LOCAL COHOMOLOGY AND $\mathcal{D}$-MODULES:

An application to arrangements of linear varieties

by

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• *Let R be a (commutative) Noetherian ring, $I \subset R$ an ideal of R, and M a R-module*. The local cohomology modules of M supported on I are defined as:

$$H_I^r(M) = R^r(\Gamma_I(M))$$

where $\Gamma_I(M) = \bigcup_{n \geq 0}(0 :_{I^n} M)$.

- It may be seen that:

$$H_I^r(M) \simeq \text{ind.lim } \mathrm{Ext}_R^r(R/I^n, M) \simeq$$

$$H^r(C(a_1, \ldots, a_t; M))$$

where $C(a_1, \ldots, a_t; M)$ denotes de Čech complex of M with respect to a (any) system of generators $a_1, \ldots, a_t$ of I, being

$$C(a_1, \ldots, a_t; M)_r = \bigoplus_{i_1 < \cdots < i_r} M_{a_{i_1} \cdots a_{i_r}}$$

- As we know, it is in general very difficult to determine the structure (vanishing or non-vanishing, support, injective dimension...) of the local cohomology modules as R-modules.

This is mainly due to the fact that they are, in general, not finitely generated.

- In this sense, *finiteness conditions on local cohomology modules* have been always considered to be important in order to get enough information about their structure.

Let us mention the following result proved some years ago:

Theorem (1993)

Let R be a regular ring containing a field and $I \subset R$ an ideal of R. Let $H^r_I(R)$ be the r-th local cohomology module of R supported on I. Then the following hold:

i) $H^j_{\mathfrak{m}}(H^r_I(R))$ is injective for any maximal ideal $\mathfrak{m}$ of R.

ii) inj.dim $_R(H^r_I(R)) \leq \dim_R H^r_I(R)$.

iii) The set of associated primes of $H^r_I(R)$ is finite.

iv) The Bass numbers of $H^r_I(R)$ are finite.

The proof of the theorem depends on the characteristic of the field, and was obtained by:

- C. Huneke and R.Y. Sharp if the field is of positive characteristic (by using the Frobenius).

- G. Lyubeznik if the field is of characteristic zero (by using the theory $\mathcal{D}$-modules).

When R is a polynomial ring or a formal power series ring over a field of characteristic zero, the local cohomology modules of R have a $\mathcal{D}$-module structure, where $\mathcal{D}$ is the ring of differential operators. In fact, they are holonomic as $\mathcal{D}$-modules.

The purpose of this talk is to show how to use some aspects of the theory of $\mathcal{D}$-modules to obtain a better understanding of the local cohomolgy modules $H^r_I(R)$, particularly in the case when I is the definition ideal of an arrangement of linear varieties.

The plan of the talk is the following:

1.- The Weyl algebra
2.- Holonomic $\mathcal{D}$-modules
3.- The characteristic variety
4.- The characteristic cycle
5.- Arrangements of linear varieties
6.- The Mayer-Vietoris spectral sequence
7.- The Betti numbers of the complement
8.- Extension problems
9.- Final comments: Monomial ideals

1.- <u>**The Weyl algebra**</u>

- Let k be a field of char$(k) = 0$.

- $R = k[x_1, \ldots, x_n]$.

- *The Weyl algebra* is the R-algebra

$$A_n(k) = R < \partial_1, \ldots, \partial_n > \subset End_k(R)$$

generated by the partial derivatives $\partial_i = \frac{d}{dx_i}$ with the relations given by:

$$\partial_i \partial_j = \partial_j \partial_i.$$

$$\partial_i r - r \partial_i = \frac{dr}{dx_i}, \text{ where } r \in R.$$

FACT: $A_n(k)$ *is left (right) Noetherian.*

We will always consider left $A_n(k)$-modules.

- For any commutative k-algebra R one may define the *ring of differential operators* of R inductively as follows:

• For $P, Q \in End_k(R)$ (k-linear operators) denote by

$$[P, Q] = P{\cdot}Q - Q{\cdot}P$$

the commutator.

