Buchsbaum rings with minimal multiplicity

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The main part of this talk is a joint work with Shiro Goto at Meiji University (see [8]). Also, some part is a joint work with Naoki Terai at Saga University (see [15]).

1. Motivation

In 1967, Abhyankar [2] proved that for a homogeneous integral domain A over an algebraically closed field k, the following inequality holds:

$$\operatorname{embdim}(A) = \dim_k A_1 \leq e(A) + \dim A - 1,$$

where $e(A)$ (resp. $\operatorname{embdim}(A)$) denotes the multiplicity (resp. embedding dimension) of A. In 1970, Sally [13] proved that the same inequality holds for any Cohen–Macaulay local ring, and called the ring A which satisfies the equality a *Cohen–Macaulay local ring with maximal embedding dimension* [1]. In 1982, Goto [5] pointed out that Sally's result can be extended to the class of Buchsbaum local rings. Namely, for any Buchsbaum local ring A, the following inequality holds:

$$\operatorname{embdim}(A) \leq e(A) + \dim A - 1 + I(A),$$

where $I(A)$ denotes the Buchsbaum invariant (I-invariant) of A. A Buchsbaum local ring A which satisfies the equality is said to be a *Buchsbaum local ring with maximal embedding dimension.* On the other hand, Goto [6] in 1983 proved that

$$e(A) \geq 1 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(H^i_{\mathfrak{m}}(A))$$

and defined the notion of *Buchsbaum local rings with minimal multiplicity.*

Recently, Terai and the author [15] defined the notion of *Buchsbaum rings with minimal multiplicity with initial degree* $q \geq 3$ *for Stanley–Reisner rings*, and proved that those rings have q-linear resolution. Their result indicates the existence of Buchsbaum homogeneous k-algebras with minimal multiplicity of higher initial degree.

The purpose of my first talk is to introduce the notion of **Buchsbaum homogeneous k-algebra with minimal multiplicity with initial degree** $q \geq 2$ and characterize those rings. Also, in my second talk, I will give several examples of those k-algebras and applications to the theory of Stanley–Reisner rings.

2. Lower bound

Throughout this talk, let $S = k[X_1, \ldots, X_n]$ be a homogeneous polynomial ring with n variables over an infinite field k with $\deg X_i = 1$. Let $A = S/I = \oplus_{n \geq 0} A_n$ be a homogeneous k-algebra with $\dim A = d$ where $I \subseteq (X_1, \ldots, X_n)^2 S$. Put $\mathfrak{m} = (X_1, \ldots, X_n)A$. Take a graded minimal free resolution over S:

$$0 \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{p,j}(A)} \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{1,j}(A)} \xrightarrow{\varphi_1} S \to A \to 0.$$

[1] The ideal I is said to have *minimal multiplicity* if $\mu_A(I) = e(I) + \dim A - l_A(A/I)$.

Then the *initial degree* of A is defined by

$$\operatorname{indeg} A = \min\{j \in \mathbb{Z} \,:\, \beta_{1,j}(A) \neq 0\}.$$

Similarly, the (*Castelnuovo–Mumford*) *regularity* (see [3]) is defined by

$$\begin{aligned} \operatorname{reg} A &= \max\{j - i \in \mathbb{Z} \,:\, \beta_{i,j}(A) \neq 0\} \\ &= \min\{n \in \mathbb{Z} \,:\, [H^i_{\mathfrak{m}}(A)]_j = 0 \quad \text{for all } i+j > n\}. \end{aligned}$$

Then $\operatorname{reg} A \geq \operatorname{indeg} A - 1$, and A has *q-linear resolution* if equality holds and $\operatorname{indeg} A = q$. Note that $\operatorname{indeg} A = 1$ and $\operatorname{reg} A = 0$ if $A(= S)$ is a polynomial ring.

In what follows, let $A = S/I$ be a homogeneous k-algebra with $d := \dim A \geq 1$, $c := \operatorname{codim} A \geq 1$ and $q := \operatorname{indeg} A \geq 2$. Also, let $x_1, \ldots, x_d \in A_1$ be a linear system of parameters of A and fix it. For an integer $\ell \geq 1$ we set

$$Q := (x_1, \ldots, x_d)A, \qquad \Sigma(\underline{x}^\ell) := \sum_{i=1}^{d} (x_1^\ell, \ldots, \widehat{x_i^\ell}, \ldots, x_d^\ell) : x_i^\ell + Q.$$

In particular, we write $\Sigma(Q) = \Sigma(\underline{x})$ for simplicity.

