

SYMBOLIC REES ALGEBRAS OF SPACE MONOMIAL PRIMES OF DEGREE 5

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ABSTRACT. Let K be a field of characteristic 0. Let $\mathfrak{p}_K(5, 103, 169)$ be the defining ideal of the space monomial curve $\{(t^5, t^{103}, t^{169}) \mid t \in K\}$. In this paper we shall prove that the symbolic Rees algebra $R_s(\mathfrak{p}_K(5, 103, 169))$ is not Noetherian, that is, is not finitely generated over K .

1. INTRODUCTION

For a prime ideal Q of a commutative ring A , the n th symbolic power of Q is defined to be $Q^n A_Q \cap A$, and denoted by $Q^{(n)}$. We call the subring

$$A[Qt, Q^{(2)}t^2, Q^{(3)}t^3, \dots]$$

of the polynomial ring $A[t]$ the symbolic Rees algebra of Q , and denoted by $R_s(Q)$. This is not necessarily finitely generated as an A -algebra. In the case where A is Noetherian, $R_s(Q)$ is Noetherian if and only if $R_s(Q)$ is finitely generated over A as a ring.

In commutative algebra, the study of the finite generation (over the base ring) of symbolic Rees algebras began with Cowsik's problem [1]. Cowsik's problem asks whether the symbolic Rees algebra of a prime ideal in a polynomial ring over a field is finitely generated. Cowsik seems to have considered this question in connection with the set-theoretic complete intersection property of affine curves. The finite generation of symbolic Rees algebras is also deeply related to other problems in mathematics.

For example, symbolic Rees algebras often appear as Cox rings of algebraic varieties or as their subrings. The finite generation of Cox rings of algebraic varieties, or of their subrings, is a fundamental problem lying at the heart of birational geometry.

In 1956, Nagata [15] gave a counterexample to Hilbert's fourteenth problem, and in that paper he proposed the following conjecture, now known as Nagata's conjecture: "Let n be a natural number with $n \geq 10$. If there exists a plane curve of degree d passing through n general points in the complex projective plane, each with multiplicity at least r , then $d > \sqrt{nr}$." Nagata proved this conjecture in the case where n is a perfect square, and thereby obtained a counterexample to Hilbert's fourteenth problem. The finite generation of the symbolic Rees algebra of the space monomial

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primes defined below is closely related to Nagata's conjecture (Proposition 5.2 in [3]).

Let K be a field. Let a, b, c be pairwise coprime positive integers. Let $S = K[x, y, z]$ be a polynomial ring over K . Let $\mathfrak{p}_K(a, b, c)$ be the kernel of the K -algebra homomorphism $S \xrightarrow{\varphi} K[T]$ defined by $\varphi(x) = T^a$, $\varphi(y) = T^b$ and $\varphi(z) = T^c$. An ideal of this form is called a space monomial prime ideal. We sometimes denote $\mathfrak{p}_K(a, b, c)$ simply by \mathfrak{p} .

If a prime ideal Q is generated by a regular sequence, then the symbolic powers of Q coincide with its ordinary powers. Hence, in this case, the symbolic Rees algebra of Q coincides with the ordinary Rees algebra of Q , and therefore the symbolic Rees algebra is finitely generated. The number of elements of a minimal generating set of a space monomial prime ideal is either 2 or 3 by a result of Herzog [9]. When the number of minimal generators is 3, a space monomial prime ideal can be regarded as the simplest example of a prime ideal in a polynomial ring over a field that is not generated by a regular sequence. For this reason, the symbolic Rees algebras of space monomial primes have been studied extensively.

Huneke [10] gave a simple necessary and sufficient condition for the symbolic Rees algebra of a space monomial prime to be finitely generated. Cutkosky [2] showed that the symbolic Rees algebra of a space monomial prime is the Cox ring of a certain algebraic surface, thereby giving a geometric interpretation of Huneke's criterion. Furthermore, using techniques from algebraic geometry and singularity theory, he provided a sufficient condition for the symbolic Rees algebra of a space monomial prime to be finitely generated. The first examples of non-finitely generated symbolic Rees algebras were given by Goto–Nishida–Watanabe [8]. For example, when the characteristic of the base field is zero, the symbolic Rees algebra corresponding to $(25, 29, 72)$ is not finitely generated. Gonzales–Gonzales–Karu [5], [6] introduced techniques from toric geometry, which made it possible to construct many examples.

In what follows, we call the minimum of a, b, c the degree of a space monomial prime $\mathfrak{p}_K(a, b, c)$. By a theorem of Cutkosky [2], it is known that the symbolic Rees algebra of a space monomial prime is finitely generated when the degree is at most 4, or equal to 6. Whether the symbolic Rees algebra of a space monomial prime of degree 5 is finitely generated has remained unknown, and this has been one of the intriguing open questions for researchers in this area. The main theorem of this paper addresses this problem.

Theorem 1.1. *Let K be a field of characteristic 0. Then the symbolic Rees algebra $R_s(\mathfrak{p}_K(5, 103, 169))$ is not Noetherian, that is, is not finitely generated over S .*

The author does not know any example of a space monomial prime of degree 8 whose symbolic Rees algebra is not Noetherian. In the case where the degree is 5 or 8, if one of the minimal generators of $\mathfrak{p}_K(a, b, c)$ gives a negative curve (defined in Definition 2.3 below), then the symbolic Rees algebra $R_s(\mathfrak{p}_K(a, b, c))$ is Noetherian [4], [13], [16]. In the case where $(a, b, c) = (7, 15, 26)$, $(9, 13, 29)$, $(10, 11, 27)$, $(11, 21, 25)$, $(12, 13, 17)$, $(13, 18, 25)$, ..., one of the minimal generators

of $\mathfrak{p}_K(a, b, c)$ gives a negative curve and the symbolic Rees algebra $R_s(\mathfrak{p}_K(a, b, c))$ is not Noetherian if the characteristic of K is 0.

Finally, we describe the structure of this paper. In Section 2, we introduce techniques from toric geometry and develop the general theory of negative curves. Using these tools, we show that in the case $(5, 103, 169)$, there exists an element f of degree 2065 in the 7th symbolic power that defines a negative curve. In Section 3, we show that if the symbolic Rees algebra of the space monomial prime $(5, 103, 169)$ over a field of characteristic zero is finitely generated, then there exists an element g of degree 17407 in the 59th symbolic power such that f and g satisfy Huneke's criterion. In Section 4, we reduce the problem to the case of characteristic 2 and show that the homogeneous component of degree 17407 of the 59th symbolic power is zero, thereby completing the proof of the theorem. The computations in Section 4 were carried out using a computer, but the results were also verified by hand without the use of a computer.

2. NEGATIVE CURVE

Suppose that a, b, c are pairwise coprime positive integers. We think that $S = K[x, y, z]$ is a graded ring with $\deg(x) = a$, $\deg(y) = b$ and $\deg(z) = c$. Then $\mathfrak{p}_K(a, b, c)$ is a homogeneous ideal of S .

We may suppose

$$\mathfrak{p}_K(a, b, c) = (x^s - y^{t_1}z^{u_1}, y^t - z^{u_2}x^{s_2}, z^u - x^{s_3}y^{t_3}) = I_2 \begin{pmatrix} x^{s_2} & y^{t_3} & z^{u_1} \\ y^{t_1} & z^{u_2} & x^{s_3} \end{pmatrix},$$

where $s = s_2 + s_3$, $t = t_1 + t_3$ and $u = u_1 + u_2$ by Herzog [9]. Let a', b', c' be integers satisfying $a'a + b'b + c'c = 1$ and put

$$(2.1) \quad T = x^{a'}y^{b'}z^{c'}.$$

Then we have

$$S[x^{-1}, y^{-1}, z^{-1}] = (S[x^{-1}, y^{-1}, z^{-1}]_0) [T^{\pm 1}].$$

Putting

$$v = \frac{z^{u_2}x^{s_2}}{y^t}, \quad w = \frac{z^u}{x^{s_3}y^{t_3}},$$

we have

$$S[x^{-1}, y^{-1}, z^{-1}]_0 = K[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]_0 = K[v^{\pm 1}, w^{\pm 1}]$$

as in the proof of Lemma 3.2 in [12]. It is easy to see that $v^\alpha w^\beta T^n \in S$ if and only if

$$\begin{cases} s_2\alpha - s_3\beta + a'n \geq 0 \\ -t\alpha - t_3\beta + b'n \geq 0 \\ u_2\alpha + u\beta + c'n \geq 0. \end{cases}$$

Let Δ be a triangle defined by

$$(2.2) \quad \begin{cases} s_2x - s_3y + a' \geq 0 \\ -tx - t_3y + b' \geq 0 \\ u_2x + uy + c' \geq 0 \end{cases}$$

in an x - y plane. We define the Erhart ring of Δ as

$$(2.3) \quad K[\Delta] := \bigoplus_{n \geq 0} \left(\bigoplus_{(\alpha, \beta) \in n\Delta \cap \mathbb{Z}^2} Kv^\alpha w^\beta \right) T^n.$$

Then $K[\Delta]$ naturally coincides with $S = K[x, y, z]$. Let \mathfrak{q} be the ideal of $K[v^{\pm 1}, w^{\pm 1}]$ generated by $v - 1$ and $w - 1$. Then we have

$$\mathfrak{p}S[x^{-1}, y^{-1}, z^{-1}] = \mathfrak{q}S[x^{-1}, y^{-1}, z^{-1}].$$

For $m \geq 0$, we have

$$\begin{aligned} \mathfrak{p}^{(m)} &= \mathfrak{p}^m S[x^{-1}, y^{-1}, z^{-1}] \cap S = \mathfrak{q}^m S[x^{-1}, y^{-1}, z^{-1}] \cap S \\ &= \bigoplus_{n \geq 0} \left(\mathfrak{q}^m \cap \bigoplus_{(\alpha, \beta) \in n\Delta \cap \mathbb{Z}^2} Kv^\alpha w^\beta \right) T^n. \end{aligned}$$

The following proposition is well-known. Here we give an outline of a proof for the reader's convenience.