- *We say that a k-linear operator P has order* 0 if $[a, P] = 0$ for any $a \in R$.

- And that P *has order* n if it does not have order less than n, and $[a, P]$ has order less than n for any $a \in R$.

• The *ring of differential operators*

$$\mathcal{D} = D(R)$$

is the set of all k-linear operators of finite order.

FACT: *The differential operators of order* 0 *are the elements of* R*. Those of order* ≤ 1 *correspond to the elements of* $Der_k(R) + R$*.*

- The ring of differential operators is a very important but difficult object, and there are many results and conjectures about it.

FACT: *If* R *is regular, the ring of differential operators is generated by* $Der_k(R)$.

- This is not true in general: $R = k[t^2, t^3]$.

Nakai's conjecture: R is regular if and only if the ring of differential operators is generated by $Der_k(R)$.

FACT: *For* $R = k[x_1, \ldots, x_n]$ *the ring of differential operators is the Weyl algebra:*

$$A_n(k) \simeq \mathcal{D}$$

What is an (algebraic) $\mathcal{D}$-module?

- Let us consider the following example:

Let $P_1, \ldots, P_n$ be a family of differential operators in $A_n(k)$ and consider the system of differential equations

$$P_1(f) = \cdots = P_m(f) = 0$$

We may then consider the $\mathcal{D}$-module defined as

$$A_n(k) / \sum_1^m A_n(k) P_i$$

FACT: *The vector space of polynomial solutions of the system is isomorphic to*

$$\mathrm{Hom}_{A_n(k)}(M, R)$$

More in general, for any pair of $\mathcal{D}$-modules M, S, one may define the solution space of M in S as $\mathrm{Hom}_{A_n(k)}(M, S)$...

2.- **Holonomic $\mathcal{D}$-modules**

- The Weyl algebra has an increasing filtration

$$\{\Sigma_v\}_{v\geq 0}$$

given by the degree (the Bernstein filtration) such that

$$gr_{\Sigma}(\mathcal{D}) = \bigoplus_{v\geq -1} \Sigma_{v+1}/\Sigma_v =$$

$$K[x_1,\ldots,x_n;y_1,\ldots,y_n] = R[y_1,\ldots,y_n]$$

where

$$y_i = \bar{\partial}_i \in \frac{\Sigma_1}{\Sigma_0}$$

(and $\Sigma_{-1} = 0$).

Note that this is a commutative ring of dimension $2n$*...*

- For any finitely generated $\mathcal{D}$-module M there exists a good filtration, that is, there is an increasing sequence of finitely generated R-submodules

$$\{\Gamma_k\}_{k\geq 0}$$

satisfying

i) $\bigcup \Gamma_k = M$,

ii) $\Sigma_v \Gamma_k \subseteq \Gamma_{v+k}$

for all $k, v \geq 0$, and such that

$$gr_{\Gamma}(M) = \bigoplus_{k\geq -1} \Gamma_{k+1}/\Gamma_k$$

is a finitely generated $gr_{\Sigma}(\mathcal{D})$-module (where $\Gamma_{-1} = 0$).

Note that the dimension of this module is $\leq 2n\ldots$

FACT (Bernstein inequality): *The dimension of the associated graded module* $gr_{\Gamma}(M)$ *is at least n.* So

$$n \leq \dim_{gr_{\Sigma}(\mathcal{D})}(gr_{\Gamma}(M)) \leq 2n$$

Definition An finitely generated $\mathcal{D}$-module is said to be *holonomic* if

$$\dim_{gr_{\Sigma}(\mathcal{D})}(gr_{\Gamma}(M)) = n$$

The definition is independent of the choosen good filtration.

Thus, in some sense, holonomic modules are the "smallest" ones in the category of $\mathcal{D}$-modules.

