INFINITELY GENERATED SYMBOLIC REES RINGS OF POSITIVE CHARACTERISTIC

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Abstract. Let X be a toric variety over a field K determined by a triangle. Let Y be the blow-up at $(1,1)$ in X. In this paper we give some criteria for finite generation of the Cox ring of Y in the case where Y has a curve C such that $C^2 \leq 0$ and $C.E = 1$ (E is the exceptional divisor). The natural surjection $\mathbb{Z}^3 \to \mathrm{Cl}(X)$ gives the ring homomorphism $K[\mathbb{Z}^3] \to K[\mathrm{Cl}(X)]$. We denote by I the kernel of the composite map $K[x,y,z] \subset K[\mathbb{Z}^3] \to K[\mathrm{Cl}(X)]$. Then Cox(Y) coincides with the extended symbolic Rees ring $R'_s(I)$. In the case where $\mathrm{Cl}(X)$ is torsion-free, this ideal I is the defining ideal of a space monomial curve.

Let Δ be the triangle (4.1) below. Then I is the ideal of $K[x,y,z]$ generated by 2-minors of the 2×3-matrix $\{\{x^7, y^2, z\}, \{y^{11}, z, x^{10}\}\}$. (In this case, there exists a curve C with $C^2 = 0$ and $C.E = 1$. This ideal I is not a prime ideal.) Applying our criteria, we prove that $R'_s(I)$ is Noetherian if and only if the characteristic of K is 2 or 3.

1. Introduction

For pairwise coprime positive integers a, b and c, let $\mathfrak{p}$ be the defining ideal of the space monomial curve (T^a, T^b, T^c) in K^3, where K is a field. The ideal $\mathfrak{p}$ is generated by at most three binomials in $K[x,y,z]$ (Herzog [13]). The symbolic Rees rings of space monomial primes are deeply studied by many authors. Huneke [14] and Cutkosky [2] developed criteria for finite generation of such rings. In 1994, Goto-Nishida-Watanabe [10] first found examples of infinitely generated symbolic Rees rings of space monomial primes. Recently, using toric geometry, González-Karu [5] found some sufficient conditions for the symbolic Rees rings of space monomial primes to be infinitely generated.

Cutkosky [2] found the geometric meaning of the symbolic Rees rings of space monomial primes. Let $\mathbb{P}(a,b,c)$ be the weighted projective surface with weight a, b, c. Let Y be the blow-up at a point in the open orbit of the toric variety $\mathbb{P}(a,b,c)$. Then the Cox ring of Y is isomorphic to the extended symbolic Rees ring of the space monomial prime $\mathfrak{p}$. Therefore the symbolic Rees ring of the space monomial prime $\mathfrak{p}$ is finitely generated if and only if the Cox ring of Y is finitely generated, that is, Y is a Mori dream space. A curve C on Y is called a negative curve if $C^2 < 0$ and C is different from the exceptional curve E. Cutkosky [2] proved that the symbolic Rees ring of the space monomial prime $\mathfrak{p}$ is finitely generated if and only if the following two conditions are satisfied:

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(1) There exists a curve C such that $C^2 \leq 0$ and $C \neq E$.[1]
(2) There exists a curve F on Y such that $C \cap F = \emptyset$.

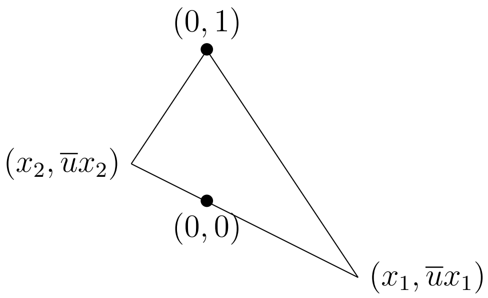
In the case of $\mathrm{ch}(K) > 0$, Cutkosky [2] proved that the symbolic Rees ring is Noetherian if there exists a negative curve. In the case of $\mathrm{ch}(K) = 0$, Inagawa-Kurano [15] developed a very simple criterion for finite generation in the case where a minimal generator of $\mathfrak{p}$ defines a negative curve C, i.e., $C.E = 1$. Examples that have a negative curve C with $C.E \geq 2$ are studied in González-AnayaGonzález-Karu [6], [7] and Kurano-Nishida [18].

The existence of negative curves is a very difficult and important problem, that is deeply related to Nagata's conjecture (Proposition 5.2 in Cutkosky-Kurano [3], Remark 2.2 (4)) and the rationality of Seshadri constant. The existence of negative curves is studied in González-AnayaGonzález-Karu [8], [9], Kurano-Matsuoka [17] and Kurano [16].

In the case of $\mathrm{ch}(K) > 0$, we do not know any example such that $R_s(\mathfrak{p})$ is infinitely generated.

In this paper, we shall discuss finite generation in a slightly broader situation than that of the symbolic Rees ring of the defining ideal of a space monomial curve. Now, we set up the situation dealt with in this paper and describe our results.

Let Δ be a triangle with three vertices $(x_2, \overline{u}x_2)$, $(x_1, \overline{u}x_1)$, $(0, 1)$

(1.1) $\Delta =$ 

where x_1 and x_2 are rational numbers such that $x_2 \leq 0 \leq x_1$, $W := x_1 - x_2 > 0$. Let $\overline{s}$, $\overline{t}$, $\overline{u}$ be the slopes of each edges, that is, $\overline{s} = \frac{\overline{u}x_2 - 1}{x_2}$ and $\overline{t} = \frac{\overline{u}x_1 - 1}{x_1}$. We assume $-\infty \leq \overline{t} \leq -1 \leq \overline{u} \leq 0 \leq \overline{s} \leq \infty$.

Let K be a field and X be the toric variety determined by Δ, that is, $X = \mathrm{Proj}\, E(\Delta)$ where

$$E(\Delta) = \bigoplus_{n \geq 0} \left(\bigoplus_{(\alpha,\beta) \in n\Delta \cap \mathbb{Z}^2} K v^{\alpha} w^{\beta} \right) T^n \subset K[v^{\pm 1}, w^{\pm 1}, T] \qquad (1.2)$$

is the Ehrhart ring of Δ. Here v, w, t are indeterminates. Let $\pi : Y \to X$ be the blow-up of X at $e = (1, 1)$, where e is the point corresponging to the prime ideal $E(\Delta) \cap (v - 1, w - 1)K[v^{\pm 1}, w^{\pm 1}, T]$.

Let E be the exceptional divisor. Let C be the proper transform of the curve of X defined by $(w - 1)T$ in $E(\Delta)$. Then C is linearly equivalent to $\pi^*\Delta - E$ and we

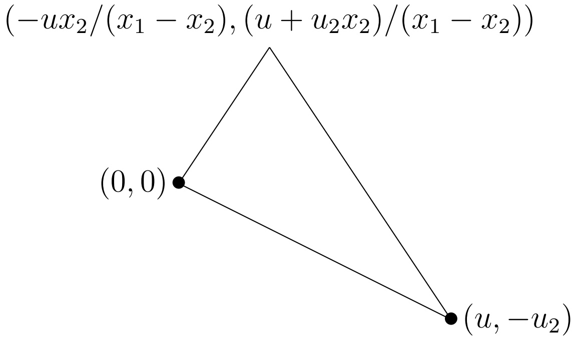
[1] If $\sqrt{abc} \notin \mathbb{Q}$, this curve satisfies $C^2 < 0$, that is, C is a negative curve.

have

$$C^2 = 2|\Delta| - 1 = W - 1.$$

Here remark that C is isomorphic to $\mathbb{P}^1_K$. Let $u_2(\geq 0)$ and $u(> 0)$ be integers such that $\overline{u} = -u_2/u$ and $(u_2, u) = 1$. Let Δ' be the triangle with three vertices $(0,0)$, $(u, -u_2)$, $(-ux_2/(x_1 - x_2), (u + u_2x_2)/(x_1 - x_2))$.

$\Delta' =$



(1.3)

Remark that the slopes of edges of Δ' are $\overline{s}$, $\overline{t}$, $\overline{u}$. We denote the Weil divisor $\pi^*\Delta' - uE$ by D. Then we have $C.D = 0$.

For a positive integer n, we think nC as a closed subscheme of Y defined by $\mathcal{O}_Y(-nC)$. We define the Cox ring of Y by

$$\mathrm{Cox}(Y) = \bigoplus_{\overline{D} \in \mathrm{Cl}(Y)} H^0(Y, \mathcal{O}_Y(D)).$$

Even if $\mathrm{Cl}(Y)$ has a torsion, we can define a ring structure on $\mathrm{Cox}(Y)$ in this case.

We shall prove the following three theorems in Section 3.

