# The Second Japan-Vietnam Joint Seminar on Commutative Algebra

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# Abstracts

### Sequentially Generalized Cohen-Macaulay Modules

by Doan Trung Cuong
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A finite module $M$ over a local ring $R$ is called a sequentially generalized Cohen-Macaulay module if it has a filtration $\mathcal{F} : M_0 \subset M_1 \subset \ldots \subset M_t = M$ of submodules of $M$ such that $\dim M_0 < \dim M_1 < \ldots < \dim M_t$ and each $M_i/M_{i-1}$ is generalized Cohen-Macaulay. The purpose of this paper is toward a theory of sequentially generalized Cohen-Macaulay modules. Various basic properties of these modules are presented and some characterizations of sequentially generalized Cohen-Macaulay property by using local cohomology modules and systems of parameters are given. We also show that the notion of dd-sequences defined in [N. T. Cuong and D. T. Cuong, dd-sequences and partial Euler-Poincaré characteristics of Koszul complex] may be useful for studying this class of modules.

### On a new invariant of finite modules over local rings

by Nguyen Tu Cuong
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Let $M$ be a finitely generated module on a local ring $R$ of dimension $d$ and $\mathcal{F} : M_0 \subset M_1 \subset \ldots \subset M_t = M$ a filtration of submodules of $M$ such that $d_o < d_1 < \cdots < d_t$, where $d_i = \dim M_i$. This paper is concerned with an integer $p_{\mathcal{F}}(M)$ which is defined as the least degree of all polynomials in $n_1, \ldots, n_d$ bounding above the function

$$\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) - \sum_{i=0}^{t} n_1 \ldots n_{d_i} e(x_1, \ldots, x_{d_i}; M_i),$$

where $(x_1, \ldots, x_d)$ is a systems of parameters of $M$ such that $(x_{d_i+1}, \ldots, x_d)M \cap M_i = 0$ for $i = 0, 1, \ldots, t-1$. We will prove that $p_{\mathcal{F}}(M)$ is independent of the choice of such systems of parameters. We also show that this invariant is just the dimension of the non-sequentially Cohen-Macaulay locus of $M$ when $R$ is a quotient of a Cohen-Macaulay ring and $\mathcal{F}$ is the dimension filtration of $M$.

**On modules of reduction number one**

by Futoshi Hayasaka
Meiji University, Japan

Let $(A, \mathfrak{m})$ be a Noetherian local ring and $N$ a parameter module in $F = A^r$and $M = N :_F \mathfrak{m}$ the socle module of $N$. In this talk, we will prove that the module $M = N :_F \mathfrak{m}$ has a reduction number at most one and hence its Rees algebra $\mathcal{R}(M)$ is Cohen-Macaulay, if the base ring $A$ is Cohen-Macaulay of dimension two and the rank of $N$ is greater than or equal to two. This result gives numerous examples of Cohen-Macaulay Rees algebras of modules, which are not integrally closed and not a parameter module.

**Some finite properties of generalized local cohomology modules**

by Nguyen Van Hoang
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Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, $I$ an ideal of $R$ and $M, N$ finitely generated $R-$modules. In this talk, we prove some finite properties such as the Artinianess, the finite property for the support, and the finiteness for associated primes of generalized local cohomology modules $H^i_I(M, N)$ defined by J. Herzog as $H^i_I(M, N) = \varinjlim_n \operatorname{Ext}^i_R(M/I^nM, N)$.

Set $I_M = \operatorname{ann}(M/IM)$. Let $r = \operatorname{depth}(I_M, N)$ and $s = \text{f-depth}(I_M, N)$, the filter depth of $N$ in $I_M$, i.e. the length of a maximal filter regular sequence of $N$ in $I_M$. The notion of generalized regular sequence was introduced as an extension of that of filter regular sequence. Set $t = \operatorname{gdepth}(I_M, N)$, the length of a maximal generalized regular sequence of $N$ in $I_M$. We prove that

1) $\operatorname{Ass} H^r_I(M, N) = \operatorname{Ass} \operatorname{Ext}^r_R(M/IM, N)$.

