_3)KM\{e^{-\gamma_1(\xi - q_1)} + e^{\beta_3(\xi + q_3)}\}
\]

\[
\leq F(\phi_1, \phi_2, \phi_3)KM\{e^{-\gamma_1q_1} + e^{\beta_3(q_3 - q_2)/2}\}
\]

for \(0 \leq \xi \leq (-q_3 - q_2)/2\); and

\[
|G(\phi_1, \phi_2, \phi_3) + H(\phi_1, \phi_2, \phi_3)|
\]

\[
\leq F(\phi_1, \phi_2, \phi_3)KM\{e^{-\gamma_1(\xi - q_1)} + e^{-\gamma_2(\xi + q_2)}\}
\]

\[
\leq F(\phi_1, \phi_2, \phi_3)KM\{e^{-\gamma_1q_1} + e^{-\gamma_2(q_3 - q_2)/2}\}
\]
for $\xi \geq (-q_3 - q_2)/2$.

4. Existence of Entire solutions - Proof of Theorem 1.1

In this section, we always assume (1.8) holds. Then $-v_0 < v_1 < 0 < v_2$ and $s_2 > s_1 > 0$.

To construct the functions $q_i$, $i = 1, 2, 3$, in §2, we consider the following initial value problems (cf. [10, 14]):

\begin{align*}
(4.1) & \quad p_1' = s_1 + Le^{\kappa p_1}, \quad -\infty < t < 0, \quad p_1(0) = p_0; \\
(4.2) & \quad p_2' = s_2 + Le^{\kappa p_1}, \quad -\infty < t < 0, \quad p_2(0) = p_0; \\
(4.3) & \quad r_1' = s_1 - Le^{\kappa r_1}, \quad -\infty < t < 0, \quad r_1(0) = r_0; \\
(4.4) & \quad r_2' = s_2 - Le^{\kappa r_1}, \quad -\infty < t < 0, \quad r_2(0) = r_0,
\end{align*}

where $L > 2KM$ is a positive constant and

$$\kappa := \min \left\{ \gamma_1, \gamma_2, \beta_2, \beta_3, \frac{(s_2 - s_1)\gamma_2}{4s_1}, \frac{(s_2 - s_1)\beta_3}{4s_1} \right\}.$$

In fact, the solutions can be written explicitly as

\begin{align*}
p_1(t) &= s_1 t - \frac{1}{\kappa} \log \left[ e^{-\kappa p_0} + \frac{L(1 - e^{\kappa s_1 t})}{s_1} \right], \\
p_2(t) &= s_2 t - \frac{1}{\kappa} \log \left[ e^{-\kappa p_0} + \frac{L(1 - e^{\kappa s_1 t})}{s_1} \right], \\
r_1(t) &= s_1 t - \frac{1}{\kappa} \log \left[ e^{-\kappa r_0} - \frac{L(1 - e^{\kappa s_1 t})}{s_1} \right], \\
r_2(t) &= s_2 t - \frac{1}{\kappa} \log \left[ e^{-\kappa r_0} - \frac{L(1 - e^{\kappa s_1 t})}{s_1} \right].
\end{align*}

Now, we take $p_0$ and $r_0$ satisfying

$$p_0 = -\frac{1}{\kappa} \log \left( e^{-\kappa r_0} - \frac{2L}{s_1} \right) < -\delta, \quad r_0 < -\frac{1}{\kappa} \log \left( \frac{2L}{s_1} + e^{\kappa \delta} \right),$$

where $\delta$ is defined as in Lemma 3.3. Then we have

\begin{align*}
\lim_{t \to -\infty} (p_1(t) - r_1(t)) &= \lim_{t \to -\infty} (p_2(t) - r_2(t)) = 0, \\
\lim_{t \to -\infty} (p_1(t) - s_1 t) &= \lim_{t \to -\infty} (p_2(t) - s_2 t) = -\frac{1}{\kappa} \log \left( e^{-\kappa p_0} + \frac{L}{s_1} \right), \\
\lim_{t \to -\infty} (r_1(t) - s_1 t) &= \lim_{t \to -\infty} (r_2(t) - s_2 t) = -\frac{1}{\kappa} \log \left( e^{-\kappa r_0} - \frac{L}{s_1} \right).
\end{align*}

Also, there exists a positive constant $N$ such that

\begin{align*}
(4.5) & \quad 0 < p_1(t) - r_1(t) = p_2(t) - r_2(t) \leq Ne^{\kappa s_1 t} \quad \text{for all } t \leq 0,
\end{align*}

and $p_1(t), p_2(t), r_1(t), r_2(t) \leq -\delta$ for all $t \leq 0$.

The next lemma shows the existence of super-sub-solutions of (3.7).
Lemma 4.1. Define the functions $\overline{U}(\xi, t)$ and $\underline{U}(\xi, t)$ by
\[
\overline{U}(\xi, t) := Q(\phi_1(\xi - p_1(t)), \phi_2(\xi + p_1(t)), \phi_3(\xi + p_2(t))),
\]
\[
\underline{U}(\xi, t) := Q(\phi_1(\xi - r_1(t)), \phi_2(\xi + r_1(t)), \phi_3(\xi + r_2(t))).
\]
Then $(\overline{U}, \underline{U})(\xi, t)$ is a pair of super-sub-solutions of (3.7) for $t \leq t_0$ with some $t_0 < 0$. Moreover,
\[
(4.6) \quad \overline{U}(\xi, t) \geq \overline{U}(\xi, t) \quad \text{for} \quad \xi \in \mathbb{R}, \ t \leq t_0,
\]
\[
(4.7) \quad \sup_{\xi \in \mathbb{R}}\{\overline{U}(\xi, t) - \underline{U}(\xi, t)\} \leq \mu e^{\kappa_{s_1} t} \quad \text{for} \quad t \leq t_0,
\]
for some constant $\mu > 0$.