Theorem 1.1. *Let K be a field. Let Δ, Δ', W, X, Y, C, D, $\overline{s}$, $\overline{t}$, $\overline{u}$, u_2, u be as above. Assume $0 < W \leq 1$. Then the following conditions are equivalent:*

- (A0) $\mathrm{Cox}(Y)$ *is finitely generated over K.*
- (A1) *There exists a curve F on Y such that $F \cap C = \emptyset$.*
- (A2) *There exists a positive integer m such that $\mathcal{O}_Y(mD)|_{\ell C} \simeq \mathcal{O}_{\ell C}$ for any positive integer ℓ.*
- (A3) *There exists a positive integer m such that $\mathcal{O}_Y(mD)|_{muC} \simeq \mathcal{O}_{muC}$.*
- (A4) *There exists a positive integer m such that ξ^m $(\in (F_{mu})^\times)$ is written as a product of elements of ${A_{mu}}^\times$ and $\psi(B_{mu})^\times$.*

We refer the reader to Section 2 for definition of ξ, F_{mu}, A_{mu} and $\psi(B_{mu})$.

If (A1) is satisfied, F is numerically equivalent to mD for some positive integer m.

For $i = 1, 2, \ldots, u$, we put

$$m_i =^\# \{(\alpha, \beta) \in \Delta' \cap \mathbb{Z}^2 \mid \alpha = i\}.$$

Note that $m_i \geq 1$ for all $i = 1, 2, \ldots, u$. We sort the sequence m_1, m_2, $\ldots$, m_u into ascending order

$$m'_1 \leq m'_2 \leq \cdots \leq m'_u.$$

We say that Δ' satisfies the <u>EMU condition</u> if

$$m'_i \geq i$$

for $i = 1, 2, \ldots, u$.

Theorem 1.2. *Let K be a field of characteristic 0. Let Δ, Δ', W, X, Y, C, D, $\overline{s}$, $\overline{t}$, $\overline{u}$, u_2, u be as above. Assume $0 < W \leq 1$. Then the following conditions are equivalent:*

- (B0) $\mathrm{Cox}(Y)$ *is finitely generated over K.*
- (B1) $\mathcal{O}_Y(D)|_{uC} \simeq \mathcal{O}_{uC}$.
- (B2) ξ $(\in (F_u)^\times)$ *is written as a product of elements of ${A_u}^\times$ and $\psi(B_u)^\times$.*
- (B3) Δ' *satisfies the EMU condition.*

Theorem 1.3. *Let K be a field of characteristic p, where p is a prime number. Let Δ, Δ', W, X, Y, C, D, $\overline{s}$, $\overline{t}$, $\overline{u}$, u_2, u be as above.*

1. *If $0 < W < 1$, then $\mathrm{Cox}(Y)$ is finitely generated over K.*
2. *Assume $W = 1$. Let σ be the minimal positive integer such that three vertices of $\sigma\Delta$ are lattice points. Then the following conditions are equivalent:*
   - (C0) $\mathrm{Cox}(Y)$ *is finitely generated over K.*
   - (C1) *There exists a non-negative integer r such that $\mathcal{O}_Y(\sigma p^r C)|_{\sigma p^r C} \simeq \mathcal{O}_{\sigma p^r C}$.*
   - (C2) *There exists a non-negative integer r such that $\mathcal{O}_Y(-\sigma p^r C)|_{\sigma p^r C} \simeq \mathcal{O}_{\sigma p^r C}$.*
   - (C3) *There exist a non-negative integer r and a positive integer j such that j is not divided by p and $H^0(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C}) \neq 0$.*
   - (C4) *There exists a non-negative integer r such that $H^0(\mathcal{O}_Y(-\sigma p^r C)|_{\sigma p^r C}) \neq 0$.*

Here, in the case $W = 1$, σC is rationally equivalent to $(\sigma/u)D$.

We shall prove the following examples in Section 4.

Example 1.4. Let g be a rational number such that $2 \leq g \leq 3$. Let Δ be a triangle with three vertices $(g-3, \frac{3-g}{2})$, $(g-2, \frac{2-g}{2})$, $(0, 1)$. This triangle satisfies the condition in (1.1). In this example, $W = 1$ is satisfied.

1. Assume that K is a field of characteristic 0. Then $\mathrm{Cox}(Y)$ is finitely generated over K if and only if $\frac{7}{3} \leq g \leq \frac{8}{3}$.
2. Assume that K is a field of characteristic p, where p is a prime number.
   - (i) If $\frac{7}{3} \leq g \leq \frac{8}{3}$, then $\mathrm{Cox}(Y)$ is finitely generated over K.
   - (ii) Suppose $g = \frac{13}{6}$. Then $\mathrm{Cox}(Y)$ is finitely generated over K if and only if $p = 2$ or 3.

In the case $g = 13/6$ in the above example, we know that $\mathrm{Cox}(Y)$ is isomorphic to the extended symbolic Rees ring $R'_s(I)$ where I is an ideal of $K[x, y, z]$ of the form

$$I = I_2 \begin{pmatrix} x^7 & y^2 & z \\ y^{11} & z & x^{10} \end{pmatrix}.$$

Here, the above ideal is not a prime ideal.

Remark that Sannai-Tanaka [21] constructed examples of prime ideals I such that symbolic Rees rings are not finitely generated over finite fields.

Remark 1.5. Assume that Δ satisfies $W = 1$. If I is a prime ideal, then we can prove that I is the defining ideal of the space monomial curve (T^1, T^1, T^1), that is, $I = (x - y, y - z)$.

Therefore, if $W = 1$, there does not exist infinitely generated $R_s(I)$ such that I is a space monomial prime ideal.

González-AnayaGonzález-Karu [9] found examples of triangles such that Y does not have a curve C such that $C^2 \leq 0$ and $C \neq E$ in the case $\mathrm{ch}(K) = 0$. In this example, W is a square of a rational number.

2. Preliminaries

Let Δ be the triangle in (1.1). Let s_2, s_3, t_3, t, u_2, u be non-negative integers such that $\overline{s} = \frac{s_2}{s_3}$, $\overline{t} = -\frac{t}{t_3}$, $\overline{u} = -\frac{u_2}{u}$, $(s_2, s_3) = (t, t_3) = (u_2, u) = 1$. (Here we put $s_2 = 1$ and $s_3 = 0$ if $\overline{s} = \infty$. Similarly we put $t = 1$ and $t_3 = 0$ if $\overline{t} = -\infty$.)

Put $\mathbf{a} = (s_2, -s_3)$, $\mathbf{b} = (-t, -t_3)$, $\mathbf{c} = (u_2, u)$. They are normal vectors of each edges of Δ. Let a, b, c be pairwise coprime positive integers such that $a\mathbf{a} + b\mathbf{b} + c\mathbf{c} = \mathbf{o}$. Let K be a field and X be the toric variety determined by Δ, that is, $X = \operatorname{Proj} E(\Delta)$ where $E(\Delta)$ is the Ehrhart ring of Δ as in (1.2). We have the following diagram such that the horizontal sequence is exact:

$$(2.1)\qquad \begin{array}{ccccccccc} & & \mathbb{Z} & & & & & & \\ & & \uparrow & \nwarrow (a\ b\ c) & & \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} & & & \\ 0 & \longleftarrow & \mathrm{Cl}(X) & \longleftarrow & \mathbb{Z}^3 & \longleftarrow & \mathbb{Z}^2 & \longleftarrow & 0 \\ & & & & \uparrow & & & & \\ & & & & {\mathbb{N}_0}^3 & & & & \end{array}$$

Here $\mathrm{Cl}(X)$ is the divisor class group of X. Take the semigroup rings of semigroups in the above diagram.

$$\begin{array}{cccc} K[T^{\pm 1}] = K[\mathbb{Z}] & & & \\ \uparrow \phi_0 & \nwarrow \psi & & \\ K[\mathrm{Cl}(X)] & \overset{\varphi}{\longleftarrow} & K[\mathbb{Z}^3] & \\ & & \uparrow \iota & \\ & & K[{\mathbb{N}_0}^3] = K[x, y, z] & \end{array}$$

The map $\psi\iota : K[x, y, z] \to K[T^{\pm 1}]$ is given by $\psi\iota(x) = T^a$, $\psi\iota(y) = T^b$ and $\psi\iota(z) = T^c$, that is, the kernel of $\psi\iota$ is the defining ideal of the space monominal curve (T^a, T^b, T^c).