2) $s$ is the least integer $i$ such that $H^i_I(M, N)$ is not Artinian, and

$$\operatorname{Ass} H^s_I(M, N) \cup \{\mathfrak{m}\} = \operatorname{Ass} \operatorname{Ext}^s_R(M/IM, N) \cup \{\mathfrak{m}\}.$$

3) $t$ is the least integer $i$ such that $H^i_I(M, N)$ is not of finite support, and

$$\operatorname{Ass} H^t_I(M, N) \cup P = \operatorname{Ass} \operatorname{Ext}^t_R(M/IM, N) \cup P,$$

where $P = \bigcup_{i=0}^{t-1} \operatorname{Supp} H^i_I(M, N)$. In particular, the sets $H^r_I(M, N)$, $H^s_I(M, N)$, $H^t_I(M, N)$ are finite sets. Moreover, $H^i_I(M, N) \cong H^i_{\mathfrak{m}}(M, N)$ for all $i < s$, and there exists an ideal $I'$ with $\dim R/I' \leqslant 1$ such that $H^i_I(M, N) \cong H^i_{I'}(M, N)$ for all $i < t$.

**Gorenstein Rees algebras and $a^*$-invariant formulas**

by Shin-ichiro Iai
Hokkaido University of Education, Japan

Let $(A, \mathfrak{m})$ be a Noetherian local ring and $I (\subsetneq A)$ an ideal of $A$. Assume that the field $A/\mathfrak{m}$ is infinite. In this talk, we study a question of when the ring $A$ is Gorenstein in case so is the Rees algebra $\mathrm{R}(I) = \bigoplus_{i \geq 0} I^i$. Let $J$ be a minimal reduction of $I$. We denote by $\mathrm{r}_J(I)$ the reduction number of $I$ with respect to $J$. The analytic spread of $I$ is $\lambda(I) = \dim A/\mathfrak{m} \otimes_A \mathrm{R}(I)$. We put $r = \mathrm{r}_J(I)$ and $\ell = \lambda(I)$. Then the main result of the talk can be stated as follows.

**Theorem 1.** *Assume that* $\mathrm{R}(I)$ *is a Gorenstein ring and* $\operatorname{grade} I \geq 2$. *Then the following two conditions are equivalent.*

(1) *$A$ is a Gorenstein ring.*

(2) *$A$ satisfies Serre's condition* $(\mathrm{S}_\ell)$ *and* $r \leq \ell - 2$.

In general, if $\mathrm{R}(I)$ is a Cohen-Macaulay ring, then the inequality $r \leq \ell - 1$ holds true. Therefore the result above means that if a non-Cohen-Macaulay ring $A$ satisfies Serre's condition $(\mathrm{S}_\ell)$, then $\mathrm{R}(I)$ is not a Gorenstein ring unless $r = \ell - 1$. The condition that $r \leq \ell - 2$ cannot be removed in the theorem. There exists an example that the Rees algebra $\mathrm{R}(I)$ whose base ring $A$ satisfies Serre's condition $(\mathrm{S}_\ell)$ is Gorenstein, but $A$ is not a Gorenstein ring.

**Example 2.** *Let $k$ be a field of* $\mathrm{char}(k) = 2$ *and let* $X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4$ *be indeterminates over $k$. Consider the local ring* $A = k[[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]]/L$ *where $L$ is an ideal generated by elements* $X_1Y_1 + X_2Y_2 + X_3Y_3, Y_1Y_2 - X_3Y_4, Y_2Y_3 - X_1Y_4, Y_1Y_3 - X_2Y_4, Y_1^2, Y_2^2, Y_3^2, Y_4^2, Y_1Y_4, Y_2Y_4, Y_3Y_4$. *Let* $\mathfrak{m}$ *denote the maximal ideal of $A$. Then* $\mathrm{R}(\mathfrak{m})$ *is Gorenstein, but $A$ is not Cohen-Macaulay, see* [I]. *Thus $A$ is a non-Gorenstein ring satisfies Serre's condition* $(\mathrm{S}_2)$. *Now, we find an ideal* $I = (X_2, X_3, Y_1)A$. *Then* $\ell = \operatorname{grade} I = 2$ *and* $\mathrm{R}(I)$ *is Gorenstein.*