The following result for Cohen–Macaulay rings is known (see, e.g., [3]).

Proposition 2.1. *Suppose that A is Cohen–Macaulay. Then*
- (1) $e(A) \geq \binom{c+q-1}{q-1}$.
- (2) $a(A) = \operatorname{reg} A - d \geq q - d - 1$.
- (3) *The following conditions are equivalent*:
  - (a) $e(A) = \binom{c+q-1}{q-1}$.
  - (b) $a(A) = q - d - 1$.
  - (c) *A has q-linear resolution.*

  When this is the case, $A/Q \cong k[Y_1, \ldots, Y_c]/(Y_1, \ldots, Y_c)^q$, where $k[Y_1, \ldots, Y_c]$ is a polynomial ring in c variables over k.

The main result in this section is the following theorem, which generalizes the inequality in Proposition 2.1.

Theorem 2.2 (**Lower bound on multiplicities for Buchsbaum rings**). *Suppose that A is Buchsbaum. Then the following inequality holds*:

$$e(A) \geq \binom{c+q-2}{q-2} + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(H^i_{\mathfrak{m}}(A)).$$

Also, the following statement holds:
- (1) $\Sigma(Q) \subseteq \mathfrak{m}^{q-1} + Q$.
- (2) $a(A) \geq q - d - 2$.
- (3) $[H^i_{\mathfrak{m}}(A)]_j = 0$ $(i < d,\ j \leq q - 2 - i)$.

Remark 2.3. (1) We can prove a similar inequality as in the above theorem for any Buchsbaum local ring.
(2) Suppose that A is Buchsbaum. If $\operatorname{indeg} A = q > q' \geq 2$, then

$$e(A) > \binom{c+q'-2}{q'-2} + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(H^i_{\mathfrak{m}}(A)).$$

(3) Herrmann and Ikeda [11, Theorem 2.2] proved that $e(A) \geq \binom{c+q-2}{q-2}$ for any Buchsbaum local ring $A = R/I$ where $(R, \mathfrak{n})$ is a regular local ring and $I \subseteq \mathfrak{n}^q$.

We need the following two lemmas to prove the above theorem.

Lemma 2.4 (Hoa–Miyazaki [10])**.** *Suppose that A is Buchsbaum. Then*

$$\operatorname{reg} A \leq a(A) + d + 1.$$

Note that $a(A) + d \leq \operatorname{reg} A$ by definition.

Lemma 2.5 (Goto [6, Theorem 4.1])**.** *Suppose that A is Buchsbaum.*

$$e(A) = e(Q) = l_A(A/\Sigma(Q)) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(H_{\mathfrak{m}}^i(A)).$$

By virtue of the above theorem, we can define the notion of Buchsbaum homogeneous k-algebras with minimal multiplicity, which generalizes the notion defined by Goto in [6].

Definition 2.6 (**Buchsbaum rings with minimal multiplicity**)**.** Suppose that A is Buchsbaum. The ring A is called a *Buchsbaum ring with minimal multiplicity of degree* q if the equality holds:

$$e(A) = \binom{c+q-2}{q-2} + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(H_{\mathfrak{m}}^i(A)).$$

We regard a polynomial ring as a Buchsbaum (Cohen–Macaulay) ring with minimal multiplicity of degree 1.

3. Characterization

In what follows, we use same notation as in the previous section. This is the main result in this talk.

Theorem 3.1 (**Characterization of Buchsbaum rings with minimal multiplicity**)**.** *Suppose that A is Buchsbaum. Then the following conditions are equivalent*:

(1) *A has minimal multiplicity of degree q, that is,*

$$e(A) = \binom{c+q-2}{q-2} + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(H_{\mathfrak{m}}^i(A)).$$

(2) $a(A) = q - d - 2$.

(3) $H_{\mathfrak{m}}^i(A) = [H_{\mathfrak{m}}^i(A)]_{q-1-i}$ $(i < d)$ *and* $[H_{\mathfrak{m}}^d(A)]_n = 0$ $(n \geq q - d - 1)$.

(4) *A has q-linear resolution with*

$$\sum_{i=0}^{d-1} \binom{d}{i} l_A(H_{\mathfrak{m}}^i(A)) = \binom{\operatorname{reg} A + c - 1}{c - 1}.$$

(5) $\Sigma(Q) = \mathfrak{m}^{q-1} + Q$.