If the order of the torion subgroup of $\mathrm{Cl}(X)$ is d, then $\mathrm{Cl}(X)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$. Therefore $K[\mathrm{Cl}(X)]$ is isomorphic to $K[T^{\pm 1}, U]/(U^d - 1)$. Put $I = \mathrm{Ker}(\varphi\iota)$. Then we know

$$(2.2)\qquad I \text{ is a prime ideal} \iff \mathrm{Cl}(X) \text{ is torsion-free} \iff \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b} + \mathbb{Z}\mathbf{c} = \mathbb{Z}^2.$$

Here suppose $K = \mathbb{C}$. We define $\phi_k : \mathbb{C}[T^{\pm 1}, U]/(U^d - 1) \to \mathbb{C}[T^{\pm 1}]$ by $\phi_k(T) = T$ and $\phi_k(U) = e^{2k\pi i/d}$ for $k \in \mathbb{Z}$. Then we have

$$I = \mathrm{Ker}(\varphi\iota) = (\varphi\iota)^{-1}(0) = (\varphi\iota)^{-1}(\cap_{k=0}^{d-1}\mathrm{Ker}(\phi_k)) = \cap_{k=0}^{d-1}\mathrm{Ker}(\phi_k\varphi\iota).$$

Here remark that each $\mathrm{Ker}(\phi_k\varphi\iota)$ is a prime ideal of $\mathbb{C}[x, y, z]$ for each k.

Definition 2.1. Let A be a commutative Noetherian ring. Let J be an ideal of A with minimal prime ideals $P_1, P_2, \ldots, P_d$. We define the nth symbolic power of J by

$$J^{(n)} = \cap_{i=1}^{d}(J^n A_{P_i} \cap A).$$

We define

$$R_s(J) = \oplus_{n\geq 0} J^{(n)}T^n \subset A[T]$$

and call it the symbolic Rees ring of J. We put

$$R'_s(J) = R_s(J)[T^{-1}] \subset A[T^{\pm 1}]$$

and call it the extended symbolic Rees ring of J.

Let $\pi : Y \to X$ be the blow-up of X at $e = (1, 1)$, where e is the point corresponging to the prime ideal $E(\Delta) \cap (v - 1, w - 1)K[v^{\pm 1}, w^{\pm 1}, T]$.

Let E be the exceptional divisor of π.

Remark 2.2. (1) Let $\overline{\Delta}$ be a triangle such that three vertices are rational points. Then there exist $M \in \mathrm{GL}(2, \mathbb{Z})$, $r \in \mathbb{Q}_{>0}$ and $\mathbf{f} \in \mathbb{Q}^2$ such that $\Delta = rM\overline{\Delta} + \mathbf{f}$, where Δ is a triangle as in (1.1). For the proof, we use a method in Herzong [13]. We do not prove this result here since we do not need it in this paper.

(2) We put $t_1 = t - t_3$ and $u_1 = u - u_2$. Since $\frac{t}{t_3} \geq 1$ and $\frac{u_2}{u} \leq 1$, t_1 and u_1 are non-negative integers. Then we have

$$I = I_2\begin{pmatrix} x^{s_2} & y^{t_3} & z^{u_1} \\ y^{t_1} & z^{u_2} & x^{s_3} \end{pmatrix} = (x^{s_2+s_3} - y^{t_1}z^{u_1}, y^{t_1+t_3} - z^{u_2}x^{s_2}, z^{u_1+u_2} - x^{s_3}y^{t_3}).$$

We give an outline of the proof of it here.

We put $J = (x^{s_2+s_3} - y^{t_1}z^{u_1}, y^{t_1+t_3} - z^{u_2}x^{s_2}, z^{u_1+u_2} - x^{s_3}y^{t_3})$. We know xyz is a non-zero divisor of S/J since S/J is a 1-dimensional Cohen-Macaulay ring by Hilbert-Burch theorem. Therefore we have

$$K[{\mathbb{N}_0}^3]/J \hookrightarrow (K[{\mathbb{N}_0}^3]/J)[(xyz)^{-1}] = K[\mathbb{Z}^3]/JK[\mathbb{Z}^3].$$

Next we shall prove $K[\mathbb{Z}^3]/JK[\mathbb{Z}^3] = K[\mathrm{Cl}(X)]$. Thus J coincides with $\mathrm{Ker}(\varphi\iota)$.

(3) We know

$$\mathrm{Cox}(Y) = R'_s(I)$$

by (2.8) in [16]. It is well-known that $R'_s(I)$ is Noetherian iff so is $R_s(I)$. Therefore Y is a Mori dream space if and only if $R_s(I)$ is finitely generated over K.[2]

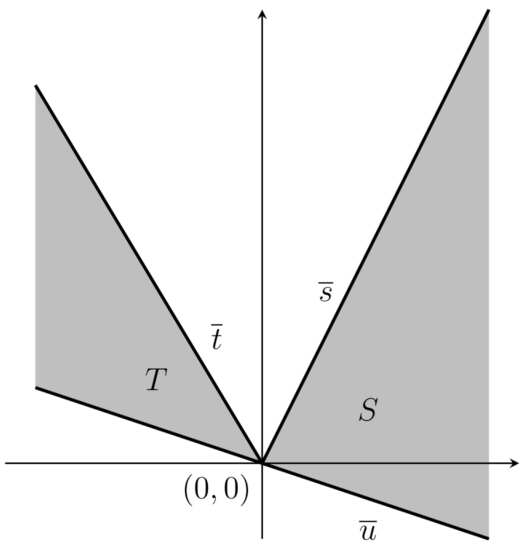
(4) Let $q_1, \ldots, q_n$ be independent generic points in $\mathbb{P}^2_{\mathbb{C}}$. Suppose that $n \geq 10$. Nagata conjectured that, if a plane curve of degree d passes through each q_i with multiplicity at least r, then $d > \sqrt{n}r$. Nagata [20] solved it affirmatively when n is a square.

In the case where I is a space monomial prime ideal, the existence of negative curves is a very difficult and important problem, that is deeply related to Nagata's conjecture (Proposition 5.2 in Cutkosky-Kurano [3]) and the rationality of Seshadri constant.

Even if I is not a prime ideal, our problem is also deeply related to Nagata's conjecture as follows; If Y does not have a curve C with $C^2 \leq 0$ except for E, Nagata's conjecture is true for $n = abcd$.

Suppose that W is not a square of a rational number. Under this condition, Y does not have a curve C with $C^2 = 0$. Assume that the characteristic of K is positive and $\mathrm{Cox}(Y)$ is not Noetherian.[3] Then there does not exists a curve C such that $C^2 \leq 0$ and $C \neq E$. Hence Nagata's conjecture is true for $n = abcd$.

In the rest of this section, let us recall a method in [15].



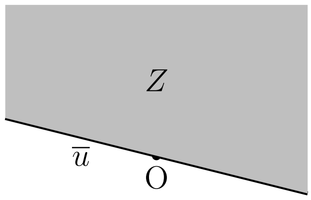
(2.3)

[2]Remark that following conditions are equivalent: (1) $R_s(I)$ is finitely generated over K, (2) $R_s(I)$ is Noetherian, (3) $R'_s(I)$ is finitely generated over K, (4) $R'_s(I)$ is Noetherian, (5) $\mathrm{Cox}(Y)$ is finitely generated over K, (6) $\mathrm{Cox}(Y)$ is Noetherian.

[3]We shall give an example such that the characteristic of K is positive and $\mathrm{Cox}(Y)$ is not Noetherian in Example 1.4 (2). However, in our example, $W = 1$ and there exists a curve C with $C^2 = 0$.

Let S and T be the cones in $\mathbb{R}^2$ defined by

$$S = \mathbb{R}_{\geq 0}(u, -u_2) + \mathbb{R}_{\geq 0}(s_3, s_2),$$
$$T = \mathbb{R}_{\geq 0}(-u, u_2) + \mathbb{R}_{\geq 0}(-t_3, t).$$



Let Z be the cone $\mathbb{R}(u, -u_2) + \mathbb{R}_{\geq 0}(0, 1)$ as above. Let v and w are indeterminates over K. Put

$$x = \frac{w-1}{v-1}.$$

Here remark

$$w = 1 - x + vx.$$

Put

$$K[Z] = \bigoplus_{(\alpha,\beta)\in Z\cap\mathbb{Z}^2} K v^\alpha w^\beta \subset K[v^{\pm 1}, w^{\pm 1}],$$
$$F = K[Z]\,[x] \subset K[v^{\pm 1}, w^{\pm 1}, \frac{1}{v-1}],$$
$$x_{\alpha,n} = v^\alpha w^{\lceil \alpha \overline{u} \rceil} x^n \in F \tag{2.4}$$

for $\alpha \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$, where $\lceil \alpha \overline{u} \rceil$ is the least integer bigger than or equal to $\alpha \overline{u}$. We refer the reader to Remark 4.3 in [15] for the product $x_{\alpha,n} x_{\alpha',n'}$. Then by Proposition 4.1 in [15], we have

$$F = \bigoplus_{\alpha\in\mathbb{Z}} \bigoplus_{n\geq 0} K x_{\alpha,n}$$
$$x^\ell F = \bigoplus_{\alpha\in\mathbb{Z}} \bigoplus_{n\geq \ell} K x_{\alpha,n} \tag{2.5}$$