Proposition 2.1. (1) *Let g_1, g_2 be an S -regular sequence with $g_1 \in [\mathfrak{p}_K(a, b, c)^{(r_1)}]_{d_1}$ and $g_2 \in [\mathfrak{p}_K(a, b, c)^{(r_2)}]_{d_2}$ for some positive integers r_1, r_2, d_1, d_2 . Then we have*

$$\frac{d_1}{r_1} \cdot \frac{d_2}{r_2} \geq abc.$$

(2) *The symbolic Rees algebra $R_s(\mathfrak{p}_K(a, b, c))$ is Noetherian if and only if there exists an S -regular sequence g_1, g_2 with $g_1 \in [\mathfrak{p}_K(a, b, c)^{(r_1)}]_{d_1}$ and $g_2 \in [\mathfrak{p}_K(a, b, c)^{(r_2)}]_{d_2}$ for some positive integers d_1, d_2, r_1, r_2 such that*

$$\frac{d_1}{r_1} \cdot \frac{d_2}{r_2} = abc.$$

Proof. (1) Take h such that g_1, g_2, h is a homogeneous system of parameters of S . Then the Poincaré series of $S/(g_1, g_2, h)$ is

$$\frac{(1 - t^{d_1})(1 - t^{d_2})(1 - t^{\deg(h)})}{(1 - t^a)(1 - t^b)(1 - t^c)}.$$

Substituting 1 for t , we obtain

$$(2.4) \quad \frac{d_1 d_2 \deg(h)}{abc} = \ell_S(S/(g_1, g_2, h)) = \mathrm{e}((h), S/(g_1, g_2)) \geq \ell_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/(g_1, g_2)S_{\mathfrak{p}}) \cdot \mathrm{e}((h), S/\mathfrak{p}) \geq r_1 r_2 \deg(h),$$

where the first inequality follows from the additive formula of multiplicities and the second one follows from Lemma 2.2 in [12].

(2) Let g_1, g_2 be a regular sequence satisfying $g_1 \in [\mathfrak{p}^{(r_1)}]_{d_1}$, $g_2 \in [\mathfrak{p}^{(r_2)}]_{d_2}$ and $\frac{d_1}{r_1} \cdot \frac{d_2}{r_2} = abc$. By (2.4), we know $\ell_S(S/(g_1, g_2, h)) = r_1 r_2 \ell_S(S/(h) + \mathfrak{p})$. Then $R_s(\mathfrak{p}_K(a, b, c))$ is Noetherian by Huneke's criterion [10].

Conversely assume that $R_s(\mathfrak{p}_K(a, b, c))$ is Noetherian. Put $X = \mathrm{Proj} S$. Let $\pi : Y \rightarrow X$ be the blow-up at the point corresponding to $\mathfrak{p}_K(a, b, c)$. Let E be

the exceptional curve. Then there exist curves C and D on Y such that $C \neq E$, $D \neq E$ and $C \cap D = \emptyset$ by Cutkosky's criterion [2]. Let g_1 and g_2 be the defining equations of $\pi(C)$ and $\pi(D)$, respectively. Suppose $g_1 \in [\mathfrak{p}^{(r_1)}]_{d_1} \setminus [\mathfrak{p}^{(r_1+1)}]_{d_1}$ and $g_2 \in [\mathfrak{p}^{(r_2)}]_{d_2} \setminus [\mathfrak{p}^{(r_2+1)}]_{d_2}$. Since $\pi(C) \cap \pi(D)$ is not empty and $C \cap D = \emptyset$, we have $\sqrt{(g_1, g_2)} = \mathfrak{p}$. Therefore the first " \geq " in (2.4) is " $=$ ". Here $\text{Spec } K[v^{\pm 1}, w^{\pm 1}]$ is an affine open subset of X . The blow-up of it at \mathfrak{q} is an open subscheme of Y . The equations of $\pi(C)$ and $\pi(D)$ in $\text{Spec } K[v^{\pm 1}, w^{\pm 1}]$ are $g'_1 := g_1/T^{\deg(g_1)}$ and $g'_2 := g_2/T^{\deg(g_2)}$, respectively, where T is an element defined in (2.1). Here we put $B = K[v^{\pm 1}, w^{\pm 1}]_{\mathfrak{q}}$. By Lemma 2.2 in [12], we obtain $\ell_B(B/(g'_1, g'_2)) = r_1 r_2$ since $C \cap D = \emptyset$. Since $\ell_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/(g_1, g_2)S_{\mathfrak{p}}) = \ell_B(B/(g'_1, g'_2))$, the second " \geq " in (2.4) is also " $=$ ". \square

Definition 2.2. We say that the Huneke's condition is satisfied if there exists an S -regular sequence g_1, g_2 with $g_1 \in [\mathfrak{p}_K(a, b, c)^{(r_1)}]_{d_1}$ and $g_2 \in [\mathfrak{p}_K(a, b, c)^{(r_2)}]_{d_2}$ for some positive integers d_1, d_2, r_1, r_2 such that $\frac{d_1}{r_1} \cdot \frac{d_2}{r_2} = abc$.

Definition 2.3. A non-zero homogeneous irreducible polynomial f in S is called a negative curve if f is contained in $[\mathfrak{p}_K(a, b, c)^{(r_0)}]_{d_0}$ for some positive integers r_0, d_0 with $d_0/r_0 < \sqrt{abc}$.

In the case where the characteristic of K is positive, Cutkosky [2] proved that $R_s(\mathfrak{p}_K(a, b, c))$ is Noetherian if there exists a negative curve. In the case where the characteristic of K is 0, $R_s(\mathfrak{p}_K(a, b, c))$ is not necessarily Noetherian even if there exists a negative curve (Goto-Nishida-Watanabe [8]).

Remark 2.4. (1) Suppose that f is a negative curve, that is, f is an irreducible element in $[\mathfrak{p}^{(r_0)}]_{d_0}$ with $d_0/r_0 < \sqrt{abc}$. Suppose $g \in [\mathfrak{p}_K(a, b, c)^{(r)}]_d$ for some positive integers r, d with $d/r < abc/(d_0/r_0)$. Then f divides g by Proposition 2.1 (1).

We know that, if a negative curve exists, it is determined uniquely up to multiplication by an element of K^{\times} .

(2) Assume that $[\mathfrak{p}_K(a, b, c)^{(r)}]_d \neq 0$ for some positive integers r, d with $d/r < \sqrt{abc}$. Suppose $0 \neq f \in [\mathfrak{p}_K(a, b, c)^{(r)}]_d$. Consider the irreducible decomposition $f = f_1 f_2 \cdots f_n$. Since the associated graded ring

$$(2.5) \quad S/\mathfrak{p} \oplus \mathfrak{p}/\mathfrak{p}^{(2)} \oplus \mathfrak{p}^{(2)}/\mathfrak{p}^{(3)} \oplus \cdots$$

is an integral domain, one of f_i 's is a negative curve. Therefore there exists a negative curve if and only if $[\mathfrak{p}_K(a, b, c)^{(r)}]_d \neq 0$ for some positive integers r, d with $d/r < \sqrt{abc}$.

(3) Let L/K be a field extension. It is easy to see

$$[\mathfrak{p}_K(a, b, c)^{(r)}]_d \otimes_K L = [\mathfrak{p}_L(a, b, c)^{(r)}]_d$$

for any r, d .

We know that $R_s(\mathfrak{p}_K(a, b, c))$ is Noetherian if and only if so is $R_s(\mathfrak{p}_L(a, b, c))$. In particular, the finite generation of $R_s(\mathfrak{p}_K(a, b, c))$ depends only on $\text{ch}(K)$, a, b, c .

There exists a negative curve in the case K, a, b, c if and only if so does in the case L, a, b, c . Furthermore assume that K is a prime field. If f is a negative curve in the case K, a, b, c , then f is also a negative curve in the case L, a, b, c . If a negative curve exists, we may assume that it is a polynomial over the prime field and it is absolutely irreducible.