(6) $\Sigma(Q) \supseteq \mathfrak{m}^{q-1}$, *that is,* $[\Sigma(Q)]_n = A_n$ $(n \geq q - 1)$.

When this is the case, we have $\operatorname{Soc}(H_{\mathfrak{m}}^d(A)) = [H_{\mathfrak{m}}^d(A)]_{q-d-2}$ *and* $\operatorname{reg} A = a(A) + d + 1 = q - 1$.

In the proof of the above theorem, the following lemma plays an important role.

Lemma 3.2 (Eisenbud–Goto [3]). *Suppose that A is Buchsbaum.*

(1) *A has q-linear resolution.*
(2) $H^i_{\mathfrak{m}}(A) = [H^i_{\mathfrak{m}}(A)]_{q-1-i}$ $(i < d)$ *and* $[H^d_{\mathfrak{m}}(A)]_n = 0$ $(n \geq q-d)$.
(3) $\mathfrak{m}^q = Q\mathfrak{m}^{q-1}$.

Remark 3.3. (1) In [12], Kamoi and Vogel proved that the following inequality:

$$\sum_{i=0}^{d-1} \binom{d}{i} l_A(H^i_{\mathfrak{m}}(A)) \leq \binom{\operatorname{reg} A + c - 1}{c-1}$$

provided A is a homogeneous Buchsbaum k-algebra.
(2) For a Buchsbaum k-algebra A, it has maximal embedding dimension if and only if it has 2-linear resolution. Also, any Buchsbaum ring with minimal multiplicity has 2-linear resolution; see [6].
(3) When $q \geq 2$, if A is a Buchsbaum ring with minimal multiplicity, then it is *not* Cohen–Macaulay.
(4) Suppose that A has minimal multiplicity of degree q. Let $x \in A_1$ be a non-zero-divisor with $e(A) = e(A/aA)$. Then A/xA has q-linear resolution, and it has minimal multiplicity if and only if $H^{d-1}_{\mathfrak{m}}(A) = 0$.

4. Examples

In this section, we give several examples of Buchsbaum homogeneous k-algebras with minimal multiplicity of higher initial degree.

4.1. **The case of** $\operatorname{depth} A = 0$. We first give examples of Buchsbaum homogeneous k-algebras with minimal multiplicity with $\operatorname{depth} A = 0$. has

Proposition 4.1. *Under the same notation as in Theorem* 3.1*, if, in addition, $A/H^0_{\mathfrak{m}}(A)$ is Cohen–Macaulay, then the following conditions are equivalent*:

(1) *A has minimal multiplicity of degree q, that is,* $e(A) = \binom{c+q-2}{q-2}$.
(2) *A has q-linear resolution and* $l_A(H^0_{\mathfrak{m}}(A)) = \binom{c+q-2}{q-1}$.
(3) $a(A) = q - d - 2$.
(4) $A/H^0_{\mathfrak{m}}(A)$ *has* $(q-1)$*-linear resolution.*

In particular, if S/J is a Cohen–Macaulay homogeneous k-algebra with $(q-1)$-linear resolution, then $A = S/\mathfrak{m}J$ is a Buchsbaum homogeneous k-algebra with minimal multiplicity of degree q with $l_A(H^0_{\mathfrak{m}}(A)) = \mu_S(J)$.

Example 4.2. Let $S = k[x,y,z,w]$ be a polynomial ring.

(1) If we set $I = (x,y,z,w)(xw - yz, y^2 - xz, z^2 - xw)$ and $A = S/I$, then $A/H^0_{\mathfrak{m}}(A) \cong k[s^3, s^2t, st^2, t^3]$ is Cohen–Macaulay with 2-linear resolution. Thus A is Buchsbaum with minimal multiplicity of degree 3. Also, we have:
$$H^0_{\mathfrak{m}}(A) = k(-2), \quad H^1_{\mathfrak{m}}(A) = 0; \quad e(A) = 3.$$
(2) If we set $I = ((x,y,z,w)(xw-yz), z^3-yw^2, y^3-x^2z, xz^2-y^2w)$ and $A = S/I$, then A is Buchsbaum with minimal multiplicity of degree 3 since $H^0_{\mathfrak{m}}(A) = k(-2)$, $H^1_{\mathfrak{m}}(A) = k(-1)$ and $e(A) = 4$. But $A/H^0_{\mathfrak{m}}(A) \cong k[s^4, s^3t, st^3, t^4]$ is *not* Cohen–Macaulay.