Proof. First, we prove $\overline{U}(\xi, t)$ is a super-solution of (3.7) for $t \leq t_0$ with some $t_0 < 0$. By (3.27)-(3.29), we have
\[
|G(\phi_1, \phi_2, \phi_3) + H(\phi_1, \phi_2, \phi_3)|
\leq
\begin{cases}
F(\phi_1, \phi_2, \phi_3)KM(e^{\beta_2 p_1} + e^{\beta_3 p_2}) & \text{for } \xi \leq 0, \\
F(\phi_1, \phi_2, \phi_3)KM(e^{\gamma_1 p_1} + e^{\beta_3 (p_2 - p_1)/2}) & \text{for } 0 \leq \xi \leq -\frac{p_1 + p_2}{2}, \\
F(\phi_1, \phi_2, \phi_3)KM(e^{\gamma_1 p_1} + e^{\gamma_2 (p_2 - p_1)/2}) & \text{for } \xi \geq -\frac{p_1 + p_2}{2}.
\end{cases}
\]
Moreover, we have
\[
p_2(t) - p_1(t) = r_2(t) - r_1(t) = (s_2 - s_1)t \to -\infty
\]
as $t \to -\infty$. Hence, by the choice of $\kappa$, there exists a $t_0 < 0$ such that
\[
(4.8) \quad \frac{\beta_3(p_2(t) - p_1(t))}{2} < \kappa p_1(t) < 0, \quad \frac{\gamma_2(p_2(t) - p_1(t))}{2} < \kappa p_1(t) < 0,
\]
\[
(4.9) \quad \frac{\beta_3(r_2(t) - r_1(t))}{2} < \kappa r_1(t) < 0, \quad \frac{\gamma_2(r_2(t) - r_1(t))}{2} < \kappa r_1(t) < 0
\]
for all $t \leq t_0$. Thus, by (4.8), we get
\[
(4.10) \quad |G(\phi_1, \phi_2, \phi_3) + H(\phi_1, \phi_2, \phi_3)| \leq 2F(\phi_1, \phi_2, \phi_3)KM e^{\kappa_{s_1}}.
\]
Then we obtain
\[
\mathcal{T}[\overline{U}] = -Q_y \phi'_1(p'_1 - s_1) + Q_z \phi'_2(p'_1 - s_1) + Q_w \phi'_3(p'_2 - s_2) - G(\phi_1, \phi_2, \phi_3) - H(\phi_1, \phi_2, \phi_3)
\geq F(\phi_1, \phi_2, \phi_3)(L - 2KM)e^{\kappa_{s_1}} \geq 0
\]
by (4.1), (4.2), (4.10) and Lemma 3.3. Hence $\overline{U}(\xi, t)$ is a super-solution of (3.7) for $t \leq t_0$.

Next, we prove $\underline{U}(\xi, t)$ is a sub-solution of (3.7) for $t \leq t_0$. By (3.27)-(3.29) and (4.9), we have
\[
(4.11) \quad |G(\phi_1, \phi_2, \phi_3) + H(\phi_1, \phi_2, \phi_3)| \leq 2F(\phi_1, \phi_2, \phi_3)KM e^{\kappa_{r_1}}.
\]
Then we obtain
\[
\mathcal{T}[\underline{U}] = -Q_y \phi'_1(r'_1 - s_1) + Q_z \phi'_2(r'_1 - s_1) + Q_w \phi'_3(r'_2 - s_2) - G(\phi_1, \phi_2, \phi_3) - H(\phi_1, \phi_2, \phi_3)
\leq -F(\phi_1, \phi_2, \phi_3)(L - 2KM)e^{\kappa_{r_1}} \leq 0
\]
by (4.3), (4.4), (4.11) and Lemma 3.3. Hence \( \hat{U}(\xi, t) \) is a sub-solution of (3.7) for \( t \leq t_0 \).

Finally, by (4.5), Lemma 3.3, the function \( F \) is bounded above and

\[
U(x + ct; t) - \hat{U}(x; t) = \int_0^1 F(\phi_1(\xi - \theta p_1 - (1 - \theta)r_1), \phi_2(\xi + \theta p_1 + (1 - \theta)r_1), \phi_3(\xi + \theta p_2 + (1 - \theta)r_2)) d\theta 
\times (p_1 - r_1),
\]

(4.6) and (4.7) hold. Hence the lemma is proved.

Now, we have a pair of super-sub-solutions of (3.7) satisfying (4.6). By using the same method as in [8, 10], the existence and uniqueness of entire solutions of (1.1) can be shown as follows.

**Theorem 4.2.** There exists a unique entire solution \( u(x, t) \) of (1.1) such that

\[
\underline{U}(x + \xi, t) \leq u(x, t) \leq \bar{U}(x + \xi, t)
\]

for all \( x \in \mathbb{R} \) and \( t \leq t_0 \) where the functions \( \underline{U} \) and \( \bar{U} \) are defined as in Lemma 4.1.

Finally, we consider the asymptotic behavior of the entire solution in Theorem 4.2 as \( t \to \pm \infty \). Since \( r_2(t) - (s_1 + s_2)t/2 \to -\infty \) and \( (s_1 + s_2)t/2 - r_1(t) \to -\infty \) as \( t \to -\infty \), there exists a constant \( T < 0 \) such that

\[
r_2(t) < \frac{s_1 + s_2}{2} t < r_1(t)
\]

for \( t < T \). Define

\[
\theta := -\frac{1}{\kappa} \log \left( e^{-\kappa r_0} - \frac{L}{s_1} \right).
\]

(4.12)

By a simple computation, there exists a constant \( \rho > 0 \) such that

\[
-\rho e^{\kappa s_1 t} < \frac{r_1(t) - s_1 t - \theta}{r_2(t) - s_2(t) - \theta} \leq 0.
\]

(4.13)

The next theorem shows the asymptotic behavior, as \( t \to -\infty \), of the entire solution obtained in Theorem 4.2.