- (4) Suppose f is a negative curve in $[\mathfrak{p}_K(a, b, c)^{(r_0)}]_{d_0}$ for some positive integers r_0, d_0 with $d_0/r_0 < \sqrt{abc}$. Then r_0 is also uniquely determined as follows: We may assume that the coefficients of f are in the prime field. Assume that $f \in [\mathfrak{p}_K(a, b, c)^{(r_0+1)}]_{d_0}$ and $d_0/r_0 < \sqrt{abc}$. It is easy to see that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are contained in $\mathfrak{p}_K(a, b, c)^{(r_0)}$. If $\frac{\partial f}{\partial x}$ is not 0, f divides it and the degree of $\frac{\partial f}{\partial x}$ is less than that of f . It is a contradiction. Thus we know $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$. Then we may assume that the characteristic of K is a prime number p . Since the coefficients of f are in the prime field, there exists a polynomial f' such that $f = f'^p$. It is a contradiction since f is irreducible.
- (5) Suppose f is a negative curve in $[\mathfrak{p}_K(a, b, c)^{(r_0)}]_{d_0}$ for some positive integers r_0, d_0 with $d_0/r_0 < \sqrt{abc}$. Suppose $r > 0$ and $d \geq 0$. If $d/r < d_0/r_0$, then $[\mathfrak{p}^{(r)}]_d = 0$ as follows: Assume $0 \neq g \in [\mathfrak{p}^{(r)}]_d$ and $d/r < d_0/r_0$. Then, by (1), f divides g . Put $g = fh$ and $h \in [\mathfrak{p}^{(r')}]_{d'} \setminus [\mathfrak{p}^{(r'+1)}]_{d'}$. Here remark that $r > r_0$ since $d \geq d_0$ and $d/r < d_0/r_0$. Then $g = fh \in [\mathfrak{p}^{(r_0+r')}]_{d_0+d'} \setminus [\mathfrak{p}^{(r_0+r'+1)}]_{d_0+d'}$ since the associated graded ring (2.5) is an integral domain and $f \notin \mathfrak{p}^{(r_0+1)}$ by (4). Then $r \leq r_0 + r'$ and $d = d_0 + d'$. Theefore

$$\frac{d'}{r'} \leq \frac{d - d_0}{r - r_0} < \frac{d_0}{r_0}.$$

Thus g is divisible by f^n for any $n > 0$. It is a contradiction.

- (6) Suppose that Huneke's condition is satisfied, that is, there exists an S -regular sequence g_1, g_2 with $g_1 \in [\mathfrak{p}_K(a, b, c)^{(r_1)}]_{d_1}$ and $g_2 \in [\mathfrak{p}_K(a, b, c)^{(r_2)}]_{d_2}$ (d_1, d_2, r_1, r_2 are positive integers) such that

$$\frac{d_1}{r_1} \cdot \frac{d_2}{r_2} = abc.$$

We may assume

$$\frac{d_1}{r_1} \leq \sqrt{abc} \leq \frac{d_2}{r_2}.$$

If a negative curve does not exist, then we have

$$\frac{d_1}{r_1} = \frac{d_2}{r_2} = \sqrt{abc}$$

by (2).

Assume that there exists a negative curve f in $[\mathfrak{p}_K(a, b, c)^{(r_0)}]_{d_0}$ for some positive integers r_0, d_0 with $d_0/r_0 < \sqrt{abc}$. We have

$$\frac{d_0}{r_0} \leq \frac{d_1}{r_1}$$

by (5). If $\frac{d_0}{r_0} < \frac{d_1}{r_1}$, then both of g_1 and g_2 are divisible by f . It is a contradiction. Therefore we obtain

$$\frac{d_0}{r_0} = \frac{d_1}{r_1}.$$

In this case, we can prove $g_1 = cf^n$ for a positive integer n and $c \in K^\times$.

(7) Let g_1, g_2 be homogeneous elements of S . Then g_1, g_2 satisfies Huneke's condition if and only if there exists positive integers ℓ_1 and ℓ_2 such that $g_1^{\ell_1}, g_2^{\ell_2}$ satisfies Huneke's condition.

Assume that there exists a negative curve f . If there exist g_1, g_2 satisfying Huneke's condition, we may assume that g_1 is a power of f by (6). Then f, g_2 also satisfies Huneke's condition.

(8) Let L/K be a field extension. Assume that there exists a negative curve f . For positive integers r, d , there exists g in $[\mathfrak{p}_K(a, b, c)^{(r)}]_d$ such that f, g satisfies Huneke's condition if and only if there exists g' in $[\mathfrak{p}_L(a, b, c)^{(r)}]_d$ such that f, g' satisfies Huneke's condition.

For a subset Q of \mathbb{Z}^2 , we put

$$KQ := \bigoplus_{(\alpha, \beta) \in Q} Kv^\alpha w^\beta \subset K[v^{\pm 1}, w^{\pm 1}].$$

For a finite set M , $\#(M)$ denotes the number of elements contained in M .

The following proposition is essentially proved in [14]. Here we give an outline of a proof for the reader's convenience.

Proposition 2.5. *Let K be a field of characteristic 0. Let Q be a subset of \mathbb{Z}^2 . Let L be a line in \mathbb{R}^2 such that L contains infinitely many points of \mathbb{Z}^2 . Put $Q' = Q \setminus (L \cap Q)$. Then, for each positive integer m , there exists a K -linear map*

$$\psi_m : KQ \cap \mathfrak{q}^m \longrightarrow KQ' \cap \mathfrak{q}^{m-1}$$

satisfying the following conditions:

- (1) The map ψ_m is surjective if $\#(L \cap Q) \geq m$.
- (2) The map ψ_m is injective if $\#(L \cap Q) \leq m$.
- (3) For $(\alpha_0, \beta_0) \in Q'$ and $\xi \in KQ \cap \mathfrak{q}^m$, the coefficient of $v^{\alpha_0} w^{\beta_0}$ in ξ is 0 if and only if that in $\psi_m(\xi)$ is 0.

Proof. First of all, remember that, for $h \in K[v^{\pm 1}, w^{\pm 1}]$, the following two conditions are equivalent:

- $h \in \mathfrak{q}^m$.
- For non-negative integers p, q with $p + q < m$,

$$(2.6) \quad \left. \frac{\partial^{p+q} h}{\partial v^p \partial w^q} \right|_{(v,w)=(1,1)} = 0.$$

Considering a multiplication of $v^{\alpha'} w^{\beta'}$ and an action of $\mathrm{GL}(2, \mathbb{Z})$, we may assume that L is the y -axis.

It is enough to construct a K -isomorphism $\psi_m : KQ \cap \mathfrak{q}^m \rightarrow KQ' \cap \mathfrak{q}^{m-1}$ satisfying the above (3) in the case $\#(L \cap Q) = m$.

Suppose

$$L \cap Q = \{(0, \beta_1), (0, \beta_2), \dots, (0, \beta_m)\},$$

where $\beta_1 < \beta_2 < \dots < \beta_m$. Remark

$$Q' = Q \setminus (L \cap Q) = \{(\alpha, \beta) \in Q \mid \alpha \neq 0\}.$$

If ξ is in \mathfrak{q}^m , then $v \frac{\partial \xi}{\partial v}$ is in \mathfrak{q}^{m-1} . We define a K -linear map $\psi_m : KQ \cap \mathfrak{q}^m \rightarrow KQ' \cap \mathfrak{q}^{m-1}$ by $\psi_m(\xi) = v \frac{\partial \xi}{\partial v}$. It is easy to check that ψ satisfies the above (3).

We shall prove that ψ_m is an isomorphism.

First we shall prove that ψ_m is injective. Take $\xi \in \text{Ker } \psi_m$. Then ξ satisfies $\frac{\partial \xi}{\partial v} = 0$. Therefore we may assume

$$\xi = \sum_{j=1}^m c_j w^{\beta_j},$$

where $c_j \in K$ for each j . Since $\xi \in \mathfrak{q}^m$, we know

$$\frac{\partial^u \xi}{\partial w^u}(1, 1) = 0$$

for $u = 0, 1, \dots, m-1$. Hence we obtain

$$(2.7) \quad 0 = \frac{\partial^u \xi}{\partial w^u}(1, 1) = \sum_{j=1}^m c_j \beta_j (\beta_j - 1) \cdots (\beta_j - u + 1)$$

for $u = 0, 1, \dots, m-1$. The matrix corresponding to the simultaneous equations (2.7) is the following:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & \cdots & \beta_m \\ \beta_1(\beta_1 - 1) & \beta_2(\beta_2 - 1) & \cdots & \beta_m(\beta_m - 1) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1(\beta_1 - 1) \cdots (\beta_1 - m + 2) & \beta_2(\beta_2 - 1) \cdots (\beta_2 - m + 2) & \cdots & \beta_m(\beta_m - 1) \cdots (\beta_m - m + 2) \end{pmatrix}$$

After some elementary transformations of rows, we obtain

$$\begin{pmatrix} 1 & \cdots & 1 \\ \beta_1 & \cdots & \beta_m \\ \vdots & & \vdots \\ \beta_1^{m-1} & \cdots & \beta_m^{m-1} \end{pmatrix}.$$

Since this matrix is invertible, (2.7) implies $c_j = 0$ for $j = 1, 2, \dots, m$.