Put

$$\begin{aligned} z_{\alpha,n} =& v^\alpha w^{\lceil (\alpha-n)\overline{u} \rceil} (x + x^2 + x^3 + \cdots)^n = v^\alpha w^{\lceil (\alpha-n)\overline{u} \rceil} x^n (1 + x + x^2 + \cdots)^n \\ =& x_{\alpha,n} w^{\lceil (\alpha-n)\overline{u} \rceil - \lceil \alpha \overline{u} \rceil} (1 + x + x^2 + \cdots)^n \in F/x^\ell F \end{aligned} \tag{2.6}$$

as in [15]. We refer the reader to (4.15) in [15] for the relation between $z_{\alpha,n}$ and $x_{\alpha,n}$. We put

$$F_\ell = F/x^\ell F = \bigoplus_{\alpha\in\mathbb{Z}} \bigoplus_{\ell>n\geq 0} Kx_{\alpha,n},$$

$$A_\ell = \bigoplus_{\alpha\geq 0} \bigoplus_{\substack{\ell > n \geq 0 \\ (\alpha, \lceil \alpha\overline{u}\rceil + n) \in S}} Kx_{\alpha,n} \ \subset \ F_\ell,$$

$$\psi(B_\ell) = \bigoplus_{\alpha\in\mathbb{Z}} \bigoplus_{\substack{\ell > n \geq 0 \\ (\alpha - n, \lceil (\alpha-n)\overline{u}\rceil + n) \in T}} Kz_{\alpha,n} \ \subset \ F_\ell$$

as in (4.16) and (4.17) in [15]. Remark that both A_ℓ and $\psi(B_\ell)$ are subrings of F_ℓ. Let C be the proper transform of the curve of X defined by $(w-1)T$ in $E(\Delta)$ (see (1.2)). Then we know that $\operatorname{Spec} A_\ell$ and $\operatorname{Spec}\psi(B_\ell)$ are affine open sets of ℓC such that $\ell C = \operatorname{Spec} A_\ell \cup \operatorname{Spec}\psi(B_\ell)$ and $\operatorname{Spec} F_\ell = \operatorname{Spec} A_\ell \cap \operatorname{Spec}\psi(B_\ell)$. Put

$$\xi = (1-x)^u(1-x+vx)^{-u_2} \in F_\ell^\times.$$

Then ξ is the transition function of the line bundle $\mathcal{O}(D)|_{\ell C}$ as in (4.18) in [15].

For integers satisfying $0 \leq m \leq \ell$, we put

$$F(m,\ell) = x^m F/x^\ell F = \bigoplus_{\alpha\geq 0} \bigoplus_{\ell>n\geq m} Kx_{\alpha,n}$$

$$A(m,\ell) = \bigoplus_{\alpha\geq 0} \bigoplus_{\substack{\ell > n \geq m \\ (\alpha, \lceil \alpha\overline{u}\rceil + n) \in S}} Kx_{\alpha,n} \subset F(m,\ell),$$

$$B(m,\ell) = \bigoplus_{\alpha\in\mathbb{Z}} \bigoplus_{\substack{\ell > n \geq m \\ (\alpha - n, \lceil (\alpha-n)\overline{u}\rceil + n) \in T}} Kz_{\alpha,n} \subset F(m,\ell).$$

Remark that $A(m,\ell)$ and $B(m,\ell)$ are ideals of A_ℓ and $\psi(B_\ell)$, respectively.

3. Proof of theorems

Proof of Theorem 1.1. The equivalence of (A0), (A1), (A2), (A3) are given in Theorem 3.1 and Theorem 3.2 in [15] in the case where I is a prime ideal (see (2.2)). We can prove the equivalence of them in the same way.

Next we shall prove the equivalence of (A3) and (A4). We shall show the following claim:

Claim 3.1. *Let m be a positive integer. Then the following conditions are equivalent:*
$(\mathrm{A3})_m$ $\mathcal{O}_Y(mD)|_{muC} \simeq \mathcal{O}_{muC}$.
$(\mathrm{A4})_m$ ξ^m $(\in (F_{mu})^\times)$ *is written as a product of elements of* ${A_{mu}}^\times$ *and* $\psi(B_{mu})^\times$.

Here we shall give an outline of the proof. We know that $\operatorname{Spec} A_{mu}$ and $\operatorname{Spec}\psi(B_{mu})$ are affine open sets of muC such that $muC = \operatorname{Spec} A_{mu} \cup \operatorname{Spec}\psi(B_{mu})$

and $\operatorname{Spec} F_{mu} = \operatorname{Spec} A_{mu} \cap \operatorname{Spec} \psi(B_{mu})$. Remark that $\mathcal{O}_Y(mD)|_{muC}$ is a line bundle for any m such that $\mathcal{O}_Y(mD)|_{\operatorname{Spec} A_{mu}}$ and $\mathcal{O}_Y(mD)|_{\operatorname{Spec} \psi(B_{mu})}$ are free. Then ξ^m is the transition function of $\mathcal{O}(mD)|_{muC}$ as in (4.18) in [15]. Thus we know that $(\mathrm{A3})_m$ is equivalent to $(\mathrm{A4})_m$.

We have completed the proof of Theorem 1.1. □

Proof of Theorem 1.2. (B1) is equivalent to (B2) by Claim 3.1.

By Theorem 1.1, (B1) implies (B0). If the condition $(\mathrm{A4})_m$ in Claim 3.1 is satisfied for some $m > 0$, then $(\mathrm{A4})_1$ holds in the case where the characteristic of the field K is 0 as in Proposition 5.9 in [15]. Thus (B0) implies (B2) by Theorem 1.1 and Claim 3.1. (In the case where $W < 1$, we can prove that (B0) implies (B1) in the same way as in Theorem 1.1 in Kurano-Nishida [16]. However this method does not work in the case where $W = 1$.)

The equivalence of (B2) and (B3) can be proved in the same way as in Theorem 1.2 in [15]. □

Proof of Theorem 1.3. One can prove (1) in the same way as Cutkosky [2].

Now we shall prove (2). In the rest of this paper, assume

$$W = x_1 - x_2 = 1.$$

In proving this theorem, we referred to Totaro's method [22] of constructing nef and non semi-ample divisors on a smooth surface over a finite field.

Remark that $\mathcal{O}_Y(\sigma C)$ is a line bundle over Y by the definition of σ. It is obvious that (C1) is equivalent to (C2).

Since σ is the width of $\sigma\Delta$, σ is a multiple of u. (Remember that the slope of the bottom edge of Δ is $-\frac{u_2}{u}$ and $(u, u_2) = 1$.) We know

$$(\sigma/u)D \sim \sigma C.$$

By Theorem 1.1, (C1) implies (C0).

In order to show that (C0) implies (C1), we shall prove that the condition (A2) in Theorem 1.1 implies (C1) in the case where the characteristic of the field K is a prime number p. Assume the condition (A2) is satisfied. Let $r \geq 0$ and $j > 0$ be integers satisfying $m = jp^r$ and $(p, j) = 1$. Then we know that $\mathcal{O}_Y(jp^r D)|_{\ell C} \simeq \mathcal{O}_{\ell C}$ for any positive integer ℓ. Therefore we have $\mathcal{O}_Y(\sigma j p^r C)|_{\ell C} \simeq \mathcal{O}_Y((\sigma/u) j p^r D)|_{\ell C} \simeq \mathcal{O}_{\ell C}$ for any positive integer ℓ. Putting $\ell = \sigma p^r$ we have $\mathcal{O}_Y(\sigma j p^r C)|_{\sigma p^r C} \simeq \mathcal{O}_{\sigma p^r C}$. Then

$$\text{(3.1)} \qquad \text{the order of } \mathcal{O}_Y(\sigma p^r C)|_{\sigma p^r C} \text{ (in } \operatorname{Pic}(\sigma p^r C)) \text{ divides } j.$$

On the other hand, without assuming (A2), we obtain

$$\text{(3.2)} \qquad \text{the order of } \mathcal{O}_Y(\sigma p^r C)|_{\sigma p^r C} \text{ (in } \operatorname{Pic}(\sigma p^r C)) \text{ is a power of } p$$

as follows. Consider the sequence of the natural maps

$$\operatorname{Pic}(\sigma p^r C) \longrightarrow \operatorname{Pic}((\sigma p^r - 1)C) \longrightarrow \operatorname{Pic}((\sigma p^r - 2)C) \longrightarrow \cdots \longrightarrow \operatorname{Pic}(C) = \mathbb{Z}.$$

The image of $\mathcal{O}_Y(\sigma p^r C)|_{\sigma p^r C}$ is $\mathcal{O}_Y(\sigma p^r C)|_C$ in $\mathrm{Pic}(C)$. It is $\mathcal{O}_C$ since $C^2 = 0$ and $C \simeq \mathbb{P}^1_K$. By the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(kC)/\mathcal{O}_Y((k+1)C) \longrightarrow \mathcal{O}^\times_{(k+1)C} \longrightarrow \mathcal{O}^\times_{kC} \longrightarrow 1,$$

we have an exact sequence

$$H^1(\mathcal{O}_Y(kC)/\mathcal{O}_Y((k+1)C)) \longrightarrow \mathrm{Pic}((k+1)C) \longrightarrow \mathrm{Pic}(kC)$$

for $k \geq 1$. Therefore each element in the kernel of $\mathrm{Pic}((k+1)C) \to \mathrm{Pic}(kC)$ vanishi by p. Consequently (3.2) holds. By (3.1) and (3.2) we know $\mathcal{O}_Y(\sigma p^r C)|_{\sigma p^r C} \simeq \mathcal{O}_{\sigma p^r C}$.