**Theorem 4.3.** Let \( u(x, t) \) be an entire solution obtained in Theorem 4.2. Then (1.9) holds for the constant \( \theta \) defined by (4.12).
\textbf{Proof.} Recall that $\xi := x + ct$. For $x \leq -(c_1 + c_2)t/2$, we have $\xi \leq 0 \leq -r_1(t)$. By (4.7), (3.3), (3.4), (3.6), (3.9) and (4.13), we derive that

$$
|u(x, t) - \phi_1(x + c_1t - \theta)| = |U(\xi, t) - \phi_1(\xi - s_1t - \theta)|
$$

\begin{align*}
&\leq |U(\xi, t) - U(\xi, t)| + |U(\xi, t) - \phi_1(\xi - s_1t - \theta)| \\
&\leq |U(\xi, t) - U(\xi, t)| + |\phi_1(\xi - r_1(t)) - \phi_1(\xi - s_1t - \theta)| + |\phi_2(\xi + r_1(t))| \\
&\leq |U(\xi, t) - U(\xi, t)| + |\phi_1(\xi - r_1(t)) - \phi_1(\xi - s_1t - \theta)| + |\phi_2(\xi + r_1(t))| \\
&\leq \mu e^{\kappa_{s_1}t} + \sup_{\xi \in \mathbb{R}} |\phi_1(\xi)| |r_1(t) - s_1t - \theta| + \eta_1 \tau K e^{\beta_2(\xi + r_1(t))} \\
&\leq \mu e^{\kappa_{s_1}t} + K \rho e^{\kappa_{s_1}t} + \eta_1 \tau K e^{\beta_2 r_1(t)}
\end{align*}

for $t \leq \min\{t_0, T\}$, where $\eta_1 = 8/a$.

Now we consider $-(c_1 + c_2)t/2 \leq x \leq -(s_1 + s_2)t/2$. This implies that $0 \leq \xi \leq -(s_1 + s_2)t/2$. Recall that $-(s_1 + s_2)t/2 \leq -r_2(t)$ for $t \leq T$. By (4.7), (3.3), (3.4), (3.5), (3.9) and (4.13), we have

\begin{align*}
|u(x, t) - \phi_2(x + c_2t + \theta)| &= |U(\xi, t) - \phi_2(\xi + s_1t + \theta)| \\
&\leq |U(\xi, t) - U(\xi, t)| + |U(\xi, t) - \phi_2(\xi + s_1t + \theta)| \\
&\leq |U(\xi, t) - U(\xi, t)| + |\phi_2(\xi + r_1(t)) - \phi_2(\xi + s_1t + \theta)| + |\phi_3(\xi + r_2(t)) - a| \\
&\leq |U(\xi, t) - U(\xi, t)| + |\phi_2(\xi + r_1(t)) - \phi_2(\xi + s_1t + \theta)| + |\phi_3(\xi + r_2(t)) - a| \\
&\leq |U(\xi, t) - U(\xi, t)| + |\phi_2(\xi + r_1(t)) - \phi_2(\xi + s_1t + \theta)| + |\phi_3(\xi + r_2(t)) - a| \\
&\leq \mu e^{\kappa_{s_1}t} + \sup_{\xi \in \mathbb{R}} |\phi_1(\xi)| |r_1(t) - s_1t - \theta| + \eta_2((|\phi_3(\xi + r_2(t)) - a| + |\phi_1(\xi - r_1(t))|) \\
&\leq \mu e^{\kappa_{s_1}t} + K \rho e^{\kappa_{s_1}t} + \eta_2 \tau K (e^{\beta_3(\xi + r_2(t))} + e^{-\gamma_1(\xi - r_1(t))}) \\
&\leq \mu e^{\kappa_{s_1}t} + K \rho e^{\kappa_{s_1}t} + \eta_2 \tau K (e^{\beta_3(-(s_1 + s_2)t/2 + r_2(t))} + e^{\gamma_1 r_1(t)})
\end{align*}

for $t \leq \min\{t_0, T\}$, where $\eta_2 = 4/(1-a)$.
For the case \( x \geq -(c_2 + c_3)t/2 \), we have \( \xi \geq -(s_1 + s_2)t/2 \). Also, for \( t \leq T \), we know that 
\[-(s_1 + s_2)t/2 \geq -r_1(t). \] 
From (4.7), (3.3), (3.4), (3.6), (3.9) and (4.13), we show that 
\[ |u(x, t) - \phi_3(x + c_3t + \theta)| = |U(\xi, t) - \phi_3(\xi + s_2t + \theta)| \]
\[ \leq |U(\xi, t) - U(\xi, t)| + |U(\xi, t) - \phi_3(\xi + s_2t + \theta)| \]
\[ \leq |U(\xi, t) - U(\xi, t)| + |\phi_3(\xi + r_2(t)) - \phi_3(\xi + s_2t + \theta)| + |a - \phi_2(\xi + r_1(t))| \cdot \]
\[ (1 - \phi_3(\xi + r_2(t)))(\phi_3(\xi - r_1(t)) - \phi_3(\xi + r_2(t)))(1 - \phi_3(\xi + r_1(t)))(1 - \phi_3(\xi + r_2(t))) \]
\[ \leq \mu e^{\kappa t} + \sup_{\xi \in \mathbb{R}}|\phi_3'(\xi)| r_2(t) - s_2t - \theta | + \eta_3|a - \phi_2(\xi + r_1(t))| \]
\[ \leq \mu e^{\kappa t} + K \rho e^{\kappa t} + \eta_3 \tau Ke^{-r_2(\xi + r_1(t))} \]
for \( t \leq \min\{t_0, T\} \), where \( \eta_3 = 8/a \).

Therefore, the theorem is proved. \( \square \)

Finally, the asymptotic behavior, as \( t \to \infty \), of the entire solution obtained in Theorem 4.2 follows directly by a result in [6]. This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

Since the proof of Theorem 1.2 is quite similar to that of Theorem 1.1, we only point out the main differences in this section.