Example 4.3. Any homogeneous non-Cohen–Macaulay Buchsbaum k-algebra A with $e(A) \le 2$ has minimal multiplicity whose degree is at most 3. Precisely speaking, we have:

(1) Suppose that $e(A) = 1$. When $\operatorname{depth} A > 0$, then A is a polynomial ring. Otherwise, A has minimal multiplicity of degree 2.

(2) Suppose that $e(A) = 2$. When $\operatorname{depth} A > 0$, A has minimal multiplicity of degree 2 (see [4]). For example, $A = k[x_1, \ldots, x_d, y_1, \ldots, y_d]/(x_1, \ldots, x_d) \cap (y_1, \ldots, y_d)$.
When $\operatorname{depth} A = 0$, A has minimal multiplicity of degree 2 or 3. For example, $A = k[x_1, \ldots, x_d, y]/(x_1y^2, \ldots, x_dy^2, y^3)$.

4.2. **The case of reduced rings.** One can find many examples of Buchsbaum reduced k-algebras with minimal multiplicity in the class of Stanley–Reisner rings. In fact, Terai and the author [15] gave another definition of Buchsbaum k-algebras with minimal multiplicity for Stanley–Reisner rings and have characterized those rings.

Let Δ be a *simplicial complex* on the vertex set $V = [n] := \{1, \ldots, n\}$, that is, Δ is a collection of subsets of V such that (a) $F \subseteq G$, $G \in \Delta \Longrightarrow F \in \Delta$ and (b) $\{i\} \in \Delta$ for $i \in V$. The dimension of $F \in \Delta$ (F is said to be a *face* of Δ) is defined by $\#(F) - 1$. Set $\dim \Delta = \max\{\dim F \,:\, F \in \Delta\}$. A complex is called *pure* if all facets (maximal faces) have the same dimension. Put $\operatorname{link}_\Delta\{v\} = \{F \in \Delta \,:\, F \cup \{v\} \in \Delta,\ v \notin F\}$, the *link* of v in Δ.

If we put $I_\Delta = (X_{i_1} \cdots X_{i_p} \,:\, 1 \le i_1 < \cdots < i_p \le n,\ \{i_1, \ldots, i_p\} \notin \Delta)S$, then $k[\Delta] = S/I_\Delta$ is called the *Stanley–Reisner ring* of Δ over k. The ring is a homogeneous reduced k-algebra with $d := \dim k[\Delta] = \dim \Delta + 1$ and $e(k[\Delta]) = f_{d-1}(\Delta)$ is equal to the number of facets F with $\dim F = d - 1$. Note that $k[\Delta]$ is Buchsbaum if and only if Δ is pure and $k[\operatorname{link}_\Delta\{v\}]$ is a Cohen–Macaulay ring (of dimension $d - 1$). Also, if $A = k[\Delta]$ is Buchsbaum if and only if $H^i_{\mathfrak{m}}(A) = [H^i_{\mathfrak{m}}(A)]_0 (\cong \widetilde{H}_{i-1}(\Delta; k))$ for all $i < d$.

In the class of Buchsbaum Stanley–Reisner rings, there exists a criterion for $k[\Delta]$ to have linear resolution in terms of h-vectors as follows:

Theorem 4.4. *Suppose that $A = k[\Delta]$ is Buchsbaum and $2 \le q \le d$. Put $h = \dim_k H^{q-1}_{\mathfrak{m}}(A)$. Then the following conditions are equivalent:*

(1) *A has q-linear resolution.*

(2) *The h-vector $h(\Delta) = (h_0, h_1, \ldots, h_{q-1}, h_q, h_{q+1}, \ldots, h_d)$ of Δ is*

$$\left(1,\, c,\, \cdots,\, \binom{c+q-2}{q-1},\, -\binom{d}{q}h,\, \binom{d}{q+1}h,\, \cdots,\, (-1)^{d-q+1}\binom{d}{d}h\right).$$

(3) *The following equalities hold:*

$$e(A) = \binom{c+q-1}{q-1} - h\binom{d-1}{q-1} \quad \text{and} \quad I(A) = h\binom{d-1}{q-1}.$$

When this is the case, $H^i_{\mathfrak{m}}(A) = 0$ ($i \ne q-1, d$) and the following inequalities hold:

$$0 \;\le\; h \;\le\; \frac{c(c+1)\cdots(c+q-2)}{d(d-1)\cdots(d-q+2)} =: h_{c,d,q}.$$

From this point of view, one can regard a *Buchsbaum simplicial complex with minimal multiplicity* as a Buchsbaum complex with linear resolution and "*maximal homology*".