The implications (C2) $\Longrightarrow$ (C4) $\Longrightarrow$ (C3) are obvious.

In the rest of this section we shall prove (C3) $\Longrightarrow$ (C2).

Remark that $\mathcal{O}_Y(nC)/\mathcal{O}_Y((n-1)C)$ is a line bundle over C since both $\mathcal{O}_Y(nC)$ and $\mathcal{O}_Y((n-1)C)$ are locally reflexive modules. Then the following are satisfied.

Lemma 3.2. *Let n be an integer.*

1. *$\mathcal{O}_Y(nC)/\mathcal{O}_Y((n-1)C)$ is periodic with period σ.*
2. *We have $\mathcal{O}_Y(nC)/\mathcal{O}_Y((n-1)C) \simeq \mathcal{O}_{\mathbb{P}^1_K}$ if σ divides n.*
3. *We have $H^0(\mathcal{O}_Y(nC)/\mathcal{O}_Y((n-1)C)) = 0$ if σ does not divide n.*

We shall prove this lemma after completing the proof of Theorem 1.3.

Let m be an integer. By the above lemma we have exact sequences

$$0 = H^0(\mathcal{O}_Y((\sigma m-n+1)C)/\mathcal{O}_Y((\sigma m-n)C)) \longrightarrow H^0(\mathcal{O}_Y(\sigma mC)|_{nC}) \longrightarrow H^0(\mathcal{O}_Y(\sigma mC)|_{(n-1)C})$$

for $n = 2, 3, \ldots, \sigma$. Thus we know that the natural map

$$H^0(\mathcal{O}_Y(\sigma mC)|_{\sigma C}) \longrightarrow H^0(\mathcal{O}_Y(\sigma mC)|_C) \text{ is injective.} \tag{3.3}$$

Assume that (C3) is satisfied.

First assume $r = 0$. Then we have the injection

$$0 \neq H^0(\mathcal{O}_Y(-\sigma jC)|_{\sigma C}) \hookrightarrow H^0(\mathcal{O}_Y(-\sigma jC)|_C) = H^0(\mathcal{O}_{\mathbb{P}^1_K}) = K$$

as in (3.3) (see Lemma 3.2). Therefore the non-zero section in $H^0(\mathcal{O}_Y(-\sigma jC)|_{\sigma C})$ does not vanish at any point of C. Thus we have $\mathcal{O}_Y(-\sigma jC)|_{\sigma C} = \mathcal{O}_{\sigma C}$ and the order of $\mathcal{O}_Y(-\sigma C)|_{\sigma C}$ divides j. Then, by (3.2), we know that the order of $\mathcal{O}_Y(-\sigma C)|_{\sigma C}$ is a power of p. Hence the order is one and $\mathcal{O}_Y(-\sigma C)|_{\sigma C} = \mathcal{O}_{\sigma C}$.

Next assume $r > 0$. We may assume that

$$H^0(\mathcal{O}_Y(-\sigma j' p^{r'} C)|_{\sigma p^{r'} C}) = 0 \tag{3.4}$$

for any integer r' such that $0 \leq r' < r$ and any positive integer j' which is not divided by p. Consider the map

$$H^0(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C}) \longrightarrow H^0(\mathcal{O}_Y(-\sigma j p^r C)|_C).$$

It is the composite map of

$$H^0(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^{r_1} C}) \longrightarrow H^0(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^{r_1 - 1} C}) \tag{3.5}$$

for $r_1 = 1, 2, \ldots, r$ and

$$H^0(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma C}) \longrightarrow H^0(\mathcal{O}_Y(-\sigma j p^r C)|_C).$$

The last map is injective by (3.3). The map (3.5) is the composition of

$$H^0(\mathcal{O}_Y(-\sigma jp^rC)|_{\sigma(p-i)p^{r_1-1}C}) \longrightarrow H^0(\mathcal{O}_Y(-\sigma jp^rC)|_{\sigma(p-i-1)p^{r_1-1}C})$$

for $i = 0, 1, 2, \ldots, p-2$. The kernel of the above map is

$$\begin{aligned} & H^0\left(\frac{\mathcal{O}_Y(-\sigma jp^rC - \sigma(p-i-1)p^{r_1-1}C)}{\mathcal{O}_Y(-\sigma jp^rC - \sigma(p-i)p^{r_1-1}C)}\right) \\ =& H^0\left(\frac{\mathcal{O}_Y(-\sigma(jp^{r-r_1+1} - p + i + 1)p^{r_1-1}C)}{\mathcal{O}_Y(-\sigma(jp^{r-r_1+1} - p + i)p^{r_1-1}C)}\right) \\ =& H^0(\mathcal{O}_Y(-\sigma(jp^{r-r_1+1} - p + i + 1)p^{r_1-1}C)|_{\sigma p^{r_1-1}C}) = 0 \end{aligned}$$

since $r_1 - 1 < r$ (see (3.4)). Therefore (3.5) is injective for $r_1 = 1, 2, \ldots, r$. Thus we obtain the injection

$$0 \neq H^0(\mathcal{O}_Y(-\sigma jp^rC)|_{\sigma p^rC}) \hookrightarrow H^0(\mathcal{O}_Y(-\sigma jp^rC)|_C) = H^0(\mathcal{O}_{\mathbb{P}^1_K}) = K.$$

Therefore any non-zero section in $H^0(\mathcal{O}_Y(-\sigma jp^rC)|_{\sigma p^rC})$ does not vanish at any point of C. Thus we have $\mathcal{O}_Y(-\sigma jp^rC)|_{\sigma p^rC} = \mathcal{O}_{\sigma p^rC}$ and the order of $\mathcal{O}_Y(-\sigma p^rC)|_{\sigma p^rC}$ divides j. Then, by (3.2), we know the order is one and $\mathcal{O}_Y(-\sigma p^rC)|_{\sigma p^rC} = \mathcal{O}_{\sigma p^rC}$.

We have completed the proof of Theorem 1.3. □

Proof of Lemma 3.2. Remember the cones S and T in (2.3). We define

(3.6)
$$\begin{aligned} a_i &=^\# \{(\alpha, \beta) \in S \cap \mathbb{Z}^2 \mid \alpha = i\} \\ b_i &=^\# \{(\alpha, \beta) \in T \cap \mathbb{Z}^2 \mid \alpha = i\}. \end{aligned}$$

By definition, we have

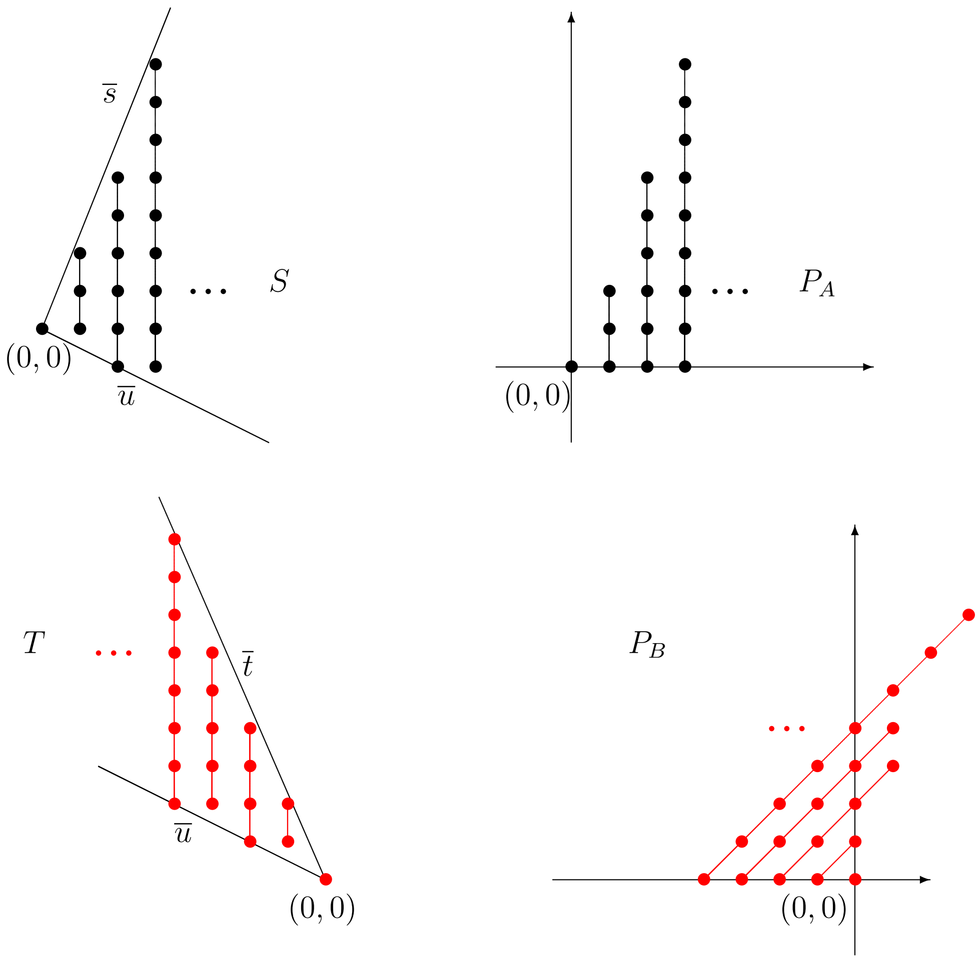
$$\cdots \geq b_{-3} \geq b_{-2} \geq b_{-1} \geq b_0 > 0 < a_0 \leq a_1 \leq a_2 \leq a_3 \leq \cdots.$$