First, as in §3, (3.1) holds for \( c_1 = -v_0, c_2 = v_1, c_3 = -\bar{v}_1 \) and \( \phi_1(s) = \psi_0(-s), \phi_2(s) = \psi_1(s), \phi_3(s) = \bar{\psi}_1(-s) \), where \( (\alpha_1, \omega_1, \alpha_2, \omega_2, \alpha_3, \omega_3) = (1, 0, 0, a, a, 0) \). The key auxiliary function we found for linking these three fronts is as follows

\[ \tilde{Q}(y, z, w) = z + \frac{(1 - y)z(a - w)(-z) + y(a - z)w(1 - z)}{(1 - y)za + (a - z)w}. \]

Similar to Lemma 3.1, we have the following lemma on some properties of this function.

Lemma 5.1. The following three statements hold:

(i) \( \tilde{Q} \) can be rewritten as

\[ \tilde{Q}(y, z, w) = \begin{cases} 
& y + (1 - y)z \frac{a(w - y)}{(1 - y)za + (a - z)w}, \\
& w + (a - z)w \frac{y - w}{(1 - y)za + (a - z)w}.
\end{cases} \]

(ii) There exist functions \( \tilde{Q}_i, i = 1, 2, 3, \) such that

\[ \tilde{Q}_y(y, z, w) = (a - z)w\tilde{Q}_1(y, z, w), \]
\[ \tilde{Q}_z(y, z, w) = (1 - y)w\tilde{Q}_2(y, z, w), \]
\[ \tilde{Q}_w(y, z, w) = (1 - y)z\tilde{Q}_3(y, z, w). \]
(iii) There exist functions $R_j$, $j = 1, \cdots, 14$, such that
\[
\begin{align*}
\tilde{Q}_{yy}(y,z,w) &= zR_1(y,z,w) = (a - z)R_2(y,z,w) = wR_3(y,z,w), \\
\tilde{Q}_{zz}(y,z,w) &= (1 - y)R_4(y,z,w) = wR_5(y,z,w) \\
&= yR_6(y,z,w) + (w - a)R_7(y,z,w), \\
\tilde{Q}_{ww}(y,z,w) &= (1 - y)R_8(y,z,w) = zR_9(y,z,w) = (a - z)R_{10}(y,z,w), \\
\tilde{Q}_{yz}(y,z,w) &= wR_{11}(y,z,w), \quad \tilde{Q}_{zw}(y,z,w) = (1 - y)R_{12}(y,z,w), \\
\tilde{Q}_{yw}(y,z,w) &= zR_{13}(y,z,w) = (a - z)R_{14}(y,z,w).
\end{align*}
\]

Now, set $\bar{\tau} = (c_1 + c_2)/2 = (v_0 + v_1)/2$, $\bar{s}_1 = (c_2 - c_1)/2 = (v_0 + v_1)/2 > 0$ and $\bar{s}_2 = c_3 - \bar{\tau} = (-2\bar{v}_1 + v_0 - v_1)/2 > \bar{s}_1$. Replacing $Q$ by $\tilde{Q}$, we have the same conclusion as in Lemma 3.2. However, to get the positivity of $\tilde{Q}_z$, we need $\phi_3 - \phi_1 > 0$ for $q_1 < \xi < -q_3$. Therefore, we replace (3.2) by
\[
\phi_1(0) = \frac{a}{4}, \quad \phi_2(0) = \phi_3(0) = \frac{a}{2}.
\]

Next, for Lemma 3.3 we need to change the definition of $F$ as follows:
\[
F(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)):=\tilde{Q}_y(\phi_1, \phi_2, \phi_3)\phi_1'(\xi - q_1) + \tilde{Q}_z(\phi_1, \phi_2, \phi_3)\phi_2'(\xi + q_2) - \tilde{Q}_w(\phi_1, \phi_2, \phi_3)\phi_3'(\xi + q_3).
\]

Then the proof of all estimates in Lemma 3.3 is similar, except the proof of (3.13). Indeed, for this $\tilde{Q}$, we do not have the positivity of $\tilde{Q}_z$ for $\xi \geq -q_3$ because of $0 \leq w \leq a$. Thus we assume the extra condition
\[
q_3 - q_2 < -\delta
\]
(5.2)
to guarantee this lemma. More precisely, when $\xi \geq -q_3$, we know that $|\phi_3'| \geq \phi_3/\tau$ and $|\phi_2'| \leq Ke^{\gamma_2(q_3 - q_2)} \leq Ke^{-\gamma_2\delta}$ by using (5.2), (3.3) and
\[
\begin{align*}
\begin{cases}
|\phi_1(s) - 1|/|\phi_1'(s)| & \leq \tau, \quad s \leq 0, \\
|\phi_2(s) - 0|/|\phi_2'(s)| & \leq \tau, \quad s \leq 0, \\
|\phi_3(s) - a|/|\phi_3'(s)| & \leq \tau, \quad s \geq 0,
\end{cases}
\end{align*}
\]
(5.3)
for some positive constant $\tau$. Then we obtain that
\[
F(\phi_1, \phi_2, \phi_3) - \frac{1}{2}\tilde{Q}_w|\phi_3'| = \tilde{Q}_y|\phi_1'| + \tilde{Q}_z|\phi_2'| + \frac{1}{2}\tilde{Q}_w|\phi_3'| \geq \tilde{Q}_z|\phi_2'| + \frac{1}{2}\tilde{Q}_w|\phi_3'|
\geq a^2(1 - \phi_1)\phi_3(\phi_3 - \phi_1)/[(1 - \phi_1)\phi_2 + (a - \phi_2)\phi_3]^2\phi_2' + \frac{1}{2\tau}144 + \frac{(a - \phi_3)^2}{2}\phi_3
\geq \phi_3 \left[ \frac{a^2(1 - 0)(0 - a/4)}{(a/4)^2}Ke^{-\gamma_2\delta} + \frac{(a - \phi_3)^2}{288\tau} \right] \geq 0
\]
and (3.13) follows. With Lemma 3.3, the same proof as before can lead to Lemma 3.4. In particular, we also have the estimates (3.27)-(3.29).