- *The category of holonomic* $\mathcal{D}$*-modules is an abelian full subcategory of the category of* $\mathcal{D}$*-modules with very good properties. In particular, they are of finite length.*

FACT: *R itself is an holonomic $\mathcal{D}$-module.*

FACT: *For any $f \in R$, the localization R_f is also an holonomic $\mathcal{D}$-module.*

Now, from the construction by means of the Čech complex of the local cohomolgy modules and the fact that the category of holonomic $\mathcal{D}$-modules is an abelian full subcategory we have that:

The local cohomology modules $H^r_I(R)$ are holonomic $\mathcal{D}$-modules too.

This is the fact on the basis of the proof by G. Lyubeznik of the finiteness results cited at the beginning of this talk. Observe that we have passed from a non-finitely generated structure to a finitely generated one (over a non commutative ring...)

3.- **The characteristic variety**

Let M be a finitely generated $\mathcal{D}$-module.

Given a good filtration $\{\Gamma_k\}_{k\geq 0}$ of M we may consider the following objects:

- *Characteristic Ideal*:

$$J(M) = \sqrt{Ann_{gr_{\Sigma}(\mathcal{D})}(gr_{\Gamma}(M))}.$$

- *Characteristic Variety*:

$$C(M) = V(J(M)) \subseteq \mathrm{Spec}\,(gr_{\Sigma}(\mathcal{D})).$$

FACT: *The characteristic ideal is independent of the chosen good filtration.*

Observe that, by the Bernstein inequality, the dimension of the characteristic variety is less or equal than n, and that for an holonomic module the dimension is exactly n.

- The characteristic variety of a finitely generated $\mathcal{D}$-module M provides a link with its structure as R-module. Namely, if

$$\pi : \text{Spec}\, R[y_1, \ldots, y_n] \to \text{Spec}\, R$$

is the natural projection, then

$$\pi(C(M)) = \text{Supp}\,(M)$$

This gives a chance to determine the support of M as R-modue if we have an adequate description of its characteristic variety.

- For instance, if $M = H^r_I(R)$ is a local cohomolgy module. In this case, it is also a *regular* holonomic module: This means that *the annihilator ideal of* $gr_\Gamma(M)$ *is itself radical*.

Characteristic varieties of regular holonomic modules have special geometric properties.

4.- <u>**The characteristic cycle**</u>

In general, it is rather difficult to compute characteristic varieties. But the following construction helps.

- Let M be a finitely generated $\mathcal{D}$-module.

The characteristic variety of M is the union of its irreducible components

$$C(M) = \bigcup_{\alpha} V_\alpha$$

- One can then attach to each irreducible component V_α a certain multiplicity m_α in the following way:

$$m_\alpha = e_{gr_\Sigma(\mathcal{D})_{P_\alpha}}(gr_\Gamma(M)_{P_\alpha})$$

where P_α is the definition ideal of V_α.

Definition *The characteristic cycle of M is defined as*

$$CC(M) = \sum_{\alpha} m_\alpha V_\alpha$$

FACT: *The characteristic cycle is additive with respect to exact sequences of finitely generated $\mathcal{D}$-modules.*

This provides one of the basic tools to compute characteristic cycles, if we are able to compute them in some basic cases.

For instance, there is a general, but not so easy, method to compute the characteristic cycle of the modules R_f, $f \in R$ (J. Briançon, P. Maisonobe and M. Merle, 1994).

FACT: (J. Àlvarez-Montaner, 2004) *The multiplicities of the characteristic cycle of $H^r_I(R)$ are invariants of R/I.*

Example

• Let $X \subset \mathbb{A}^n_k$ be a linear variety of codimension r defined by the prime ideal $\mathfrak{p}$.

- X could be any of the coordinate hyperplanes: then $\mathfrak{p} = (x_i)$ for some i.

- Or X could be any of their intersections: then $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_r})$ for some $i_1, \ldots, i_r$.

FACT: *The characteristic variety of* $H^r_{\mathfrak{p}}(R)$ *is of the form*

$$C(H^r_{\mathfrak{p}}(R)) = T^*_X \mathbb{A}^n_k$$

the conormal bundle of A^n_k with respect to X. It is an irreducible variety and

$$CC(H^r_{\mathfrak{p}}(R)) = T^*_X \mathbb{A}^n_k$$

Moreover, $\pi(T^*_X \mathbb{A}^n_k) = X$.