Remark that $a_0 = a_1 = \cdots = \infty$ if $\overline{s} = \infty$, and $b_0 = b_{-1} = \cdots = \infty$ if $\overline{t} = -\infty$.

We put

$$P_A := \left\{ (\alpha, n) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} \alpha \geq 0,\ n \geq 0, \\ (\alpha, \lceil \alpha \overline{u} \rceil + n) \in S \end{array} \right\} = \left\{ (\alpha, n) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} \alpha \geq 0,\ n \geq 0, \\ a_\alpha \geq n+1 \end{array} \right\},$$

$$P_B := \left\{ (\alpha, n) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} \alpha \in \mathbb{Z},\ n \geq 0, \\ (\alpha - n, \lceil (\alpha - n) \overline{u} \rceil + n) \in T \end{array} \right\} = \left\{ (\alpha, n) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} \alpha \in \mathbb{Z},\ n \geq 0, \\ b_{\alpha - n} \geq n+1 \end{array} \right\}.$$



Put

$$\theta = \tfrac{-x_2\sigma}{x_1-x_2} = -x_2\sigma \text{ and } \theta' = \tfrac{x_1\sigma}{x_1-x_2} = x_1\sigma. \tag{3.7}$$

Then θ and θ' are non-negative integers such that $\sigma = \theta + \theta'$.

Claim 3.3. (1) *$P_A \cap P_B = \{m(\theta, \sigma) \mid m \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ denotes the set of non-negative integers.*
(2) *For $\alpha \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, $(\alpha, n) \in P_A$ if and only if $(\alpha + \theta, n + \sigma) \in P_A$.*
(3) *For $\alpha \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, $(\alpha, n) \in P_B$ if and only if $(\alpha + \theta, n + \sigma) \in P_B$.*

First we shall prove (2). If $\overline{s} = \infty$, then $\theta = 0$ and $a_0 = a_1 = \cdots = \infty$. In this case, $P_A = \{(\alpha, n) \mid \alpha, n \in \mathbb{N}_0\}$. The assertion is obvious in this case. Assume $\overline{s} < \infty$. Remark that $(\theta, \overline{u}\theta)$ and $(\theta, \overline{s}\theta)$ are lattice points and $\overline{s}\theta = \overline{u}\theta + \sigma$. Therefore, for any $i \in \mathbb{N}_0$, we have

$$a_{i+\theta} = a_i + \sigma. \tag{3.8}$$

For $\alpha \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$,

$$(\alpha, n) \in P_A \Leftrightarrow a_\alpha \geq n + 1 \Leftrightarrow a_{\alpha+\theta} \geq n + 1 + \sigma \Leftrightarrow (\alpha + \theta, n + \sigma) \in P_A.$$

For $\alpha < 0$ and $n \in \mathbb{N}_0$, then (α, n) and $(\alpha+\theta, n+\sigma)$ are not in P_A.

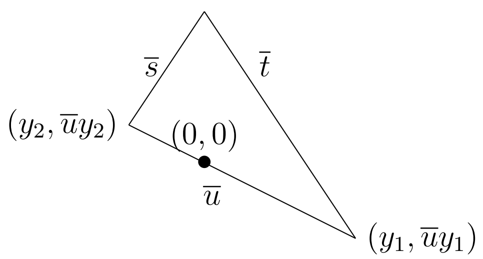
Next we prove (3). Put

$$P'_B = \left\{ (\alpha, n) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} \alpha \le 0,\ n \ge 0, \\ (\alpha, \lceil \alpha \overline{u} \rceil + n) \in T \end{array} \right\}.$$

By the same way as in (2), we can prove that, for $\alpha \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, $(\alpha, n) \in P'_B$ if and only if $(\alpha - \theta', n + \sigma) \in P'_B$. Then, for $\alpha \in \mathbb{Z}$ and $n \in \mathbb{N}_0$,

$$(\alpha, n) \in P_B \Leftrightarrow (\alpha - n, n) \in P'_B \Leftrightarrow (\alpha - n - \theta', n + \sigma) \in P'_B \Leftrightarrow (\alpha + \theta, n + \sigma) \in P_B.$$

Now we start to prove (1). Suppose $(\alpha, n) \in P_A \cap P_B$. By (2) and (3), we may assume $0 \le n < \sigma$. Since $(\alpha, n) \in P_A$, we know $\alpha \ge 0$, $a_\alpha \ge n+1$. Since $(\alpha, n) \in P_B$, we have $b_{\alpha-n} \ge n+1$. Therefore we know $\alpha - n \le 0$. Put $y_1 = n - \alpha \ge 0$ and $y_2 = -\alpha \le 0$. Consider the triangle $\overline{\Delta}$ such that the slopes of edges are $\overline{s}$, $\overline{t}$, $\overline{u}$, respectively, and $(y_1, \overline{u}y_1)$, $(y_2, \overline{u}y_2)$ are the endpoints of the bottom edge as below.

$\overline{\Delta} =$ 

Since $a_\alpha \ge n+1$ and $b_{\alpha-n} \ge n+1$, the point $(0, n)$ is contained in $\overline{\Delta}$. Since the area of $\overline{\Delta}$ is $n^2/2$, the point $(0, n)$ is a vertex of $\overline{\Delta}$. Then we know that $(y_1, \overline{u}y_1)$ and $(y_2, \overline{u}y_2)$ are lattice points. Then σ divides n. Thus we obtain $n = \alpha = 0$. We have completed the proof of Claim 3.3.

Let's go back to the proof of Lemma 3.2. Assume $n = -q < 0$. We have $(q+1)C = \operatorname{Spec} A_{q+1} \cup \operatorname{Spec} \psi(B_{q+1})$ and $\operatorname{Spec} F_{q+1} = \operatorname{Spec} A_{q+1} \cap \operatorname{Spec} \psi(B_{q+1})$. Then we have

$$\left. \frac{\mathcal{O}_Y(-qC)}{\mathcal{O}_Y(-(q+1)C)} \right|_{\operatorname{Spec} F_{q+1}} = F(q, q+1) = \bigoplus_{\alpha \in \mathbb{Z}} K x_{\alpha, q},$$

$$\left. \frac{\mathcal{O}_Y(-qC)}{\mathcal{O}_Y(-(q+1)C)} \right|_{\operatorname{Spec} A_{q+1}} = A(q, q+1) = \bigoplus_{(\alpha, q) \in P_A} K x_{\alpha, q} \subset F(q, q+1),$$

$$\left. \frac{\mathcal{O}_Y(-qC)}{\mathcal{O}_Y(-(q+1)C)} \right|_{\operatorname{Spec} \psi(B_{q+1})} = B(q, q+1) = \bigoplus_{(\alpha, q) \in P_B} K x_{\alpha, q} \subset F(q, q+1).$$

Here remark that $x_{\alpha,q} = z_{\alpha,q}$ in $F(q, q+1)$ by (4.15) in [15]. Thus we know

$$H^0\left(\frac{\mathcal{O}_Y(-qC)}{\mathcal{O}_Y(-(q+1)C)}\right) = \left(\bigoplus_{(\alpha,q)\in P_A} Kx_{\alpha,q}\right) \cap \left(\bigoplus_{(\alpha,q)\in P_B} Kx_{\alpha,q}\right),$$

$$H^1\left(\frac{\mathcal{O}_Y(-qC)}{\mathcal{O}_Y(-(q+1)C)}\right) = \frac{\bigoplus_{\alpha\in\mathbb{Z}} Kx_{\alpha,q}}{\left(\bigoplus_{(\alpha,q)\in P_A} Kx_{\alpha,q}\right) + \left(\bigoplus_{(\alpha,q)\in P_B} Kx_{\alpha,q}\right)}.$$

Therefore we know

$$H^0\left(\frac{\mathcal{O}_Y(-qC)}{\mathcal{O}_Y(-(q+1)C)}\right) = \begin{cases} K & (\text{if } \sigma|q), \\ 0 & (\text{otherwise}) \end{cases}$$

by Claim 3.3. We have proved (2) and (3) in Lemma 3.2 in the case $n < 0$.