Now, we consider the the functions $\overline{U}(\xi, t)$ and $\underline{U}(\xi, t)$ by

$$
\overline{U}(\xi, t) := \overline{Q}(\phi_1(\xi - \tilde{p}_1(t)), \phi_2(\xi + \tilde{p}_1(t)), \phi_3(\xi + \tilde{p}_2(t))), \\
\underline{U}(\xi, t) := \overline{Q}(\phi_1(\xi - \tilde{r}_1(t)), \phi_2(\xi + \tilde{r}_1(t)), \phi_3(\xi + \tilde{r}_2(t))).
$$

Here $\tilde{p}_i$ and $\tilde{r}_i$, $i = 1, 2$, are the solutions of the following initial value problems:

\[
\begin{aligned}
\tilde{p}_1' &= s_1 + Le^{\kappa \tilde{p}_1}, \quad -\infty < t < 0, \quad \tilde{p}_1(0) = \tilde{p}_0, \\
\tilde{r}_1' &= s_1 - Le^{\kappa \tilde{r}_1}, \quad -\infty < t < 0, \quad \tilde{r}_1(0) = \tilde{r}_0, \\
\tilde{p}_2' &= s_2 - Le^{\kappa \tilde{p}_1}, \quad -\infty < t < 0, \quad \tilde{p}_2(0) = \tilde{r}_0, \\
\tilde{r}_2' &= s_2 + Le^{\kappa \tilde{r}_1}, \quad -\infty < t < 0, \quad \tilde{r}_2(0) = \tilde{r}_0,
\end{aligned}
\]

where $\tilde{p}_0 = p_0$, $\tilde{r}_0 = r_0$ are the same as in §4, $L > 2KM$ is a positive constant and

$$
\kappa := \min \left\{ \gamma_1, \gamma_2, \beta_2, \beta_3, \frac{(s_2 - s_1)\gamma_2}{4s_1}, \frac{(s_2 - s_1)\beta_3}{4s_1} \right\}.
$$

It is easy to show that

\[
\begin{aligned}
\tilde{p}_1(t) &= s_1t - \frac{1}{\kappa} \log \left[ e^{-\kappa \tilde{p}_0} + \frac{L(1 - e^{\kappa s_1t})}{s_1} \right], \\
\tilde{p}_2(t) &= s_2t + \frac{1}{\kappa} \log \left[ e^{-\kappa \tilde{p}_0} + \frac{L(1 - e^{\kappa s_1t})}{s_1} \right] + \tilde{p}_0 + \tilde{r}_0, \\
\tilde{r}_1(t) &= s_1t - \frac{1}{\kappa} \log \left[ e^{-\kappa \tilde{r}_0} - \frac{L(1 - e^{\kappa s_1t})}{s_1} \right], \\
\tilde{r}_2(t) &= s_2t + \frac{1}{\kappa} \log \left[ e^{-\kappa \tilde{r}_0} - \frac{L(1 - e^{\kappa s_1t})}{s_1} \right] + \tilde{p}_0 + \tilde{r}_0.
\end{aligned}
\]

Notice that, since we have $\tilde{p}_2(t) - \tilde{p}_1(t) \to -\infty$ and $\tilde{r}_2(t) - \tilde{r}_1(t) \to -\infty$ as $t \to -\infty$, there exists a $t_0 < 0$ such that

$$
\tilde{p}_2(t) - \tilde{p}_1(t) < -\delta \quad \text{and} \quad \tilde{r}_2(t) - \tilde{r}_1(t) < -\delta
$$

for all $t \leq t_0$. Also, there exists a positive constant $N$ such that

$$
0 < \tilde{p}_1(t) - \tilde{r}_1(t) = \tilde{r}_2(t) - \tilde{p}_2(t) \leq Ne^{\kappa s_1t} \quad \text{for all} \quad t \leq 0.
$$

Then Lemma 4.1 can be easily proved.

With Lemma 4.1, we then have the existence of entire solution connecting three fronts $\phi_i$, $i = 1, 2, 3$ as that of Theorem 4.2. The asymptotic behavior, as $t \to -\infty$, of this entire solution (namely, (1.11)) can be proved similarly as that of Theorem 4.3 by defining

$$
\theta_1 := -\frac{1}{\kappa} \log \left( e^{-\kappa \tilde{r}_0} - \frac{L}{s_1} \right), \quad \theta_2 := \frac{1}{\kappa} \log \left( e^{-\kappa \tilde{r}_0} - \frac{L}{s_1} \right) + \tilde{p}_0 + \tilde{r}_0.
$$

Note that the fact that

$$
|\tilde{r}_1(t) - s_1t - \theta_1| \leq pe^{\kappa s_1t}, \quad |\tilde{r}_2(t) - s_2t - \theta_2| \leq pe^{\kappa s_1t}
$$
for some positive constant \( \rho \) was used. Finally, the asymptotic behavior, as \( t \to \infty \), of this entire solution follows directly by a result of [6]. This completes the proof of our second main theorem, Theorem 1.2.