A proof of Theorem 1 is based on a discussion of $a^*$-invariant formulas. To state our result of the $a^*$-invariant, we need further notation. Let $S = \bigoplus_{i \geq 0} S_i$ be a Noetherian graded algebra over the local ring $(S_0, \mathfrak{n})$ with infinite residue field, generated by elements of degree 1. Let $S_+ = \bigoplus_{i > 0} S_i$ and $\mathfrak{M} = \mathfrak{n}S + S_+$. We denote by $\mathrm{H}^i_{\mathfrak{M}}(*)$ the $i$th graded local cohomology functor of $S$ with respect to $\mathfrak{M}$. Let $[\mathrm{H}^i_{\mathfrak{M}}(S)]_j$ denote the homogeneous componen t of the graded module $\mathrm{H}^i_{\mathfrak{M}}(S)$ of degree $j$. We define

$$\mathrm{a}_i(S) := \max\{n \in \mathbb{Z} \mid [\mathrm{H}^i_{\mathfrak{M}}(S)]_n \neq (0)\} \text{ and}$$

$$\mathrm{a}^*(S) := \max\{\mathrm{a}_i(S) \mid i \in \mathbb{Z}\},$$

and call them respectively the $i$th $a$-invariant and the $a^*$-invariant of $S$. Put $\mathrm{a}(S) = \mathrm{a}_{\dim S}(S)$. Let $Z$ be a minimal reduction of $S_+$ generated by elements of degree 1.Let $\mathrm{r}(S)$=min$\{\mathrm{r}_L(S_+) \mid L$ is a minimal reduction of $S_+$ generated by elements of degree 1$\}$ and $\ell(S) = \lambda(S_+)$. Put $\mathcal{A} = \{\mathfrak{p} \in \operatorname{Spec} S_0 \mid \ell(S_\mathfrak{p}) = \dim S_\mathfrak{p}\}$. With this notation, we can state the following result, which is a generalization of a theorem due to [U].

**Proposition 3.** *Assume that* $\operatorname{depth} S_\mathfrak{p} \geq \min\{\dim S_\mathfrak{p}, \ell(S)\}$ *for all* $\mathfrak{p} \in \operatorname{Spec} S_0$. *Then the following equalities hold true.*

$$\begin{aligned} \mathrm{a}^*(S) &= \max\{\mathrm{r}(S_\mathfrak{p}) - \ell(S_\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{A}, \dim S_\mathfrak{p} < \ell\} \cup \{\mathrm{r}_Z(S) - \ell(S)\} \\ &= \max\{\mathrm{r}(S_\mathfrak{p}) - \ell(S_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} S_0\}. \end{aligned}$$

*Moreover when the ring $S$ is quasi-unmixed, we have the equality*

$$\mathrm{a}^*(S) = \max\{\mathrm{a}(S), \mathrm{r}_Z(S) - \ell(S)\}.$$

# References


[I] S. Ikeda, On the Gorensteinness of Rees algebras over local rings, Nagoya Math. J. 102 (1986), 135-154.

[U] B. Ulrich, Cohen-Macaulayness of Associated Graded Rings and Reduction Numbers of Ideals, Workshop on Commutative Algebra and Its Relations to Combinatorics and Computer Algebra, ICTP, Trieste, May 1994.


**Remark on the polynomial type Poincare series**


by Yuji Kamoi

Meiji University, Japan


We consider a homogeneous graded ring $A$ over a local Aritinian ring $A_0$ with a Poincare series of the form $P(A, T) = \frac{e_0}{(1-t)^d}$, where $e_0$ is a multiplicity of $A$. Such a ring is expected to be a polynomial ring over $A_0$. However, this is not true in general.

In this talk, we give some equivalent conditions for $A$ to be a polynomial ring under a certain assumption on $a$-invariant. Also we give some example of non polynomial $A$ without a $a$-invariant condition.

It's a piece of cake, but a little bit funny taste.

**On Faltings' annihilator theorem**

by Takesi Kawasaki
Tokyo Metropolitan University, Japan

Let $A$ be a Noetherian ring. We say that the annihilator theorem holds for $A$ if it satisfies the following proposition.