Next we shall prove that ψ_m is surjective. Take

$$\eta = \sum_{(\alpha, \beta) \in Q'} b_{(\alpha, \beta)} v^\alpha w^\beta \in kQ' \cap \mathfrak{q}^{m-1},$$

where $b_{(\alpha,\beta)} \in K$. We define

$$\tilde{\eta} = \sum_{(\alpha,\beta) \in Q'} \frac{1}{\alpha} b_{(\alpha,\beta)} v^\alpha w^\beta \in KQ'.$$

Put

$$(2.8) \quad \xi(c_1, c_2, \dots, c_m) = \tilde{\eta} + \sum_{j=1}^m c_j w^{\beta_j} \in KQ$$

for $c_1, c_2, \dots, c_m \in K$. Since

$$(2.9) \quad \frac{\partial}{\partial v} \xi(c_1, c_2, \dots, c_m) = \frac{\partial \tilde{\eta}}{\partial v} = v^{-1} \eta \in \mathfrak{q}^{m-1},$$

we have

$$\eta = v \frac{\partial}{\partial v} \xi(c_1, c_2, \dots, c_m).$$

In order to show $\eta \in \text{Im}(\psi_m)$, it is sufficient to show $\xi(c_1, c_2, \dots, c_m) \in \mathfrak{q}^m$ for some $c_1, c_2, \dots, c_m \in K$.

By (2.9) and the equivalence (2.6), we know

$$\left. \frac{\partial^{i+j} \xi(c_1, c_2, \dots, c_m)}{\partial v^i \partial w^j} \right|_{(v,w)=(1,1)} = 0$$

for i and j satisfying $i > 0$, $j \geq 0$, and $0 \leq i + j < m$. Therefore it is enough to show

$$\frac{\partial^u \xi(c_1, c_2, \dots, c_m)}{\partial w^u}(1, 1) = \frac{\partial^u \tilde{\eta}}{\partial w^u}(1, 1) + \sum_{j=1}^m c_j \beta_j (\beta_j - 1) \cdots (\beta_j - u + 1) = 0$$

for $u = 0, 1, \dots, m-1$ and for some $c_1, c_2, \dots, c_m \in K$. That is to say, it is enough that c_1, c_2, \dots, c_m satisfies

$$\begin{aligned} & \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & \cdots & \beta_m \\ \beta_1(\beta_1 - 1) & \beta_2(\beta_2 - 1) & \cdots & \beta_m(\beta_m - 1) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1(\beta_1 - 1) \cdots (\beta_1 - m + 2) & \beta_2(\beta_2 - 1) \cdots (\beta_2 - m + 2) & \cdots & \beta_m(\beta_m - 1) \cdots (\beta_m - m + 2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_m \end{pmatrix} \\ &= - \begin{pmatrix} \tilde{\eta}(1, 1) \\ \frac{\partial \tilde{\eta}}{\partial w}(1, 1) \\ \frac{\partial^2 \tilde{\eta}}{\partial w^2}(1, 1) \\ \vdots \\ \frac{\partial^{m-1} \tilde{\eta}}{\partial w^{m-1}}(1, 1) \end{pmatrix}. \end{aligned}$$

Since the above $m \times m$ matrix is invertible, there exist c_1, c_2, \dots, c_m satisfying the above equation. \square

Put $X = \text{Proj } S$. Let Y be the blow-up at the point corresponding to $\mathfrak{p}_K(a, b, c)$. Assume that f is a negative curve in $[\mathfrak{p}_K(a, b, c)]_{d_0}^{(r_0)}$ with $d_0/r_0 < \sqrt{abc}$. Let C be the proper transform of $V_+(f)$. Then C is a curve on Y satisfying

$$C^2 = \frac{d_0^2}{abc} - r_0^2 < 0.$$

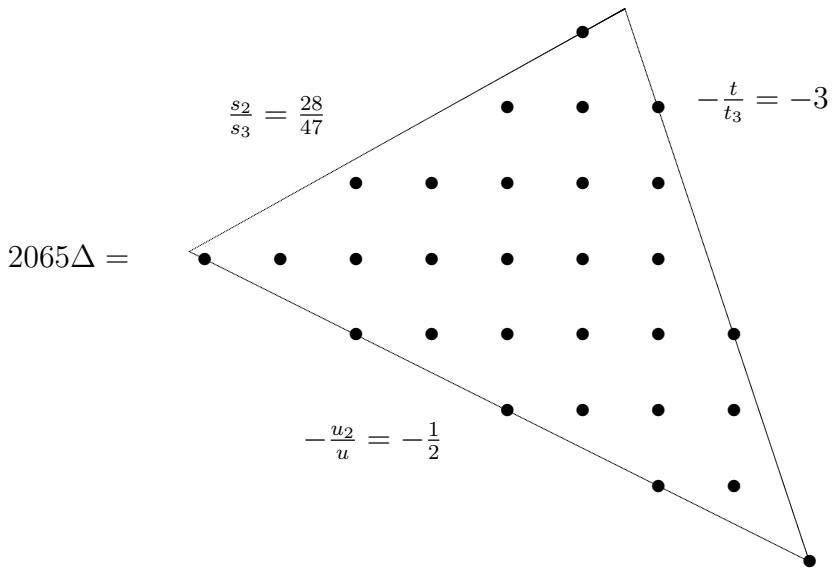
Proposition 2.6. *Let K be a field of characteristic 0.*

- (1) *There exists a negative curve $f \in [\mathfrak{p}_K(5, 103, 169)]_{2065}^{(7)}$, that is, there exists an irreducible polynomial $f \in [\mathfrak{p}_K(5, 103, 169)]_{2065}^{(7)}$. (Here $2065/7 < \sqrt{5 \cdot 103 \cdot 169}$.) The coefficient of x^{413} in f is not zero.*
- (2) *Furthermore assume that K is algebraically closed. Let C be the proper transform of $V_+(f)$ where f is the negative curve as in (1). Then C is isomorphic to \mathbb{P}_K^1 .*

Proof. We shall prove (1). Remark that

$$\mathfrak{p}_K(5, 103, 169) = I_2 \begin{pmatrix} x^{28} & y & z \\ y^2 & z & x^{47} \end{pmatrix},$$

where $I_2(\)$ stands for the ideal generated by the 2-minors of the given matrix. Consider the triangle 2065Δ , where Δ is the triangle defined as in (2.2).



(2.10)

Here recall

$$S_{2065} = K(2065\Delta \cap \mathbb{Z}^2)T^{2065}$$

as in (2.3), where $K(2065\Delta \cap \mathbb{Z}^2)$ is the K -vector space spanned by $\{v^\alpha w^\beta \mid (\alpha, \beta) \in 2065\Delta \cap \mathbb{Z}^2\}$. The bottom lattice point corresponds to x^{413} . The top lattice point corresponds to y^2z^{11} . Here remark

$$[\mathfrak{p}_K(5, 103, 169)^{(7)}]_{2065} = (K(2065\Delta \cap \mathbb{Z}^2) \cap \mathfrak{q}^7) T^{2065}.$$

Counting from the top, the numbers of lattice points in each row are 1, 3, 5, 7, 6, 4, 2, 1. We know that, by Proposition 2.5, there exists a polynomial f such that

$$[\mathfrak{p}_K(5, 103, 169)^{(7)}]_{2065} = Kf,$$

where the coefficients of x^{413} and y^2z^{11} in f are both non-zero. (First we set L to be the line containing 7 lattice points in the fourth row from the top and apply Proposition 2.5. Next, we set L to be the line containing 6 lattice points in the fifth row from the top, and so on.) Hence f is not divided by x, y and z . Put $f = \tilde{f}T^{2065}$, where

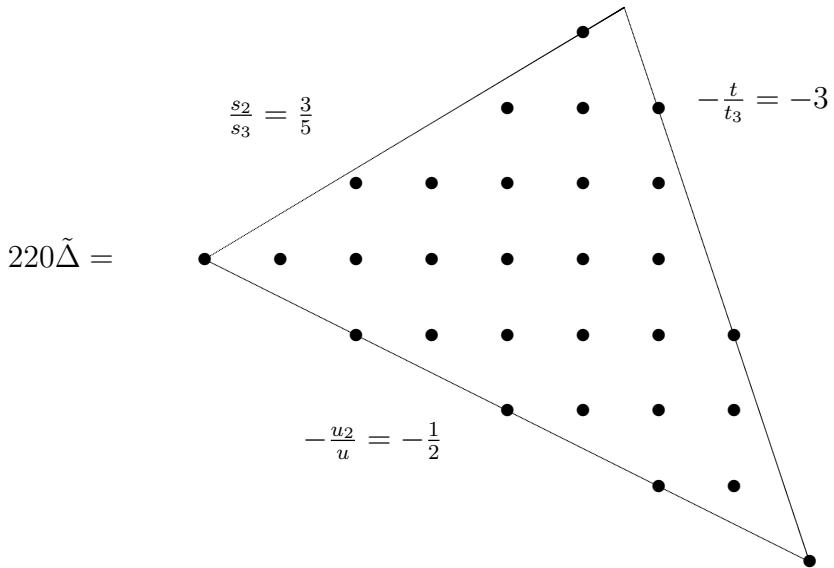
$$\tilde{f} \in K(2065\Delta \cap \mathbb{Z}^2) \cap \mathfrak{q}^7 = \bigoplus_{(\alpha, \beta) \in 2065\Delta \cap \mathbb{Z}^2} K v^\alpha w^\beta \cap \mathfrak{q}^7$$