6. Proof of Theorem 1.3

First, recall the rational function \( Q \) defined by (3.5) and \( \tilde{p}_1 \) defined as in §5. Let

\[
\tilde{c} = \frac{-v_0 + v_1}{2} < 0
\]

and take \( \psi_0(0) = a/4 \) and \( \psi_1(0) = a/2 \). Then, according to [14, Proposition 3.2 and §4] or a proof similar to Theorem 1.1, the function

\[
\bar{U}_1(x, t) := Q(\psi_0(-x - \tilde{c}t + \tilde{p}_1(t)), \psi_1(x + \tilde{c}t + \tilde{p}_1(t)), a)
\]

is a supersolution of (1.1) for \( t \ll -1 \). Note that

\[
Q(y, z, a) = z + \frac{y(1 - z)(a - z)}{a - yz}.
\]

Lemma 6.1. Set \( \bar{U}(x, t) := \bar{U}_1(-|x|, t) \). Then \( \bar{U} \) is a supersolution of (1.1) for \( t \ll -1 \).

Proof. We have

\[
\bar{U}_{1,x}(x, t) = -Q_y \cdot \psi'_0(-x - \tilde{c}t + \tilde{p}_1(t)) + Q_z \cdot \psi'_1(x + \tilde{c}t + \tilde{p}_1(t))
\]

where

\[
Q_y = Q_y(\psi_0(-x - \tilde{c}t + \tilde{p}_1(t)), \psi_1(x + \tilde{c}t + \tilde{p}_1(t)), a),
\]

\[
Q_z = Q_z(\psi_0(-x - \tilde{c}t + \tilde{p}_1(t)), \psi_1(x + \tilde{c}t + \tilde{p}_1(t)), a).
\]

Recall that \( \psi_0(-\tilde{c}t + \tilde{p}_1(t)) \to 0, \psi_1(\tilde{c}t + \tilde{p}_1(t)) \to a \) as \( t \to -\infty \). Using

\[
Q_y(y, z, a) = \frac{a(a - z)(1 - z)}{(a - yz)^2}, \quad Q_z(y, z, a) = \frac{a(a - y)(1 - y)}{(a - yz)^2},
\]

and (5.3), we have

\[
\bar{U}_x(0^-, t) \geq 0
\]

for \( t \ll -1 \). Therefore, \( \bar{U} \) is a supersolution of (1.1) for \( t \ll -1 \).

Proof of Theorem 1.3. Because we already have a supersolution, we need to construct a subsolution. To construct a subsolution, we borrow \( \bar{Q} \) in the proof of Theorem 1.2, namely,

\[
\bar{Q}(y, z, w) = y + (1 - y)z \frac{a(w - y)}{(1 - y)za + (a - z)w}
\]

and define

\[
\bar{U}(x, t) := \bar{Q}(\psi_0(-x - \tilde{c}t + \tilde{r}_1), \psi_1(x + \tilde{c}t + \tilde{r}_1), \psi_1(-x - \tilde{c}t - \tilde{r}_3)),
\]

where \( \tilde{r}_3(t) := \tilde{r}_2(t) - \tilde{p}_0 - \tilde{r}_0 + 1 \) with \( \tilde{r}_i(\cdot) := \tilde{r}_i(t) \), \( i = 1, 2 \), defined as in §5 and here we take \( c_3 = -c_2, \phi_3(s) = \phi_2(-s) \). Then this \( \bar{U} \) is a subsolution of (1.1) for \( t \ll -1 \).

We claim that \( \bar{U}(x, t) - \bar{U}(x, t) \geq 0 \) for \( x \in \mathbb{R} \) and \( t < -T \) with some sufficiently large \( T \).
First, we consider \( x \leq 0 \). Because \( Q(y, z, a) - \tilde{Q}(y, z, w) > 0 \) for \( y \in (0, 1) \), \( z \in (0, a) \) and \( w \in (0, a) \), we can easily check that
\[
Q(\psi_0(-x - ct + \tilde{r}_1), \psi_1(x + ct + \tilde{r}_1), a) \\
- \tilde{Q}(\psi_0(-x - ct + \tilde{r}_1), \psi_1(x + ct + \tilde{r}_1), \psi_1(-x - ct - \tilde{r}_3)) > 0.
\]

Then we obtain
\[
\overline{U}(x, t) - \underline{U}(x, t) \\
> Q(\psi_0(-x - ct + \tilde{p}_1), \psi_1(x + ct + \tilde{p}_1), a) - Q(\psi_0(-x - ct + \tilde{r}_1), \psi_1(x + ct + \tilde{p}_1), a) \\
+ Q(\psi_0(-x - ct + \tilde{r}_1), \psi_1(x + ct + \tilde{p}_1), a) - Q(\psi_0(-x - ct + \tilde{r}_1), \psi_1(x + ct + \tilde{r}_1), a) \\
= \int_0^1 J \, d\theta \cdot (\tilde{p}_1 - \tilde{r}_1)
\]

where
\[
J := Q_y(\psi_0(-x - ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1), \psi_1(x + ct + \tilde{p}_1), a) \psi'_0(-x - ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1) \\
+ Q_z(\psi_0(-x - ct + \tilde{r}_1), \psi_1(x + ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1), a) \psi'_1(x + ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1).
\]

Using this, we divide our discussion into two cases: \( \psi_0(-x - ct + \tilde{r}_1) \leq a \) and \( \psi_0(-x - ct + \tilde{r}_1) > a \).

When \( \psi_0(-x - ct + \tilde{r}_1) \leq a \), we have \( Q_x \geq 0 \). Then it is easy to see that \( \overline{U} - \underline{U} \geq 0 \), since we always have \( Q_y \geq 0 \).