Recall that $\mathcal{O}_Y(-\sigma C)$ is a line bundle over Y such that $\mathcal{O}_Y(-\sigma C)|_C \simeq \mathcal{O}_{\mathbb{P}^1_k}$. Therefore, for any $n \in \mathbb{Z}$, we have

$$\mathcal{O}_Y((n-\sigma)C) = \mathcal{O}_Y(nC) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-\sigma C)$$

and

$$\begin{aligned} \frac{\mathcal{O}_Y((n-\sigma)C)}{\mathcal{O}_Y((n-\sigma-1)C)} &= \frac{\mathcal{O}_Y(nC)}{\mathcal{O}_Y((n-1)C)} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-\sigma C) \\ =&\frac{\mathcal{O}_Y(nC)}{\mathcal{O}_Y((n-1)C)} \otimes_{\mathcal{O}_C} \mathcal{O}_Y(-\sigma C)|_C \simeq \frac{\mathcal{O}_Y(nC)}{\mathcal{O}_Y((n-1)C)} \end{aligned}$$

by (2). We have completed the proof of Lemma 3.2. □

4. Examples

We shall prove (1) and (2) in Example 1.4 in this section.

Let Δ_g be the triangle with three vertices $(1,0)$, $(0,0)$, $(g,4)$. By the affine transformation $\frac{1}{2}\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g-2 \\ \frac{2-g}{2} \end{pmatrix}$, Δ_g is transformed to the triangle Δ with three vertices $(g-3, \frac{3-g}{2})$, $(g-2, \frac{2-g}{2})$, $(0,1)$. Remark that if $2 \le g \le 3$, then Δ satisfies the conditions in (1.1). In this case the triangle Δ' in (1.3) has three vertices $(0,0)$, $(2,-1)$, $(6-2g, g-1)$.

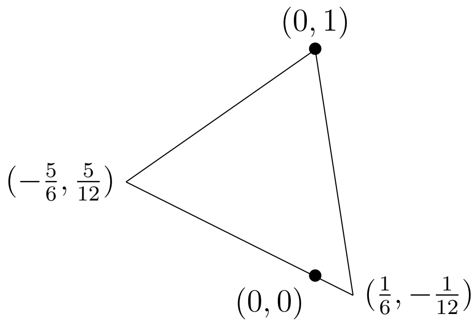
First we shall prove (1) in Example 1.4. We apply Theorem 1.2 here. Checking the EMU condition, we know that $\mathrm{Cox}(Y)$ is Noetherian if and only if the point $(1,1)$ is in Δ'. By an easy calculation we know that it is equivalent to $7/3 \le g \le 8/3$.

Next we shall prove (2) in Example 1.4.

Fix a triangle satisfying (1.1). We have $\mathrm{Cl}(Y) = \mathrm{Cl}(X) \oplus \mathbb{Z}$, and it is independent of the base field K (see (2.1)). For a Weil divisor F on Y, let $h^0(F)_K$ denotes the dimension of $H^0(Y, \mathcal{O}_Y(F))$ in the case where the base field is K. It is easy to see that $h^0(F)_K$ depends only on the characteristic of K. This fact implies that finite generation depends only on the characteristic of K. It is easy to check $h^0(F)_{\mathbb{F}_p} \ge h^0(F)_{\mathbb{Q}}$ for any prime number p. Using this inequality, we obtain that, if Cox(Y) is finitely generated in the case where the base field is of characteristic 0, so is for any base field K. The assertion (2) (i) follows from this.

In the rest of this paper, we shall prove (2) (ii) in Example 1.4. Suppose $g = \frac{13}{6}$. Then Δ is the triangle with three vertices $(-\frac{5}{6}, \frac{5}{12})$, $(\frac{1}{6}, -\frac{1}{12})$, $(0, 1)$.

(4.1) $\Delta =$



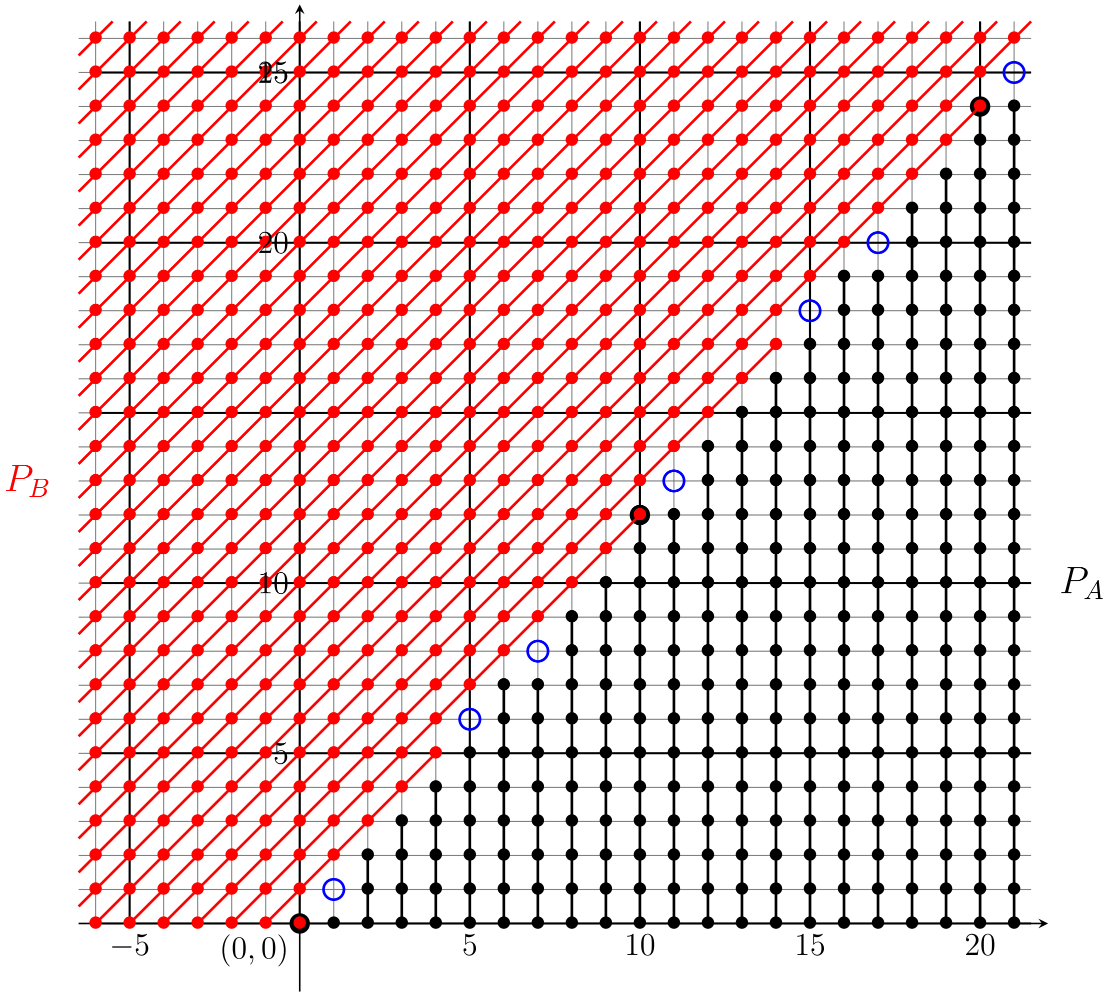
In this case, $x_1 = \frac{1}{6}$, $x_2 = \frac{-5}{6}$, $u = 2$, $u_2 = 1$, $\overline{t} = \frac{-13}{2}$, $\overline{u} = \frac{-1}{2}$, $\overline{s} = \frac{7}{10}$ in (1.1). In this case, $(a, b, c) = (1, 1, 6)$ and $d = 24$. We know $\sigma = 12$ in Theorem 1.3, and $\theta = 10$ and $\theta' = 2$ in (3.7). The sequences in (3.6) are

$$a_0 = 1,\ a_1 = 1,\ a_2 = 3,\ a_3 = 4,\ a_4 = 5,\ a_5 = 6,\ a_6 = 8,\ a_7 = 8,\ a_8 = 10,\ a_9 = 11,\ a_{10} = 13,\ \ldots$$

and

$$b_0 = 1,\ b_{-1} = 6,\ b_{-2} = 13,\ \ldots .$$

Here we have $a_{i+10} = a_i + 12$ for $i \geq 0$ as in (3.8) and $b_{-i-2} = b_{-i} + 12$ for $i \geq 0$. The sets P_A and P_B are the following:
(4.2)



The set $\{(\alpha, n) \mid \alpha \in \mathbb{Z}\}$ in the above picture corresponds to a K-basis of $F(n, n+1)$. The set $\{(\alpha, n) \in P_A \mid \alpha \in \mathbb{Z}\}$ corresponds to a K-basis of $A(n, n+1)$. The set $\{(\alpha, n) \in P_B \mid \alpha \in \mathbb{Z}\}$ corresponds to a K-basis of $B(n, n+1)$.