5.- **Arrangements of linear varieties**

- Assume now that k is any field.

- Let $\{X_i,\ i = 1, \ldots, m\}$ be a finite collection of linear varieties in the affine space $\mathbb{A}^n_k$.

- For each i, let $\mathfrak{p}_i$ be the defining prime ideal of X_i.

Their union $X = \bigcup_{i=1}^m X_i$ is said to be an

Definition *Arrangement of linear varieties.*

It is an affine variety defined by the ideal

$$I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$$

- Observe that all the linear varieties contain the origin (central arrangement) if and only if I is a square free monomial ideal.

Now, we want to apply the theory explained before in order to study the local cohomology modules $H_I^r(R)$.

Our purpose is twofold:

1.- *To compute and describe explicitly their characteristic cycles.*

2.- *To give an interpretation of the multiplicities in terms of...*

This will be possible thanks to the existence of a *spectral sequence* that will provide a very good filtration of the local cohomology modules.

By means of this filtration and the formal properties of the characteristic cycle we shall be able to reach our target.

A good way to organize part of the information contained in the arrangement is by means of a poset:

- The arrangement X defines in a natural way a *partially ordered set* $P(X)$ in the following way:

Consider all possible intersections of the linear varieties X_i, $i = 1, \ldots, m$:

$$X_{i_1} \cap \cdots \cap X_{i_k}$$

for $1 \leq i_1 \leq \cdots \leq i_k \leq m$, with the order given by the inclusion.

- We may also realize $P(X)$ as the *poset associated to the family of ideals*

$$\mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_k}$$

for $1 \leq i_1 \leq \cdots \leq i_k \leq m$, ordered by reverse inclusion.

6.- **The Mayer-Vietoris spectral sequence**

The following construction follows closely a similar one by A. Björner and T. Ekedahl (1997) to compute the l-adic cohomology of the complement of an arrangement of linear varieties over a finite field.

• To any poset $(P, \leq)$, abelian category $\mathcal{C}$ and covariant functor $F : P \to C$ one may attach a complex: *the Roos complex of* F

$$\text{Roos}_*(F)$$

that allows to compute the ith left-derived functors of the direct limit functor of F evaluated at P. Namely,

$$H_i(\text{Roos}_*(F)) = \text{ind.lim}_P^{(i)}(F(P))$$

Now, let $P(X)$ the poset attached to the arrangement of linear varieties X.

• For any $p \in P(X)$ denote by I_p its definition ideal, and by $h(p)$ the height of I_p.

- Then, for any R-module M and $i \geq 0$ we may consider the following functor over the natural poset $P(X)$:

$$H^i_{[*]}(M) : P \longrightarrow R\text{-mod}$$

$$p \mapsto H^i_{I_p}(M)$$

Let us consider the following easy example:

• $X = X_1 \cup X_2$, where X_1 and X_2 are the coordinate axis of the affine plane $\mathbb{A}^2_k$. Then,

$$I_X = (x{\cdot}y), \quad I_{X_1} = (x)$$

$$I_{X_2} = (y), \ \ I_{X_1 \cap X_2} = (x, y)$$

For any R-module M, the inclusions

$$(x) \hookrightarrow (x, y), \;\; (y) \hookrightarrow (x, y)$$

induce natural morphisms

$$f : H^i_{(x,y)}(M) \to H^i_{(x)}(M)$$

$$f : H^i_{(x,y)}(M) \to H^i_{(y)}(M)$$

In this case, the Roos complex for the functor $F = H^i_{[*]}(M)$ is

$$d_1 : K_1(F) \to K_0(F)$$

where

$$K_1(F) = H^i_{(x,y)}(M) \oplus H^i_{(x,y)}(M)$$

$$K_0(F) = H^i_{(x)}(M) \oplus H^i_{(x,y)}(M) \oplus H^i_{(y)}(M)$$

and the differential

$$d_1 : K_1(F) \to K_0(F)$$

is given by

$$d_1 = (f - Id) \oplus (g - Id)$$

The following fact is crucial for our purposes:

FACT: *Let E be an injective R-module. Then, the augmented Roos complex*

$$\mathrm{Roos}_*(H^0_{[*]}(E)) \to H^0_I(E) \to 0$$

is exact.