Theorem 4.5 (Terai and Y- [15]). *Suppose that* $A = k[\Delta]$ *is Buchsbaum. Then*

$$e(A) \geq \frac{c+d}{d}\binom{c+q-2}{q-2}$$

holds, and the following statements are equivalent:

(1) *A has minimal multiplicity of initial degree q, that is,* $e(A) = \frac{c+d}{d}\binom{c+q-2}{q-2}$.
(2) *A has minimal multiplicity of degree q in our sense.*
(3) *A has q-linear resolution with* $l_A(H_{\mathfrak{m}}^{q-1}(A)) = h_{c,d,q}$.
(4) $k[\mathrm{link}_\Delta\{v\}]$ *has* $(q-1)$*-linear resolution for all* $v \in V$.
(5) *The h-vector of A can be written as in the shape of Theorem* 4.4(2) *and* $h = h_{c,d,q}$.
(6) $k[\Delta^*]$ *has pure and almost linear resolution and* $a(k[\Delta^*]) = 0$, *where* $\Delta^* = \{F \in \Delta : V \setminus F \notin \Delta\}$ *denotes*
the Alexander dual complex of Δ.

Example 4.6. Let us pick up examples of Buchsbaum complexes with minimal multiplicity.

(1) Δ is a finite disjoint union of $(d-1)$-simplexes if and only if $k[\Delta]$ has minimal multiplicity of degree 2.
(2) Let q, d be given integers with $2 \leq q \leq d$. Put $n = 2d - q + 2$ and $f = 2(d - q + 1)$. Let Δ be the Alexander dual of the boundary complex Γ of a cyclic polytope $C(n, f)$ with n vertices. Then $k[\Delta]$ is a d-dimensional Buchsbaum Stanley–Reisner ring with minimal multiplicity of degree q with $h = h_{c,d,q} = 1$.
(3) For a given integer $n \geq 3$, there exists a 3-dimensional Buchsbaum complex on $[n]$ with minimal multipicity of degree 3 if and only if $n \equiv 0, 2 \pmod 3$; see Hanano's examples in [9].

In general, the following question remains open when $d \geq 4$.

Problem 4.7. Let c, d, q, h be integers with $c \geq 2$, $2 \leq q \leq d$ and $0 \leq h \leq h_{c,d,q}$. Construct $(d-1)$-dimensional Buchsbaum complexes Δ with q-linear resolution such that $\operatorname{codim} k[\Delta] = c$ and $\dim_k H_{\mathfrak{m}}^{q-1}(k[\Delta]) = h$. See also [16].

4.3. **The case of integral domains.** We have no examples of Buchsbaum homogeneous integral domain over $k = \overline{k}$ with minimal multiplicity of degree $q \geq 2$. The following proposition (in the case of $q = 2$) immediately follows from Abhyankar's result mentioned as above.

Proposition 4.8. *Let A be a Buchsbaum homogeneous k-algebra with minimal multiplicity*
of degree at most 2. *Also, suppose that* $k = \overline{k}$ *and A is an integral domain. Then A is isomorphic to a polynomial ring.*

So it is natural to ask the following question:

Question 4.9. *Is there a Buchsbaum homogenbeous integral domain* (*over* $k = \overline{k}$) *with minimal multiplicity of degree* $q \geq 2$?

Remark 4.10. Let $d \geq 2$, h^i $(1 \leq i \leq d-1)$, $s \geq 0$ be integers. Then there exist Buchsbaum homogeneous integral domains A over an algebraically closed field k which has q-linear resolution such that $\dim A = d$, $\operatorname{codim} A = 2$ and such that $\dim_k H^i_{\mathfrak{m}}(A) = h^i$ for all $i = 1, \ldots, d-1$, where

$$q = \sum_{i=1}^{d-1} \left\{ \sum_{j=1}^{d+2-i} (-1)^j \cdot (j-2) \cdot \binom{d+2}{i+j} \right\} h^i + s - 3.$$

But this number "q is too big"! See [3, 7] and [1].

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