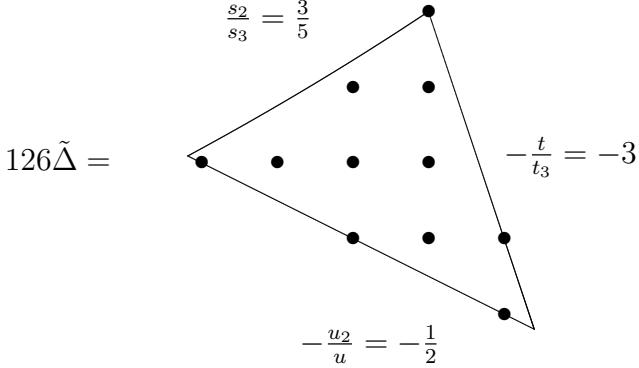
as in (2.3). It is enough to show that \tilde{f} is irreducible in $K[v^{\pm 1}, w^{\pm 1}]$.

Here consider

$$\mathfrak{p}_K(5, 11, 18) = I_2 \begin{pmatrix} x^3 & y & z \\ y^2 & z & x^5 \end{pmatrix}.$$

Let $\tilde{\Delta}$ be the triangle as in (2.2) in the case $(a, b, c) = (5, 11, 18)$. Then $220\tilde{\Delta}$ and $126\tilde{\Delta}$ are as follows:





The bottom lattice point in $220\tilde{\Delta}$ corresponds to x^{44} . The bottom (resp. top, left) lattice point in $126\tilde{\Delta}$ corresponds to xy^{11} (resp. z^7 , $x^{23}y$). We know that, by Proposition 2.5, there exists a polynomial g_1 such that

$$[\mathfrak{p}_K(5, 11, 18)^{(7)}]_{220} = Kg_1,$$

where the coefficient of x^{44} in g_1 is non-zero. Put $g_1 = \tilde{g}_1 T^{220}$, where

$$\tilde{g}_1 \in \bigoplus_{(\alpha, \beta) \in 220\tilde{\Delta} \cap \mathbb{Z}^2} Kv^\alpha w^\beta \cap \mathfrak{q}^7$$

as in (2.3). Then there exists a unit ϵ in $K[v^{\pm 1}, w^{\pm 1}]$ such that $\tilde{f} = \epsilon \tilde{g}_1$. By Proposition 2.5, there exists a polynomial g_2 such that

$$[\mathfrak{p}_K(5, 11, 18)^{(4)}]_{126} = Kg_2,$$

where the coefficients of xy^{11} , z^7 and $x^{23}y$ in g_2 are non-zero. Therefore g_2 is not divisible by x , y , z . Put $g_2 = \tilde{g}_2 T^{126}$, where $\tilde{g}_2 \in K(126\Delta \cap \mathbb{Z}^2) \cap \mathfrak{q}^4$. Then \tilde{g}_2 is irreducible by Lemma 2.3 in [7] since the coefficients of the top and left lattice points are not 0. Hence g_2 is irreducible. Since the coefficient of x^{44} in g_1 is non-zero, the sequence g_1, g_2 is S -regular. Since

$$\frac{220}{7} \cdot \frac{126}{4} = 5 \cdot 11 \cdot 18,$$

g_1, g_2 satisfies Huneke's condition. Since

$$\frac{220}{7} < \frac{126}{4},$$

g_1 is congruence to a power of the negative curve in the case $(a, b, c) = (5, 11, 18)$ by Remark 2.4 (6). Since 220 is not a multiple of 7, there does not exist a negative curve with $r_0 = 1$. Therefore we know that g_1 itself is a negative curve in the case $(a, b, c) = (5, 11, 18)$. Hence g_1 and \tilde{g}_1 are irreducible. Then \tilde{f} is irreducible. Therefore f is a negative curve in the case $(a, b, c) = (5, 103, 169)$.

Next we shall prove (2). Since K is algebraically closed, it is enough to show $H^1(\mathcal{O}_C) = 0$. By the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-C) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

we have an isomorphism

$$H^1(\mathcal{O}_C) \simeq H^2(\mathcal{O}_Y(-C))$$

since $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$ by Leray's spectral sequence. By the Serre duality, we obtain

$$H^2(\mathcal{O}_Y(-C)) = H^0(\mathcal{O}_Y(C + K_Y)).$$

Let H be the pullback of $\mathcal{O}_X(1)$. Then H and E are minimal generators of $\text{Cl}(Y)$. Then C is linearly equivalent to $d_0H - r_0E = 2065H - 7E$. Since K_Y is linearly equivalent to $(-5 - 103 - 169)H + E$, we have $\mathcal{O}_Y(C + K_Y) \simeq \mathcal{O}_Y(1788H - 6E)$. Hence

$$H^0(\mathcal{O}_Y(1788H - 6E)) = K(1788\Delta \cap \mathbb{Z}^2) \cap \mathfrak{q}^6.$$

Here remark that $1788\Delta \cap \mathbb{Z}^2$ corresponds to the lattice points in the interior of 2065Δ in (2.10). Counting from the top, the numbers of lattice points in each row in the interior of 2065Δ are 2, 5, 6, 4, 3, 1. By Proposition 2.5, we obtain

$$K(1788\Delta \cap \mathbb{Z}^2) \cap \mathfrak{q}^6 = 0.$$

We have completed the proof of Proposition 2.6. \square

3. HUNEKE'S CONDITION

Let a, b, c be pairwise coprime positive integers. Remember $X = \text{Proj } S$ and Y is the blow-up of X at $\mathfrak{p}_K(a, b, c)$.

In this section, we shall prove the following theorem:

Theorem 3.1. *Let K be an algebraically closed field of characteristic 0. Let a, b, c be pairwise coprime positive integers. Assume that there exists a negative curve $f \in [\mathfrak{p}_K(a, b, c)^{(r_0)}]_{d_0}$ with $d_0/r_0 < \sqrt{abc}$. Let d_2 and r_2 be positive integers satisfying the following four conditions*

- (a) $\frac{d_0}{r_0} \cdot \frac{d_2}{r_2} = abc$.
- (b) $f \notin (y, z)S$ or a divides d_2 .
- (c) $f \notin (z, x)S$ or b divides d_2 .
- (d) $f \notin (x, y)S$ or c divides d_2 .

Assume the following three conditions:

- (1) $R_s(\mathfrak{p}_K(a, b, c))$ is Noetherian.
- (2) Let C be the proper transform of $V_+(f)$. Then C is isomorphic to \mathbb{P}_K^1 .
- (3) There exists a positive integer ℓ such that $H^1(\mathcal{O}_Y(d_2H - r_2E - \ell C)) = 0$.

Then there exists $g_2 \in [\mathfrak{p}_K(a, b, c)^{(r_2)}]_{d_2}$ such that f, g_2 satisfies Huneke's condition.

Remark 3.2. We can describe Theorem 3.1 using only algebraic language. The above condition (2) is equivalent to $[\mathfrak{p}_K(a, b, c)^{(r_0-1)}]_{d_0-a-b-c} = 0$ as in the proof of Proposition 2.6 (2). For the condition (3), we know $H^1(\mathcal{O}_Y(d_2H - r_2E - \ell C)) \neq 0$ if $r_2 - \ell r_0 \leq -2$, and

$$H^1(\mathcal{O}_Y(d_2H - r_2E - \ell C)) = \begin{cases} 0 & r_2 - \ell r_0 = -1, 0 \\ H_{\mathfrak{m}}^2(\mathfrak{p}_K(a, b, c)^{(r_2-\ell r_0)})_{d_2-\ell d_0} & r_2 - \ell r_0 > 0, \end{cases}$$

where $\mathfrak{m} = (x, y, z)S$ by Leray's spectral sequence.

The author has two different proofs of the above theorem. The first proof uses the technique of reduction mod p , and this method is employed in [8] and [12]. The second proof examines the transition functions of line bundles; this approach is used in [11]. We shall present the second proof below.

Before starting to prove the above theorem, we need to prove the following easy lemma:

Lemma 3.3. *Let K be a field of characteristic 0. Let m be a positive integer. Let A be a K -algebra and I be a nilpotent ideal of A .*

Let $\pi : A \rightarrow A/I$ be the natural surjective ring homomorphism. Assume that $s \in A$ and $r \in (A/I)^\times$ satisfy $r^m = \pi(s)$.

Then there uniquely exists $u \in A$ such that $\pi(u) = r$ and $u^m = s$.