• Let us consider now an injective resolution of R in R-mod:

$$0 \to R \to E^*$$

- One then gets a double complex:

$$\mathrm{Roos}_{-i}(H^0_{[*]}(E^j))$$

for $i \leq 0, j \geq 0$, which is a second quadrant complex with only a finite number of non-zero columns.

- So it gives a *spectral sequence* that, because of the lemma, converges to $H_I^*(R)$:

$$E_1^{-i,j} = \text{Roos}_i(H^j_{[*]}(R)) \Rightarrow H_I^{j-i}(R)$$

$i, j \geq 0$, that we call

the Mayer-Vietoris spectral sequence.

- Note that, since the differential d_1 is that of the Roos complex, we have by the above lemma that the second term of this spectral sequence is

$$E_2^{-i,j} = \text{ind.lim}_P^{(i)} H^j_{[*]}(R)$$

Remark: *If* $\text{char} k = 0$ *then the Mayer-Vietoris spectral sequence can be regarded as a spectral sequence in the category of* $\mathcal{D}$*-modules.*

We may now state our main result (J. Àlvarez Montaner, R. García López and S. Z. A., 2003).

Theorem

(1) *There is an isomorphism of R-modules*

$$\text{ind.lim}_P^{(i)} H^j_{[*]}(R) \simeq \bigoplus_{h(p)=j} (H^j_{I_p}(R))^{\oplus b_{i,p}}$$

where $b_{i,p} = \textit{dim}_k(\widetilde{H}_{i-1}(K(> p); k)$

(and we agree that $\widetilde{H}_{-1}(K(> p); k) = k$ *and the reduced homology of the empty simplicial complex is* k *in degree* -1 *and* 0 *otherwise).*

(2) *The Mayer-Vietoris spectral sequence degenerates at the* E_2*-term.*

As a consequence of the degeneration of the Mayer-Vietoris spectral sequence at the E_2-term we obtain the following useful consequence:

Corollary

For any $r \geq 0$ there exists a filtration of the local cohomology module $H_I^r(R)$:

$$\{F_j^r\}_{r \leq j \leq n}$$

such that

$$F_j^r / F_{j-1}^r \simeq \bigoplus_{h(p)=j} (H_{I_p}^j(R))^{\oplus m_{r,p}}$$

where $m_{r,p} = dim_k \widetilde{H}_{h(p)-r-1}(K(> p); k)$.

Remark: *The simple existence of a Mayer-Vietoirs spectral sequence may be stated more in general (G. Lyubeznik, 2005).*

This is true for any commutative ring A, family of ideals $I_1, \ldots, I_n$, and A-module M, by following a similar construction as above. Nevertheless, even applied to our case, the spectral sequence obtained by G. Lyubeznik is not exactly the same as ours.

Question (G. Lyubeznik) *Does the general Mayer-Vietoris spectral sequence always degenerate? Does it degenerate at E_2 at least when A is regular and $M = A$?*

7.- **The Betti numbers of the complement**

- Assume char$k = 0$.

- Taking into account the *additivity of the characteristic cycle*, the *value of the characteristic cycle of the local cohomology modules supported at the definition ideal of a linear variety*, and the above *filtration* we may obtain the characteristic cycle of the local cohomolgy modules $H^r_I(R)$.

Corollary

$$CC(H^r_I(R)) = \sum_{p \in P(X)} m_{r,p}\, T^*_{X_p}\mathbb{A}^n_k$$

where $m_{r,p} =$ *dim*$_k(\widetilde{H}_{h(p)-r-1}(K(> p); k)$.

In this way we obtain a first combinatorial description of the multiplicities.