**The Annihilator Theorem.** *Let $M$ be a finitely generated $A$-module, $Y$, $Z$ subsets of* $\operatorname{Spec} A$ *which are stable under specialization and $t$ a positive integer. Then the following are equivalent*:

(1) $\operatorname{ht} \mathfrak{q}/\mathfrak{p} + \operatorname{depth} M_{\mathfrak{p}} \geq t$ *for any* $\mathfrak{p} \notin Y$ *and* $\mathfrak{q} \in V(\mathfrak{p}) \cap Z$;

(2) *there is an ideal $\mathfrak{b}$ such that $V(\mathfrak{b}) \subset Y$ and $\mathfrak{b}$ annihilates local cohomology modules $H^0_Z(M), \ldots, H^{t-1}_Z(M)$.*

Faltings [1] showed that the annihilator theorem holds for $A$ if $A$ has a dualizing complex or if $A$ is a homomorphic image of a regular ring.

In this talk, the speaker want to show the following theorem.

**Theorem 1.** *The annihilator theorem holds for $A$ if*

(C1) *$A$ is universally catenary*;

(C2) *all the formal fibers of all the localizations of $A$ are Cohen-Macaulay*; *and*

(C3) *the Cohen-Macaulay locus of each of finite type $A$-algebra is open.*

# References

[1] Gerd Faltings, *Über die Annulatoren lokaler Kohomologiegruppen*, Arch. Math. (Basel) **30** (1978), no. 5, 473–476. MR 58 #22058

**An elementary proof of the formula** $\mathrm{cl}({}^eA) = \frac{(p^{de} - p^{(d-1)e})}{2}\mathrm{cl}(\omega_A)$

by Kazuhiko Kurano
Meiji University, Japan

Let $A$ be a F-finite Noetherian normal local ring containing a field of characteristic $p > 0$ with perfect residue class field. Using a theorem of Huneke-McDermott-Monsky and the formula $\mathrm{cl}({}^eA) = \frac{(p^{de} - p^{(d-1)e})}{2}\mathrm{cl}(\omega_A)$, it is proved in AC/0506492 that the second term of the Hilbert-Kunz function of $A$ vanishes if $A$ is $\mathbf{Q}$-Gorenstein ring. (Here, $\omega_A$ is the canonical module of $A$ and ${}^eA$ is the $e$-th Frobenius.) The formula is proved in AC/0506492 using the singular Riemann-Roch theory.

In the lecture, we give an elementary proof to the formula. We need neither algebraic geometry nor algebraic K-theory.

### Uniform bounds in generalized Cohen-Macaulay rings

by Cao Huy Linh
College of Education, Hue University, Vietnam

We establish a uniform bound for the Castelnuovo-Mumford regularity of associated graded rings of parameter ideals in a generalized Cohen-Macaulay ring. As consequences, we obtain uniform bounds for the relation type and the postulation number. Moreover, we show that generalized Cohen-Macaulay rings can be characterized by the existence of such uniform bounds.

### The Rees algebras of ideals in two dimensional regular local rings

by Naoyuki Matsuoka
Meiji University, Japan

Let $(A, \mathfrak{m})$ be a regular local ring of dimension 2 and $I$ be a $\mathfrak{m}$-primary ideal in $A$. In this talk, I will show the following two theorems.

**Theorem 1.** *The following conditions are equivalent.*

$(i)$ *The Rees algebra* $R = \mathcal{R}(I) = A[It] \subseteq A[t]$ *is Buchsbaum.*

$(ii)$ *The extended Rees algebra* $R' = \mathcal{R}'(I) = A[It, t^{-1}] \subseteq A[t, t^{-1}]$ *is Buchsbaum.*

$(iii)$ *The associated graded ring* $G = \mathrm{G}(I) = R'/t^{-1}R'$ *is Buchsbaum.*

*When this is the case, the equality* $I^3 = QI^2$ *holds true and* $\mathbb{I}(R) = \mathbb{I}(R') = \mathbb{I}(G) = \ell_A(I^2/QI)$ *where* $(*)$ *denotes the Buchsbaum invariant.*

Let $\overline{I}$ be the integral closure of $I$ and $\widetilde{I} = \bigcup_{n>0}[I^{n+1} : I^n]$ be the Ratliff-Rush closure of $I$. The inclusions $I \subseteq \widetilde{I} \subseteq \overline{I}$ hold in this situation.