Next we consider the case where \( \psi_0(-x - ct + \tilde{r}_1) > a \). Note that \( \tilde{p}_1(t) > \tilde{r}(t) \) for \( t \leq 0 \). Because \( \psi_0(-x - ct + \tilde{r}_1) > a > a/2 = \psi_0(0) \) and \( \psi_0 \) is increasing, we have
\[
0 < -x - ct + \tilde{r}_1 \leq -x - ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1 \leq -x - ct + \tilde{p}_1,
\]
\[
x + ct + \tilde{r}_1 \leq x + ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1 \leq x + ct + \tilde{p}_1 \leq \tilde{r}_1 + \tilde{p}_1 < 0.
\]

for any \( \theta \in [0, 1] \). This implies that
\[
\psi_0(-x - ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1) \geq \frac{a}{2},
\]
\[
0 < \psi_1(x + ct + \tilde{p}_1), \psi_1(x + ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1) < \frac{a}{2}.
\]

Thus we have
\[
J = \frac{a[a - \psi_1(x + ct + \tilde{p}_1)][1 - \psi_1(x + ct + \tilde{p}_1)]}{[a - \psi_0(-x - ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1)\psi_1(x + ct + \tilde{p}_1)]^2} \psi'_0(-x - ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1) \\
+ \frac{a[a - \psi_1(x + ct + \tilde{r}_1)][1 - \psi_0(-x - ct + \tilde{r}_1)\psi_1(x + ct + \tilde{r}_1)]}{[a - \psi_0(-x - ct + \tilde{r}_1)\psi_1(x + ct + \tilde{r}_1)]^2} \psi'_1(x + ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1) \\
\geq \frac{a(a - a/2)(1 - a/2)}{a^2} \psi'_0(-x - ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1) \\
- \frac{4a(1 - a)}{a^2} \psi'_0(-x - ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1).
\]
Also, by the facts $\psi_0'(s)/(1-\psi_0(s)) \leq \lambda$ for $s \geq 0$ for some positive constant $\lambda$ and $\tilde{p}_1-\tilde{r}_1 \leq N$, we know
\[
\frac{1 - \psi_0(-x - ct + \tilde{r}_1)}{1 - \psi_0(-x - ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1)} = \exp \left\{ \int_{-x - ct + \tilde{r}_1}^{x - ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1} \frac{\psi_0'(\zeta)}{1 - \psi_0(\zeta)} d\zeta \right\} \leq e^{N\lambda}.
\]
Using (3.3) and (3.4) yields
\[
J \geq \left( a - a/2 \right) \left( 1 - a/2 \right) \left[ 1 - \psi_0(-x - ct + \theta \tilde{p}_1 + (1 - \theta)\tilde{r}_1) \right] \frac{a}{a} - \frac{4(1 - a)\left[ 1 - \psi_0(-x - ct + \tilde{r}_1) \right]}{a} Ke^{-3e^{s_T}}.
\]
by taking $\delta := -\tilde{r}_1(-T) - \tilde{p}_1(-T)$ sufficiently large. Hence we obtain that $\mathcal{U} - \mathcal{U} \geq 0 \text{ when }$ 
\[
\psi_0(-x - ct + \tilde{r}_1) > a \text{ and } t < -T.
\]
Next, we show $\mathcal{U}(x, t) - \mathcal{U}(x, t) \geq 0 \text{ for } x > 0 \text{ and } t < -T \text{ with some sufficiently large } T.$
For this, we write
\[
\mathcal{U}(x, t) - \mathcal{U}(x, t) = \psi_1(-x + ct + \tilde{p}_1) - \psi_1(-x - ct - \tilde{r}_3) + \psi_0(-x + ct + \tilde{p}_1)a - \psi_1(-x - ct + \tilde{r}_1)\psi_1(-x + ct + \tilde{p}_1) - \psi_1(-x - ct + \tilde{r}_1)\psi_1(-x - ct + \tilde{r}_3)
\]
(6.1)
It is easy to check that
\[
0 < -ct - \tilde{r}_2 + \tilde{p}_0 + \tilde{r}_0 \leq ct + \tilde{p}_1 < ct - \tilde{p}_1
\]
for $t < -T$ with some sufficiently large $T$ by the facts $\tilde{p}_1 < 0,$
\[
-ct - \tilde{r}_2 + \tilde{p}_0 + \tilde{r}_0 = -c_3t - \frac{1}{\kappa} \log \left( e^{-\kappa\tilde{r}_0} - \frac{L(1 - e^{s_1t})}{s_1} \right)
\]
\[
\tilde{c}t + \tilde{p}_1 - (-ct - \tilde{r}_2 + \tilde{p}_0 + \tilde{r}_0) = \frac{1}{\kappa} \log \left( \frac{e^{-\kappa\tilde{r}_0} - L(1 - e^{s_1t})}{e^{-\kappa\tilde{r}_0} - \frac{L(1 - e^{s_1t})}{s_1}} \right).
\]
Also, since $\psi_1$ is increasing, we know that $\psi_1(-x + ct + \tilde{p}_1) \geq \psi_1(-x - ct - \tilde{r}_3).$
For $0 \leq x \leq -ct - \tilde{r}_2 + \tilde{p}_0 + \tilde{r}_0$, we have $\psi_1(-x - ct - \tilde{r}_3) \geq \psi_1(-1) > 0, 0 \leq \psi_1 \leq a$ and $\psi_0(-x - ct + \tilde{r}_1) \to 0$ as $t \to -\infty$. Then it follows from (6.1) that $\mathcal{U}(x, t) - \mathcal{U}(x, t) \geq 0$ for $t < -T \text{ with some sufficiently large } T.$
For $-ct - \tilde{r}_2 + \tilde{p}_0 + \tilde{r}_0 \leq x \leq ct + \tilde{p}_1$, we have $\psi_1(-x + ct + \tilde{p}_1) \geq \psi_1(0), \psi_1(-x - ct - \tilde{r}_3) \leq \psi_1(-1) \text{ and }$ $\psi_0(-x - ct + \tilde{r}_1) \to 0$ as $t \to -\infty$. Again, by (6.1), we obtain that $\mathcal{U} - \mathcal{U} \geq 0$ for $t < -T \text{ with some sufficiently large } T.$
For \( x \geq \tau t + \bar{p}_1 \), we know \( \psi_1(-x + \tau t + \bar{p}_1) \leq \psi_1(0) = a/2 \) and \( \psi_0(x - \tau t + \bar{p}_1) \geq \psi_0(-x - \tau t + \bar{r}_1) \), by \( x - \tau t + \bar{p}_1 \geq -x - \tau t + \bar{r}_1 \) and \( \psi_0 \) and \( \psi_1 \) are increasing. Also, as \( t \to -\infty \), we have \( \psi_1(x + \tau t + \bar{r}_1) \to a \), since

\[
x + \tau t + \bar{r}_1 \geq 2\tau t + \bar{p}_1 + \bar{r}_1
= 2c_2 t - \frac{1}{\kappa} \log \left[ e^{-\eta_0} + \frac{L(1 - e^{x_{s_1} t})}{s_1} \right] - \frac{1}{\kappa} \log \left[ e^{-\eta_0} - \frac{L(1 - e^{x_{s_1} t})}{s_1} \right] \to \infty.
\]