• Now, assume that $k = \mathbb{R}$.

- M. Goreski and R. MacPherson (1988) gave the following formula for the homology of the complement of X in $\mathbb{A}_k^n$.

Theorem

$$\widetilde{H}_i(\mathbb{A}_{\mathbb{R}}^n - X; \mathbb{Z}) \simeq$$

$$\bigoplus_{p \in P(X)} H^{h(p)-i-1}(K(\geq p), K(> p); \mathbb{Z}),$$

where $\widetilde{H}_i(\cdot; \mathbb{Z})$ denotes the ith reduced singular homology with integer coefficients $\mathbb{Z}$ and $H^j(\cdot, \cdot; \mathbb{Z})$ the jth relative simplicial cohomology with coefficients $\mathbb{Z}$.

(In fact, the extension of this formula to the l-adic cohomology was the motivation of the work by A. Björner and T. Ekedahl.)

- Now, we may compare the terms that appear in the formula of Goresky and MacPherson and our terms in the formula for the characteristic cycle.

As a consequence, for $k = \mathbb{R}$, we obtain an *expression for the Betti numbers of the complement of* X *in terms of the multiplicities of the local cohomology modules* $H^r_I(R)$, which is a purely algebraic interpretation.

<u>**Corollary**</u>

$$\dim_{\mathbb{Q}} \widetilde{H}_i(\mathbb{A}^n_{\mathbb{R}} - X, \mathbb{Q}) = \sum_{p \in P(X)} m_{i+1,p}$$

And regarding a complex arrangement in $\mathbb{A}^n_{\mathbb{C}}$ as a real arrangement in $\mathbb{A}^{2n}_{\mathbb{R}}$, the formula in this case becomes

$$\dim_{\mathbb{Q}} \widetilde{H}_i(\mathbb{A}^n_{\mathbb{C}} - X, \mathbb{Q}) = \sum_{p \in P(X)} m_{i+1-h(p),p}$$

8.- **Extension problems**

- Let k be any field.

For a given local cohomology module $H_I^r(R)$ we may consider the exact sequences

$$0 \to F_{j-1} \to F_j \to F_j/F_{j-1} \to 0$$

for $r \leq j \leq n$.

In general, these exact sequences are not split, as the following easy example shows:

- Let $R = k[x, y]$ and $I = (x \cdot y)$.

Let us denote by

- $I_1 = (x)$, $I_2 = (y)$ and $\mathfrak{m} = (x, y)$.

It's not hard to see that in the case of two ideals, the Mayer-Vietoris spectral sequence is given by the usual Mayer-Vietoris long exact sequence.

- For $H^1_I(R)$ we then have

- $F_1 = H^1_{I_1}(R) \oplus H^1_{I_2}(R)$,

- $F_2 = H^1_I(R)$

and the corresponding exact sequence is

$$0 \to H^1_{I_1}(R) \oplus H^1_{I_2}(R) \to H^1_I(R) \to H^2_{\mathfrak{m}}(R) \to 0$$

which is not split because the maximal ideal $\mathfrak{m}$ is not an associated prime ideal of $H^1_I(R)$.

Problem *Determine these extensions.*

9.- **Final comments: Monomial ideals**

- To compute the local cohomology supported at a monomial ideal we may assume that the ideal is reduced, that is, a *square free monomial ideal*.

- A square free monomial ideal defines a *central arrangement* of linear varieties, and so we may apply all the above results.

But the *local cohomolgy modules supported on monomial ideals* have been studied from other points of view by several authors. For instance,

- J. Àlvarez Montaner (computational, 2000, 2004),

- M. Mustaţā (topological, 2000),

- K. Yanagawa (categorical, 2001)

...

The following fact makes the difference between the monomial case and the general case:

• Let $I \subset R$ be a square free monomial ideal.