**Theorem 2.** *Assume that the Rees algebra* $\mathcal{R}(I)$ *of* $I$ *is Buchsbaum and the equality* $\overline{I} = \widetilde{I}$ *holds. Let* $B = A[[X]]$ *be the formal power series ring over* $A$ *and* $J = IB + XB$. *Then* $e_2(J) = 0$ *and* $e_3(J) = -\ell_A(\overline{I}/I) \leq 0$.

*Moreover, if* $\mathcal{R}(I)$ *is* <u>*not*</u> *Cohen-Macaulay then* $e_3(J) < 0$

**Remark 3.** *There exists ideals* $I$ *such that the Rees algebra* $\mathcal{R}(I)$ *is Buchsbaum and the equality* $\overline{I} = \widetilde{I}$ *holds true in* $A = k[x, y]_{(x,y)}$ *where* $k[x, y]$ *denotes the polynomial ring over a field* $k$.

### Adjoint ideals in two dimensional regular local rings

by Yukio Nakamura
Meiji University, Japan

This is a joint work with E.Hyry and L. Ojala.

Let $(A, \mathfrak{m})$ be a two dimensional regular local ring and $I$ an ideal in $A$. Let $X \to \operatorname{Spec} A$ be a proper birational morphism where $X$ is regular and $I\mathcal{O}_X$ is invertible. Then the adjoint of $I$ is defined as the global section $\Gamma(X, I\omega_X)$. The adjoint of $I$ always contains $I$ and is an integrally closed ideal.

In my talk, for a given integrally closed ideal, we discuss when it is the adjoint of some ideal. And we give its applications.

**On a length formula for the j-multiplicity of an ideal**

by Koji Nishida
Chiba University, Japan

This is a joint work with B. Ulrich.

Let $(R, \mathfrak{m})$ be a $d$-dimensional Noetherian local ring of $\dim R = d > 0$ and let $I$ be a proper ideal of $R$. We set

$$\mathrm{j}\,(I) = \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}} \ \mathrm{length}_R\, \mathrm{H}^0_{\mathfrak{m}}(I^n/I^{n+1})$$

and call it the j-multiplicity of $I$, where $\mathrm{H}^0_{\mathfrak{m}}(\ \cdot\ )$ denotes the 0-th local cohomology functor. It is obvious that $\mathrm{j}\,(I)$ coincides with the usual multiplicity $\mathrm{e}_I(R)$ if $I$ is an $\mathfrak{m}$-primary ideal. Moreover, the j-multiplicity enjoys a lot of properties similar to those of the usual multiplicity (see [1], [2], [3], [4]).

In this talk we will give a length formula of the j-multiplicity which generalize the formula of Flenner and Manaresi given in [2]. Our result can be stated as follows: If $A/\mathfrak{m}$ is an infinite field, then choosing sufficiently general elements $a_1, \dots, a_{d-1}, a_d$ of $I$, we have

$$\mathrm{j}\,(I) = \mathrm{length}_R\, R\,/\,((a_1, \dots, a_{d-1}) :_R I^n) + a_d R$$

for $n \gg 0$. Let $\mathfrak{a} = (a_1, \dots, a_{d-1})$ and $\mathfrak{b} = \mathfrak{a} :_R I^n$ for $n \gg 0$. Then the equality above means $\mathrm{j}\,(I) = \mathrm{e}_{a_d R}(R/\mathfrak{b})$ since $R/\mathfrak{b}$ is a Cohen-Macaulay ring of dimension one. If $R$ itself is Cohen-Macaulay and $I$ satisfies certain Artin-Nagata property (cf. [5]), then $I \cap \mathfrak{b} = \mathfrak{a}$, and so we have an exact sequence $0 \to I/\mathfrak{a} \to R/\mathfrak{b} \to R/I + \mathfrak{b} \to 0$, which implies $\mathrm{e}_{a_d R}(R/\mathfrak{b}) = \mathrm{e}_{a_d R}(I/\mathfrak{a}) = \mathrm{length}_R\, I/\mathfrak{a} + a_d I$ . Thus we get

$$\mathrm{j}\,(I) = \mathrm{length}_R\, I/\mathfrak{a} + a_d I\,,$$

which is the length formula of Flenner and Manaresi. We will investigate some concrete examples in order to illustrate how to use our lenght formula.