From (6.1), we obtain that

\[
\overline{U}(x, t) - \underline{U}(x, t) \\
\geq \psi_1(-x + \tau t + \bar{p}_1) - \psi_1(-x - \tau t - \bar{r}_3) + \psi_0(-x - \tau t - \bar{r}_1) \left[ (a - \psi_1(-x + \tau t + \bar{p}_1))[1 - \psi_1(-x + \tau t + \bar{p}_1)] - a \psi_0(x - \tau t + \bar{p}_1) \psi_1(-x + \tau t + \bar{p}_1) \right] - \left[ a - \psi_1(x + \tau t + \bar{r}_1) \right] \psi_1(-x - \tau t - \bar{r}_3) + \left[ a - \psi_1(x + \tau t + \bar{r}_1) \right] \psi_1(-x - \tau t - \bar{r}_3) \geq 0
\]

for \( t < -T \) with some sufficiently large \( T \).

Since we have the supersolution \( \overline{U}(x, t) = U_1(-|x|, t) \), the subsolution \( \underline{U}(x, t) \) and \( \overline{U}(x, t) \geq \underline{U}(x, t) \) for \( x \in \mathbb{R} \) and \( t < -T_0 \) with some sufficiently large \( T_0 \). For \( T > T_0 \), let us denote the solution of (1.1) with the initial function \( u_0 \) by \( u(x, t; u_0) \). Consider the following solution \( u^T(x, t) := u(x, t + T; \overline{U}(\cdot, -T)) \), we see that

\[
\overline{U}(x, t) \leq u^T(x, t) \leq \overline{U}(x, t)
\]

for any \( x \in \mathbb{R} \) and \( t \geq -T \). By the uniqueness and \( u^T(x, -T) = \overline{U}(x, -T) = \overline{U}(-x, -T) = u^T(-x, -T) \), we have \( u^T(x, t) = u^T(x, t) \) any \( x \in \mathbb{R} \) and \( t \geq -T \). Note that, by comparison, \( u^{T_1} \geq u^{T_2} \) if \( T_1 < T_2 \). Thus \( u^\infty := \lim_{T \to \infty} u^T \) is well-defined. Then we obtain that \( u^\infty \) is an entire solution of (1.1) which satisfies

\[
\overline{U}(x, t) \leq u^\infty(x, t) \leq \overline{U}(x, t)
\]

and \( u^\infty(-x, t) = u^\infty(x, t) \).

Finally, we claim that \( u(x, t) = u^\infty(x, t) \) satisfies (1.12). For \( x \leq 0 \) and \( t \leq 0 \), we have

\[
-x - \tau t - \bar{r}_3 \geq -\tau t - (-v_1 - \tau) t - \frac{1}{\kappa} \log \left[ e^{-\eta_0} - \frac{L(1 - e^{x_{s_1} t})}{s_1} \right] - 1 \geq v_1 t + \bar{r}_0 - 1
\]

and

\[
a - \psi_1(-x - \tau t - \bar{r}_3) \leq \tau |\psi'_1(-x - \tau t - \bar{r}_3)| \\
\leq \tau K \exp(-\gamma_2(-x - \tau t - \bar{r}_3)) \leq \tau K e^{-\gamma_2 v_1 t},
\]
where $K_1 = K e^{-\gamma_2(\hat{r}_0-1)}$. It follows that
\[
Q(\psi_0(-x - ct + \hat{r}_1), \psi_1(x + ct + \hat{r}_1), a) \\
- \bar{Q}(\psi_0(-x - ct + \hat{r}_1), \psi_1(x + ct + \hat{r}_1), \psi_1(-x - ct - \hat{r}_3)) \\
\leq a(1-0)[a - \psi_1(-x - ct - \hat{r}_3)](a-0)[1 + a(1-0)a] \\
(a^2/4)(a/2) \\
\leq K_2 e^{-\gamma_2 v_1 t},
\]

where
\[
K_2 := \frac{8(1+a)\tau K_1}{a}.
\]

Now we choose $\kappa_1 = \min\{\kappa, -\gamma_2 v_1/s_1\}$. By the facts $Q_y, Q_z, \psi_0', \psi_1'$ are bounded, (4.5) and the last inequality, we obtain that
\[
\bar{U}(x,t) - \underline{U}(x,t) \leq K_3 e^{\kappa s_1 t} + K_2 e^{-\gamma_2 v_1 t} \leq (K_2 + K_3)e^{\kappa s_1 t}, \quad x \leq 0, \ t \leq 0,
\]
for some positive constant $K_3$. By a similar argument as in the proof of Theorem 4.3, we obtain that
\[
\lim_{t \to -\infty} \left\{ \sup_{x \leq w_1(t)} |u(x,t) - \psi_0(-x + v_0 t + \theta_1)| + \sup_{w_1(t) \leq x \leq 0} |u(x,t) - \psi_1(x + v_1 t + \theta_2)| \right\} = 0.
\]

Since $u(x,t) = u(-x,t)$, (1.12) is proved.

Finally, the asymptotic behavior (1.10) can be derived as before and thereby the proof of Theorem 1.3 is completed.

\begin{thebibliography}{99}
\end{thebibliography}


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