FACT: *The local cohomology modules* $H_I^r(R)$ *have a natural* $\mathbb{Z}^n$*-graduation coming from the one in* R *given by*

$$\deg(x_i) = \varepsilon_i$$

where ε_i *denotes the canonical basis of* $\mathbb{Z}^n$*.*

Now, if $\mathrm{char}(k) = 0$ we have two structures for the local cohomology modules $H_I^r(R)$:

- As $\mathcal{D}$-modules.

- As $\mathbb{Z}^n$-graded R-modules.

In fact, both structures can be compared in the following way:

• Set $\Omega = \{-1, 0\}^n$.

• Let ${}^*\mathcal{P}$ the set homogeneous prime ideals of R. They are of the form

$$\mathfrak{p} = (x_{i_1}, \dots, x_{i_k})$$

for $1 \leq i_1 \leq \cdots \leq i_k \leq n$.

- So there there is a bijection

$$\Omega \to^* \mathcal{P} : \alpha \mapsto \mathfrak{p}_\alpha = \langle x_{i_k} \mid \alpha_{i_k} = -1 \rangle$$

Proposition

$$m_{r,p_\alpha} = \dim_k H^r_I(R)_\alpha$$

for all $r \geq 0$ *and* $\alpha \in \Omega$.

On the other hand, *the graded Betti numbers in the minimal free resolution of the Alexander dual* $I^\vee$ *of* I *may be completely described by means of the graded structure of the local cohomology modules* $H_I^r(R)$.

FACT: *By M. Mustaţă (2000) one has that*

$$\beta_{i,-\alpha}(I^\vee) = \dim_k H_I^{|\alpha|-i}(R)_\alpha$$

for $\alpha \in \Omega$, *and all the other graded Betti numbers are zero.*

As a consequence we get:

Corollary

Let I *be a square free monomial ideal. Then, for any* $\alpha \in \Omega$ *we have*

$$\beta_{i,-\alpha}(I^\vee) = m_{|\alpha|-i,p_\alpha}$$

- Let us consider again the example

$$I = (x \cdot y) \subset k[x,y]$$

and use the same notation as before. From the exact sequence

$$0 \to H^1_{I_1}(R) \oplus H^1_{I_2}(R) \to H^1_I(R) \to H^2_{\mathfrak{m}}(R) \to 0$$

we have that the characteristic cycle is

$$CC(H^1_I(R)) = T^*_{X_1}\mathbb{A}^2_k + T^*_{X_2}\mathbb{A}^2_k + T^*_{(0,0)}\mathbb{A}^2_k$$

- On the other hand, the Alexander dual of I is $I^\vee = (x,y)$ that has the following minimal free $\mathbb{Z}^2$-graded resolution:

$$0 \to R(-1,-1) \to R(-1,0) \oplus R(0,-1) \to 0$$

and the following equalities hold

$$\beta_{0,(1,0)} = 1 = m_{1,I_1}$$

$$\beta_{0,(0,1)} = 1 = m_{1,I_2}$$

$$\beta_{1,(1,1)} = 1 = m_{1,\mathfrak{m}}$$

Thus if

• $k = \mathbb{R}$ or $k = \mathbb{C}$, I is a square free monomial ideal, and X is the arrangement of linear varieties defined by I in $\mathbb{A}^n_k$,

there is an "equivalence" between the sets of numbers given by

- *the multiplicities of the local cohomology modules $H^r_I(R)$ as $\mathcal{D}$-modules;*

- *the (algebraic) graded Betti numbers of Alexander dual of I;*

- *the (topological) Betti numbers of the complement of X.*

Also, in the case of a square free monomial ideal I it is possible to do something more. For instance,

• *To solve (for any field k) the extension problems given by the filtration of the local cohomology modules* $H_I^r(R)$ (by using the category of straight modules introduced by K. Yanagawa);

• If the characteristic of k is 0, *to describe the smallest full abelian subcategory of the category of the category of $\mathcal{D}$-modules that contains the local cohomology modules* $H_I^r(I)$ (that it turns out to be equivalent to the category straight-modules);

...

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Of course, each reference in the list provides more references on the subject. In particular, the book by S. C. Coutinho contains the list of the classical and most important references for $\mathcal{D}$-modules theory.