Proof. It is enough to show it in the case where $I^2 = 0$. Take $u' \in A^\times$ such that $\pi(u') = r$. Suppose $\gamma \in I$. Then $\pi(u' + \gamma) = r$ is satisfied. We have

$$\pi((u' + \gamma)^m) = \pi(u')^m = r^m = \pi(s).$$

Therefore there exists $\delta \in I$ such that

$$\delta = (u' + \gamma)^m - s = ((u')^m - s) + m(u')^{m-1}\gamma.$$

Here, since $\pi((u')^m - s) = \pi(u')^m - \pi(s) = 0$, we know $(u')^m - s \in I$. If $\gamma = -m^{-1}((u')^{-1})^{m-1}((u')^m - s)$, we have $\delta = 0$. Therefore we obtain

$$(u' - m^{-1}((u')^{-1})^{m-1}((u')^m - s))^m = s.$$

□

Proof of Theorem 3.1. If g_1, g_2 satisfy Huneke's condition, then we may assume that one of g_1 and g_2 is f by Remark 2.4 (7). Assume that there exists $g \in [\mathfrak{p}_K(a, b, c)^{(r)}]_d$ such that f, g satisfies Huneke's condition. Then we have

$$\frac{d_0}{r_0} \cdot \frac{d}{r} = abc.$$

Therefore we have $d/r = d_2/r_2$ by (a). Replacing r, d by suitable multiples, we may assume that there exists a positive integer m such that $r = mr_2$ and $d = md_2$.

We can take finite number of affine open subsets U_1, \dots, U_n of Y satisfying

- $\{C \cap U_i\}_{i=1}^n$ is an affine open covering of C ,
- $\mathcal{O}_Y(d_2H - r_2E)|_{U_i} \simeq \mathcal{O}_{U_i}$ for $i = 1, 2, \dots, n$,

for the following reasons: The singular points of Y come from those of $X = \text{Proj } S$. The singular points of X are contained in $\{V_+(x, y), V_+(y, z), V_+(z, x)\}$. If $f \notin (y, z)S$, then C does not pass through $V_+(y, z)$. Suppose $f \in (y, z)S$. Since a divides d_2 , $\mathcal{O}_Y(d_2H - r_2E)$ is locally free at $V_+(y, z)$. Thus $\mathcal{O}_Y(d_2H - r_2E)$ is locally free near C .

The divisorial sheaf $\mathcal{O}_Y(d_2H - r_2E)$ is a line bundle on $U := \cup_i U_i$. Let

$$\{t_{ij} \in \Gamma(\mathcal{O}_Y(U_i \cap U_j))^\times\}_{i,j}$$

be the transition function of $\mathcal{O}_Y(d_2H - r_2E)|_U$.

Let D_2 be the proper transform of $V_+(y)$. Since D_2 is an effective divisor such that $D_2 \sim md_2H - mr_2E$ and $D_2 \cap C = \emptyset$, we obtain $\mathcal{O}_Y(md_2H - mr_2E)|_{\ell C} \simeq \mathcal{O}_{\ell C}$ for any $\ell > 0$. Here ℓC is the closed subscheme defined by the ideal sheaf $\mathcal{O}_Y(-\ell C)$. Therefore there exists $s_i \in \Gamma(\mathcal{O}_Y(U_i \cap \ell C))^\times$ for $i = 1, 2, \dots, n$ such that

$$s_i = (t_{ij}|_{\ell C})^m s_j$$

in $\Gamma(\mathcal{O}_Y(U_i \cap U_j \cap \ell C))^\times$ for any i, j . Therefore we obtain

$$s_i|_C = (t_{ij}|_C)^m s_j|_C$$

in $\Gamma(\mathcal{O}_Y(U_i \cap U_j \cap C))^\times$ for any i, j .

Since $(d_2H - r_2E).C = 0$ and $C \simeq \mathbb{P}_K^1$, we have $\mathcal{O}_Y(d_2H - r_2E)|_C \simeq \mathcal{O}_C$. Therefore there exists $r_i \in \Gamma(\mathcal{O}_Y(U_i \cap C))^\times$ for $i = 1, 2, \dots, n$ such that

$$r_i = (t_{ij}|_C)r_j$$

in $\Gamma(\mathcal{O}_Y(U_i \cap U_j \cap C))^\times$ for any i, j . Hence we obtain

$$r_i^m = (t_{ij}|_C)^m r_j^m$$

in $\Gamma(\mathcal{O}_Y(U_i \cap U_j \cap C))^\times$ for any i, j . Since $\mathcal{O}_Y(md_2H - mr_2E)|_C \simeq \mathcal{O}_C$ and $\Gamma(\mathcal{O}_C) = K$, there exists $q \in K^\times$ such that

$$qr_i^m = s_i|_C$$

in $\Gamma(\mathcal{O}_Y(U_i \cap C))^\times$ for $i = 1, 2, \dots, n$. Replacing r_i by $q^{1/m}r_i$, we may assume $q = 1$. Then, by Lemma 3.3, there exists $u_i \in \Gamma(\mathcal{O}_Y(U_i \cap \ell C))^\times$ for $i = 1, 2, \dots, n$ such that $u_i^m = s_i$ and $u_i|_C = r_i$. By the uniqueness in Lemma 3.3, $u_i = (t_{ij}|_{\ell C})u_j$ is satisfied for any i, j . Thus we know

$$(3.1) \quad \mathcal{O}_Y(d_2H - r_2E)|_{\ell C} \simeq \mathcal{O}_{\ell C}$$

for any $\ell > 0$.

Suppose that ℓ is a positive integer satisfying the condition (3) in Theorem 3.1. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-\ell C) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{\ell C} \longrightarrow 0.$$

Since $\mathcal{O}_Y(d_2H - r_2E)$ is a line bundle near C , we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(d_2H - r_2E - \ell C) \longrightarrow \mathcal{O}_Y(d_2H - r_2E) \longrightarrow \mathcal{O}_Y(d_2H - r_2E)|_{\ell C} \longrightarrow 0.$$

Then we obtain an exact sequence

$$H^0(\mathcal{O}_Y(d_2H - r_2E)) \longrightarrow H^0(\mathcal{O}_Y(d_2H - r_2E)|_{\ell C}) \longrightarrow H^1(\mathcal{O}_Y(d_2H - r_2E - \ell C)).$$

By the condition (3) in Theorem 3.1, we know $H^1(\mathcal{O}_Y(d_2H - r_2E - \ell C)) = 0$. Since $H^0(\mathcal{O}_Y(d_2H - r_2E)) \rightarrow H^0(\mathcal{O}_Y(d_2H - r_2E)|_{\ell C}) = H^0(\mathcal{O}_{\ell C}) \rightarrow H^0(\mathcal{O}_C) = K$ by (3.1), there exists a section in $H^0(\mathcal{O}_Y(d_2H - r_2E))$ which does not vanish at each point of C . Therefore there exists an effective divisor D such that $D \sim d_2H - r_2E$ and $C \cap D = \emptyset$. The defining equation of D satisfies our requirement. \square

4. THE PROOF OF INFINITE GENERATION

In this section, we shall prove Theorem 1.1.

We shall prove Theorem 1.1 by contradiction. Assume that $R_s(\mathfrak{p}_K(5, 103, 169))$ is Noetherian, where K is an algebraically closed field of characteristic 0.

There exists a negative curve f as in Proposition 2.6 (1) with $f \notin (y, z)S$. Therefore $d_2 = 103 \cdot 169$ and $r_2 = 59$ satisfy the conditions (a), (b), (c), (d) in Theorem 3.1. The condition (2) in Theorem 3.1 is satisfied by Proposition 2.6 (2). We shall prove that the condition (3) in Theorem 3.1 is satisfied with $\ell = 8$. Here we have

$$d_2H - r_2E - \ell C = 103 \cdot 169H - 59E - 8(2065H - 7E) = 887H - 3E.$$

$$887\Delta = \frac{s_2}{s_3} = \frac{28}{47} \quad -\frac{t}{t_3} = -3$$

$$-\frac{u_2}{u} = -\frac{1}{2}$$

We know $\dim_K S_{887} = 6$ and $\dim_K [\mathfrak{p}_K(5, 103, 169)^{(3)}]_{887} = 0$ by Proposition 2.5. Therefore we obtain $H^1(\mathcal{O}_Y(887H - 3E)) = 0$.¹ Hence, if $R_s(\mathfrak{p}_K(5, 103, 169))$ is Noetherian where K is an algebraically closed field of characteristic 0, we know that there exists $g_2 \in [\mathfrak{p}_K(5, 103, 169)^{(59)}]_{17407}$ such that f, g_2 satisfies Huneke's condition by Theorem 3.1.

Let \mathbb{F}_2 be the prime field of characteristic 2. It is easy to see

$$\dim_K[\mathfrak{p}_K(5, 103, 169)^{(59)}]_{17407} = \dim_{\mathbb{Q}}[\mathfrak{p}_{\mathbb{Q}}(5, 103, 169)^{(59)}]_{17407} \leq \dim_{\mathbb{F}_2}[\mathfrak{p}_{\mathbb{F}_2}(5, 103, 169)^{(59)}]_{17407}.$$

Then the infinite generation of $R_s(\mathfrak{p}_K(5, 103, 169))$ immediately comes from the following lemma:

¹For pairwise coprime positive integers a, b, c , a field K , and positive integers r, d , we have

$$\frac{r(r+1)}{2} = \dim_K S_d - \dim_K H^0(\mathcal{O}_Y(dH - rE)) + \dim_K H^1(\mathcal{O}_Y(dH - rE)).$$

Lemma 4.1. $[\mathfrak{p}_{\mathbb{F}_2}(5, 103, 169)]_{17407}^{(59)} = 0$.