# References

[1] R. Achilles and M. Manaresi, Multiplicity for ideals of maximal analytic spread and intersection theory, J. Math. Kyoto Univ., **33** (1993), 1029–1046.

[2] H. Flenner and M. Manaresi, A numerical characterization of reduction ideals, Math. Z., **238** (2001), 205–214.

[3] H. Flenner and M. Manaresi, A length formula for the multiplicity of distinguished components of intersections, J. Pure Appl. Algebra, **165** (2001), 155–168.

[4] H. Flenner, L. O'Carroll and W. Vogel, Joins and intersections, Springer Monographs in Math., Springer, Berlin (1999).

[5] B. Ulrich, Artin-Nagata property and reductions of ideals, Contemp. Math., **159** (1994), 373–400.

**Remarks on the Cohen-Macaulay type of modules**

by Hideto Sakurai
Meiji University, Japan

Let $A$ be a Noetherian local ring with the maximal ideal $\mathfrak{m}$, and $M$ be a finitely generated $A$-module with $d = \dim M$. This talk is aimed at exploring the index $\ell_A((QM : \mathfrak{m})/QM)$ of reducibility of parameter ideals $Q$ for $M$ and the Cohen-Macaulay type $\mathrm{r}(M)$ of $M$, where, in this talk, we call

$$\sup_Q \ell_A((QM : \mathfrak{m})/QM)$$

the Cohen-Macaulay type of $M$, where $Q$ runs over the parameter ideals for $M$.

We shall give a lower bound of the Cohen-Macaulay type of $M$ and consider the Cohen-Macaulay property of $M$ which has a low Cohen-Macaulay type. This lower bound is expressed in terms of the lengths of the socles of local cohomology modules of $M$. Moreover we shall give that this lower bound is equal to the supremum of the index of reducibility of *distinguished* parameter ideals for $M$ when $M$ is sequentially Cohen-Macaulay.

**Uncountably many totally reflexive modules**

by Ryo Takahashi
Meiji University, Japan

A few years ago, Huneke and Leuschke proved a theorem which solved a conjecture of Schreyer. It asserts that an excellent Cohen-Macaulay local ring of countable Cohen-Macaulay type which is complete or has uncountable residue field has at most a one-dimensional singular locus. In this talk, we verify that the excellent property can be removed, and consider the theorem over an arbitrary local ring. The main purpose of this talk is to prove that the existence of a certain prime ideal and a certain totally reflexive module implies the existence of an uncountably infinite number of isomorphism classes of indecomposable totally reflexive modules.

**Stability of associated primes of integral closures of monomial ideals**

by Tran Nam Trung
Hanoi Institute of Mathematics, Hanoi, Vietnam

This paper is devoted to study the following question: Given a monomial ideal $I$ of a polynomial ring $R$, determine an explicit number, such that the sequence $\{\mathrm{Ass}(R/\overline{I^n})\}_{n\in\mathbb{N}}$ is stationary when $n$ exceeds it?

**On the structure of the fiber cone of ideals with analytic spread one**

by Santiago Zarzuela
Universitat de Barcelona, Spain

Let $(A, \mathfrak{m})$ be a local ring and $I \subset A$ an ideal with positive grade. Assume that the residue field of $A$ is infinite. We study the fiber cone $F(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$ when the analytic spread of $I$ is one. In this case, for any minimal reduction $J \subset I$ of $I$, $F(I)$ has a structure as a finitely generated module over $F(J)$, which is a polynomial ring in one variable over the residue field. Thus, we may apply the structure theorem for (graded) modules over a (graded) principal domain to get a complete description of $F(I)$ as a module over $F(J)$.

We analyze this structure in order to study and characterize in terms of the ideal itself several arithmetical properties of the fiber cone, and other numerical invariants such as multiplicity, reduction number or Castelnuovo-Mumford regularity. For that, we describe the set of invariants provided by the structure of $F(I)$ as a module over $F(J)$ in terms of lengths of colon ideals. In principle, this information is weaker than the one provided by the structure of $F(I)$ as algebra, and even it may depend on the chosen minimal reduction, but it is easier to compute, and rich enough to give interesting information about the fiber cone. Some applications to ideals of higher analytic spread are also given.

*This is a joint work with Teresa Cortadellas.*