The calculations in the proof of the above lemma were carried out using a computer; however, the author also verified them by hand without using a computer.

Proof. We assume that the base field is \mathbb{F}_2 .

Consider the following homogeneous elements of the symbolic powers of $\mathfrak{p} = \mathfrak{p}_{\mathbb{F}_2}(5, 103, 169)$:

$$\begin{aligned}
A01 &= y^3 - x^{28}z \in [\mathfrak{p}]_{309} \\
B01 &= z^2 - x^{47}y \in [\mathfrak{p}]_{338} \\
C01 &= y^2z - x^{75} \in [\mathfrak{p}]_{375} \\
D02 &= (yA01B01 - C01^2)/x^{28} \\
&= yz^3 + x(\dots) \in [\mathfrak{p}^{(2)}]_{610} \\
D03 &= (C01D02 - A01B01^2)/x^{19} \\
&= y^7z + x(\dots) \in [\mathfrak{p}^{(3)}]_{890} \\
D04 &= (B01D03 - A01^2D02)/x^9 \\
&= z^7 + x(\dots) \in [\mathfrak{p}^{(4)}]_{1183} \\
D07 &= (D02^2D03 - A01^3D04)/x^9 \\
&= y^2z^{11} + x(\dots) \in [\mathfrak{p}^{(7)}]_{2065} \\
D09 &= (A01^2D07 - D02D03D04)/x \\
&= y^{26} + x(\dots) \in [\mathfrak{p}^{(9)}]_{2678} \\
D15 &= (D03D04^3 - A01D07^2)/x^2 \\
&= y^{43} + x(\dots) \in [\mathfrak{p}^{(15)}]_{4429} \\
D25 &= (D02D03^3D07^2 - D04^4D09)/x^2 \\
&= y^{62}z^6 + x(\dots) \in [\mathfrak{p}^{(25)}]_{7400} \\
D29 &= (D02D04^3D15 - D03^2D07^2D09)/x^2 \\
&= y^{80}z^2 + x(\dots) \in [\mathfrak{p}^{(29)}]_{8578} \\
D32 &= (D03^3D04^2D15 - D07D25)/x \\
&= y^{82}z^6 + x(\dots) \in [\mathfrak{p}^{(32)}]_{9460} \\
D37 &= (D03^5D07D15 - D04^2D29)/x \\
&= y^{98}z^5 + x(\dots) \in [\mathfrak{p}^{(37)}]_{10939} \\
D41 &= (y^2A01^2D04D07^5 - zB01D02D03D07^5 - x^9yB01^2D04D07^5 - x^{10}yB01D04^6D07D09 \\
&\quad - x^{14}yD02^2D37 - x^{15}yA01^2D09D15^2 - x^{38}D04^8D09 - x^{39}D02D04^6D15 - x^{42}D04D37)/x^{43} \\
&= y^{116}z + x(\dots) \in [\mathfrak{p}^{(41)}]_{12117}
\end{aligned}$$

$$\begin{aligned}
D43 = & (y^2 A01^2 D04 D07^4 D09 - z B01 D02 D03 D07^4 D09 - x^9 y B01^2 D04 D07^4 D09 \\
& - x^{10} z A01 D02 D04 D07^3 D15 - x^{11} y A01 B01 D02 D07^2 D25 - x^{12} y A01 B01 D02 D07 D32 - \\
& x^{13} y B01 D02 D03 D37 - x^{14} y A01^4 D09 D15^2 - x^{22} y D02 D04 D37 - x^{27} y A01 D03 D04 D07^5 \\
& - x^{29} y D03 D04 D07^3 D15 - x^{39} D02^2 D07^2 D25 - x^{40} D02^2 D07 D32 - x^{41} A01^2 D04 D37 \\
& - x^{42} A01^2 D41 - x^{45} D02 D04^5 D07^3 - x^{46} D02 D03^2 D07^5 - x^{47} D04^7 D15 - \\
& x^{48} D03^2 D04^2 D07^2 D15) / x^{51} \\
= & y^{115} z^5 + x(\dots) \in [\mathfrak{p}^{(43)}]_{12690}
\end{aligned}$$

$$\begin{aligned}
D49 = & (y B01 D04^4 D07 D25 - z D03^2 D07^4 D15 - x y B01 D04^4 D32 - x^4 y A01^2 D03^2 D41 \\
& - x^7 y A01^3 D04 D07^6 - x^9 y A01^2 D04 D07^4 D15 - x^{13} y D03^2 D43 - x^{16} y D07^7 \\
& - x^{18} B01^2 D04 D07^4 D15 - x^{19} B01 D02 D07^3 D25 - x^{20} B01 D02 D07^2 D32 \\
& - x^{21} A01 D02^2 D07 D37 - x^{22} B01 D02 D03 D43 - x^{23} A01^4 D15^3 - x^{28} D04^6 D25 \\
& - x^{29} D04^5 D29 - x^{30} D04^3 D37 - x^{31} D02 D04 D43) / x^{32} \\
= & y^{134} z^4 + x(\dots) \in [\mathfrak{p}^{(49)}]_{14478}
\end{aligned}$$

$$\begin{aligned}
D53 = & (y B01 D04^5 D07 D25 - z D03^2 D04 D07^4 D15 - x y B01 D04^5 D32 - x^4 y D02^2 D49 - \\
& x^6 y D02 D04^{11} D07 - x^7 y B01 D03 D07^7 - x^{10} y D03 D04 D07^3 D25 - x^{13} y D03^2 D04 D43 \\
& - x^{16} y D04 D07^7 - x^{18} B01^2 D04^2 D07^4 D15 - x^{19} B01 D02 D04 D07^3 D25 \\
& - x^{20} B01 D02 D04 D07^2 D32 - x^{21} A01 B01 D07^2 D37 - x^{25} B01 D04^6 D07^4 \\
& - x^{28} D04^7 D25 - x^{29} D04^6 D29 - x^{32} D04 D49) / x^{33} \\
= & y^{152} + x(\dots) \in [\mathfrak{p}^{(53)}]_{15656}
\end{aligned}$$

$$\begin{aligned}
D56 = & (y B01 D04^4 D07^2 D25 - z D03^2 D07^5 D15 - x y B01 D04^4 D07 D32 - x^4 y A01 D02 D04 D49 \\
& - x^5 y A01 D02 D53 - x^9 y A01^2 D04 D07^5 D15 - x^{13} y D03^2 D07 D43 - x^{16} y D07^8 \\
& - x^{18} B01^2 D04 D07^5 D15 - x^{19} B01 D02 D07^4 D25 - x^{20} B01 D02 D07^3 D32 \\
& - x^{21} A01 D02^2 D07^2 D37 - x^{22} B01 D02 D03 D07 D43 - x^{23} A01^3 D04 D49 - x^{24} A01^3 D53 \\
& - x^{28} D04^6 D07 D25 - x^{29} D04^6 D32) / x^{33} \\
= & y^{154} z^4 + x(\dots) \in [\mathfrak{p}^{(56)}]_{16538}
\end{aligned}$$

Let I be a homogeneous ideal contained in $\mathfrak{p}^{(59)}$ generated by the following 105 elements:

$B01^3 D04^{14},$	$B01 D02 D04^{14},$	$D04^{13} D07,$	$B01 D04^{11} D07^2,$	$B01^2 D04^9 D07^3,$
$D02 D04^9 D07^3,$	$B01 D02 D04^7 D07^4,$	$D04^6 D07^5,$	$B01 D04^4 D07^6,$	$B01^2 D04^2 D07^7,$
$D02 D04^2 D07^7,$	$B01 D02 D07^8,$	$C01 D02 D07^8,$	$A01 D02 D07^8,$	$B01 A01^2 D07^8,$
$D03 D07^8,$	$A01^3 D07^8,$	$D02 D04^{12} D09,$	$D03^2 D04 D07^7,$	$D04^9 D07^2 D09,$
$B01 D04^7 D07^3 D09,$	$D03^3 D04^2 D07^6,$	$D02 D04^5 D07^4 D09,$	$B01 D02 D04^3 D07^5 D09,$	$D04^2 D07^6 D09,$
$B01 D07^7 D09,$	$C01 D07^7 D09,$	$D04^{11} D15,$	$B01 D04^9 D07 D15,$	$B01^2 D04^7 D07^2 D15,$
$D02 D04^7 D07^2 D15,$	$B01 D02 D04^5 D07^3 D15,$	$D04^4 D07^4 D15,$	$B01 D04^2 D07^5 D15,$	$B01^2 D07^6 D15,$
$D02 D07^6 D15,$	$A01 B01 D07^6 D15,$	$A01 C01 D07^6 D15,$	$A01^2 D07^6 D15,$	$D02 D04^8 D25,$
$B01 D02 D04^6 D07 D25,$	$D04^5 D07^2 D25,$	$B01 D04^3 D07^3 D25,$	$B01^2 D04 D07^4 D25,$	$D02 D04 D07^4 D25,$
$A01 B01 D04 D07^4 D25,$	$D03^3 D07^5 D15,$	$A01^2 D04 D07^4 D25,$	$B01 D02 D03 D07^4 D25,$	$D07^5 D09 D15,$
$D02 D04^7 D29,$	$B01 D02 D04^6 D32,$	$D04^5 D07 D32,$	$B01 D04^3 D07^2 D32,$	$B01^2 D04 D07^3 D32,$
$D02 D04 D07^3 D32,$	$A01 B01 D04 D07^3 D32,$	$D04^2 D07^3 D15^2,$	$A01^2 D04 D07^3 D32,$	$B01 D02 D03 D07^3 D32,$
$A01 D07^4 D15^2,$	$D02 D04^5 D37,$	$B01 D02 D04^3 D07 D37,$	$D04^2 D07^2 D37,$	$B01 D07^3 D37,$
$C01 D07^3 D37,$	$A01 D07^3 D37,$	$B01 D03 D04 D07^2 D37,$	$C01 D03 D04 D07^2 D37,$	$A01 D03 D04 D07^2 D37,$
$A01^4 D04 D07^2 D37,$	$D04^4 D43,$	$B01 D04^2 D07 D43,$	$B01^2 D07^2 D43,$	$D02 D07^2 D43,$
$A01 B01 D07^2 D43,$	$A01 C01 D07^2 D43,$	$A01^2 D07^2 D43,$	$A01 B01 D03 D04 D07 D43,$	$A01 C01 D03 D04 D07 D43,$
$A01^2 D03 D04 D07 D43,$	$A01^5 D04 D07 D43,$	$D07^2 D15^3,$	$D02 D04^2 D49,$	$B01 D02 D07 D49,$
$D03^3 D07 D43,$	$A01 D02 D07 D49,$	$A01^2 B01 D07 D49,$	$D03 D07 D49,$	$A01^3 D07 D49,$
$A01^2 B01 D03 D04 D49,$	$D03^2 D04 D49,$	$A01^3 D03 D04 D49,$	$B01^2 D04 D53,$	$D02 D04 D53,$
$B01 D02 D56,$	$C01 D02 D56,$	$A01 D02 D56,$	$A01^2 B01 D56,$	$D03 D56,$
$A01^3 D56,$	$A01^2 B01 D03 D53,$	$D03^2 D53,$	$A01^3 D03 D53,$	$A01^6 D53$

Let L be a monomial ideal of $k[y, z]$ as follows:

$$L := \left(\begin{array}{ccccc} z^{104}, & yz^{103}, & y^2 z^{102}, & y^4 z^{101}, & y^6 z^{100}, \\ y^7 z^{99}, & y^9 z^{98}, & y^{10} z^{97}, & y^{12} z^{96}, & y^{14} z^{95}, \\ y^{15} z^{94}, & y^{17} z^{93}, & y^{19} z^{92}, & y^{20} z^{91}, & y^{22} z^{90}, \\ y^{23} z^{89}, & y^{25} z^{88}, & y^{27} z^{87}, & y^{28} z^{86}, & y^{30} z^{85}, \\ y^{32} z^{84}, & y^{33} z^{83}, & y^{35} z^{82}, & y^{37} z^{81}, & y^{38} z^{80}, \\ y^{40} z^{79}, & y^{42} z^{78}, & y^{43} z^{77}, & y^{45} z^{76}, & y^{47} z^{75}, \\ y^{48} z^{74}, & y^{50} z^{73}, & y^{51} z^{72}, & y^{53} z^{71}, & y^{55} z^{70}, \\ y^{56} z^{69}, & y^{58} z^{68}, & y^{60} z^{67}, & y^{61} z^{66}, & y^{63} z^{65}, \\ y^{65} z^{64}, & y^{66} z^{63}, & y^{68} z^{62}, & y^{70} z^{61}, & y^{71} z^{60}, \\ y^{73} z^{59}, & y^{74} z^{58}, & y^{76} z^{57}, & y^{78} z^{56}, & y^{79} z^{55}, \\ y^{81} z^{54}, & y^{83} z^{53}, & y^{84} z^{52}, & y^{86} z^{51}, & y^{88} z^{50}, \\ y^{89} z^{49}, & y^{91} z^{48}, & y^{92} z^{47}, & y^{94} z^{46}, & y^{96} z^{45}, \\ y^{97} z^{44}, & y^{99} z^{43}, & y^{101} z^{42}, & y^{102} z^{41}, & y^{104} z^{40}, \\ y^{106} z^{39}, & y^{107} z^{38}, & y^{109} z^{37}, & y^{111} z^{36}, & y^{112} z^{35}, \\ y^{114} z^{34}, & y^{115} z^{33}, & y^{117} z^{32}, & y^{119} z^{31}, & y^{120} z^{30}, \\ y^{122} z^{29}, & y^{124} z^{28}, & y^{125} z^{27}, & y^{127} z^{26}, & y^{129} z^{25}, \\ y^{130} z^{24}, & y^{132} z^{23}, & y^{133} z^{22}, & y^{135} z^{21}, & y^{137} z^{20}, \\ y^{138} z^{19}, & y^{140} z^{18}, & y^{142} z^{17}, & y^{143} z^{16}, & y^{145} z^{15}, \\ y^{147} z^{14}, & y^{148} z^{13}, & y^{150} z^{12}, & y^{152} z^{11}, & y^{153} z^{10}, \\ y^{155} z^9, & y^{157} z^8, & y^{158} z^7, & y^{160} z^6, & y^{161} z^5, \\ y^{163} z^4, & y^{165} z^3, & y^{166} z^2, & y^{168} z, & y^{170} \end{array} \right)$$

Let D be the i, j th entry of the 21×5 matrix of generators of I . If $y^k z^l$ is the i, j th entry of the 21×5 matrix of generators of L , then we have

$$D \equiv y^k z^l \pmod{x}.$$

Thus we have $(L, x)S = I + xS$.

Furthermore we have

$$\ell(S/I+xS) = \ell(k[y, z]/L) = 8850 = \left(\frac{59 \times 60}{2} \right) \times 5 = e((x), S/\mathfrak{p}^{(59)}) = \ell(S/\mathfrak{p}^{(59)}+xS).$$

The proof of the second equality in the above equation requires some calculation. Then we have $I + xS = \mathfrak{p}^{(59)} + xS$. Since $\mathfrak{p}^{(59)} = I + (xS \cap \mathfrak{p}^{(59)}) = I + x\mathfrak{p}^{(59)}$, we obtain $I = \mathfrak{p}^{(59)}$ by Nakayama's lemma.

The matrix whose entries are the degrees of the generators of I is the following:

$$\left(\begin{array}{cccccc} 17576, & 17510, & 17444, & 17481, & 17518, \\ 17452, & 17489, & 17423, & 17460, & 17497, \\ 17431, & 17468, & 17505, & 17439, & 17476, \\ 17410, & 17447, & 17484, & 17418, & 17455, \\ 17492, & 17426, & 17463, & 17500, & 17434, \\ 17471, & 17508, & 17442, & 17479, & 17516, \\ 17450, & 17487, & 17421, & 17458, & 17495, \\ 17429, & 17466, & 17503, & 17437, & 17474, \\ 17511, & 17445, & 17482, & 17519, & 17453, \\ 17490, & 17424, & 17461, & 17498, & 17432, \\ 17469, & 17506, & 17440, & 17477, & 17514, \\ 17448, & 17485, & 17419, & 17456, & 17493, \\ 17427, & 17464, & 17501, & 17435, & 17472, \\ 17509, & 17443, & 17480, & 17517, & 17451, \\ 17488, & 17422, & 17459, & 17496, & 17430, \\ 17467, & 17504, & 17438, & 17475, & 17512, \\ 17446, & 17483, & 17417, & 17454, & 17491, \\ 17425, & 17462, & 17499, & 17433, & 17470, \\ 17507, & 17441, & 17478, & 17515, & 17449, \\ 17486, & 17523, & 17457, & 17494, & 17428, \\ 17465, & 17502, & 17436, & 17473, & 17510 \end{array} \right).$$

Then we know the minimal degree of generators of I is 17410, which is the degree of $D03D07^8$.

Thus we know

$$[\mathfrak{p}^{(59)}]_{17407} = 0.$$

□

Remark 4.2. Consider the subring

$$A := S[A01t, B01t, C01t, D02t^2, D03t^3, \dots, D56t^{56}]$$

of the symbolic Rees algebra

$$R_s(\mathfrak{p}_{\mathbb{F}_2}(5, 103, 169)) = S[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3, \dots].$$

If $n \leq 59$, then we have

$$A \cap St^n = \mathfrak{p}_{\mathbb{F}_2}(5, 103, 169)^{(n)}t^n.$$

We don't need this equation in this paper.

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