[I] First suppose $\mathrm{ch}(K) = 2$. We shall show that the condition (A4) in Theorem 1.1 is satisfied with $m = 2$. We have

$$\xi^2 = (1-x)^4(1-x+vx)^{-2} = (1-x_{0,4})(1-x_{0,2}+(x_{1,1})^2)^{-1} = 1 + x_{0,2} - (x_{1,1})^2$$

in F_4. Here remark that $(x^iF)(x^jF) = x^{i+j}F$, $x_{\alpha,n} \in x^nF$ by (2.5). We know $F(2,4) = A(2,4) + B(2,4)$ by (4.2). Take $f_A \in A(2,4)$ and $f_B \in B(2,4)$ such that $x_{0,2} - (x_{1,1})^2 = f_A + f_B$. Then we know $1 + f_A \in A_4^\times$, $1 + f_B \in \psi(B_4)^\times$ and

$$(1+f_A)(1+f_B) = 1 + f_A + f_B = 1 + x_{0,2} - (x_{1,1})^2$$

in F_4. Thus the condition (A4) in Theorem 1.1 is satisfied with $m = 2$. We know that $\mathrm{Cox}(Y)$ is Noetherian by Theorem 1.1.

[II] Next suppose $\mathrm{ch}(K) = 3$. We shall show that the condition (A4) in Theorem 1.1 is satisfied with $m = 3$. We have

$$\begin{aligned}\xi^3 =&(1-x)^6(1-x+vx)^{-3} = (1-x_{0,3})^2(1-x_{0,3}+(x_{1,1})^3)^{-1} \\ =&(1-2x_{0,3})(1+x_{0,3}-(x_{1,1})^3) = 1-x_{0,3}-(x_{1,1})^3\end{aligned}$$

in F_6. We have $F(3,6) = A(3,6) + B(3,6)$ by (4.2). Take $f'_A \in A(3,6)$ and $f'_B \in B(3,6)$ such that $-x_{0,3} - (x_{1,1})^3 = f'_A + f'_B$. Then we know $1 + f'_A \in A_6^\times$, $1 + f'_B \in \psi(B_6)^\times$ and

$$(1+f'_A)(1+f'_B) = 1 + f'_A + f'_B = 1 - x_{0,3} - (x_{1,1})^3$$

in F_6. Therefore the condition (A4) in Theorem 1.1 is satisfied with $m = 3$. We know that $\mathrm{Cox}(Y)$ is Noetherian by Theorem 1.1.

[III] Assume that the characteristic of K is p, where p is a prime number such that $p \geq 5$. In the rest of this paper, we shall prove that $\mathrm{Cox}(Y)$ is not Noetherian. It is enough to show that the condition (C3) in Theorem 1.3 is not satisfied, that is, we want to show

$$\begin{array}{l} H^0(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C}) = 0 \text{ for any non-negative integer } r \\ \text{and a positive integer } j \text{ such that } (j,p)=1. \end{array} \tag{4.3}$$

Let $\chi(\mathcal{F})$ denotes the Euler characteristic of a coherent sheaf $\mathcal{F}$ over nC, that is,

$$\chi(\mathcal{F}) = \dim_K H^0(\mathcal{F}) - \dim_K H^1(\mathcal{F}).$$

By Claim 3.3 and (4.2), we know

$$\mathcal{O}_Y(-nC)/\mathcal{O}_Y(-(n+1)C) \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1_K} & (n \equiv 0 \mod 12) \\ \mathcal{O}_{\mathbb{P}^1_K}(-2) & (n \equiv 1,6,8 \mod 12) \\ \mathcal{O}_{\mathbb{P}^1_K}(-1) & (\text{otherwise}). \end{cases}$$

Therefore we have

$$\chi(\mathcal{O}_Y(-nC)/\mathcal{O}_Y(-(n+1)C)) \simeq \begin{cases} 1 & (n \equiv 0 \mod 12) \\ -1 & (n \equiv 1,6,8 \mod 12) \\ 0 & (\text{otherwise}). \end{cases}$$

Since χ is an additive function for short exact sequences, we know

$$\chi(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C}) = -2p^r. \tag{4.4}$$

Put

$$\text{①}_d = x_{10d+1,12d+1}, \quad \text{②}_d = x_{10d+5,12d+6}, \quad \text{③}_d = x_{10d+7,12d+8}$$

and

$$C_{p^r,j} = \{\text{ⓘ}_d \mid \mathrm{i} = 1,2,3;\ d = jp^r, jp^r+1, \ldots, (j+1)p^r - 1\}.$$

By the exact sequence

$$0 \longrightarrow A(\sigma j p^r, \sigma(j+1)p^r) + B(\sigma j p^r, \sigma(j+1)p^r) \longrightarrow F(\sigma j p^r, \sigma(j+1)p^r) \longrightarrow H^1(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C}) \longrightarrow 0,$$

$C_{p^r,j}$ spans $H^1(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C})$ as a K-vector space.

We define the subset $D_{p^r,j}$ of $C_{p^r,j}$ as

$$D_{p^r,j} = \{x_{\alpha,n} \in C_{p^r,j} \mid x_{\alpha,n} \equiv 0 \text{ modulo } A(\sigma j p^r, n+1) + B(\sigma j p^r, n+1) \text{ in } F(\sigma j p^r, n+1)\}\ .$$

Lemma 4.1. *The set $C_{p^r,j} \setminus D_{p^r,j}$ is a K-basis of $H^1(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C})$.*

Proof. First we shall prove that $C_{p^r,j} \setminus D_{p^r,j}$ spans $H^1(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C})$ as a K-vector space. Let $< C_{p^r,j} \setminus D_{p^r,j} >_K$ be the K-vector subspace of $F(\sigma j p^r, \sigma(j+1)p^r)$ spanned by $C_{p^r,j} \setminus D_{p^r,j}$. We shall prove

(4.5)

$$A(\sigma j p^r, \sigma(j+1)p^r) + B(\sigma j p^r, \sigma(j+1)p^r) + < C_{p^r,j} \setminus D_{p^r,j} \underset{K}{>} + F(n, \sigma(j+1)p^r) = F(\sigma j p^r, \sigma(j+1)p^r)$$

for $n = \sigma j p^r, \sigma j p^r + 1, \ldots, \sigma(j+1)p^r$ by induction on n. It is obvious in the case $n = \sigma j p^r$. It is enough to show

$$A(\sigma j p^r, \sigma(j+1)p^r) + B(\sigma j p^r, \sigma(j+1)p^r) + < C_{p^r,j} \setminus D_{p^r,j} \underset{K}{>} + F(n+1, \sigma(j+1)p^r) \supset F(n, \sigma(j+1)p^r)$$

for $n = \sigma j p^r, \sigma j p^r + 1, \ldots, \sigma(j+1)p^r - 1$. We have only to show that each $x_{\alpha,n}$ ($\alpha \in \mathbb{Z}$) is contained in the left-hand side. If $(\alpha, n) \in P_A$, then $x_{\alpha,n}$ is contained in $A(\sigma j p^r, \sigma(j+1)p^r)$. If $(\alpha, n) \in P_B$, then $x_{\alpha,n}$ is contained in $B(\sigma j p^r, \sigma(j+1)p^r) + F(n+1, \sigma(j+1)p^r)$ by (4.15) in [15]. If $x_{\alpha,n} \in C_{p^r,j} \setminus D_{p^r,j}$, then $x_{\alpha,n}$ is contained in $< C_{p^r,j} \setminus D_{p^r,j} >_K$. If $x_{\alpha,n} \in D_{p^r,j}$, then $x_{\alpha,n}$ is contained in the left-hand side by definition of $D_{p^r,j}$. Therefore (4.5) is satisfied for $n = \sigma(j+1)p^r$. Hence $C_{p^r,j} \setminus D_{p^r,j}$ spans $H^1(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C})$ as a K-vector space.

Next we shall prove that $C_{p^r,j} \setminus D_{p^r,j}$ are linearly independent in $H^1(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C})$. Assume the contrary. There exist x_{α_1,n_1}, x_{α_2,n_2}, $\ldots$, $x_{\alpha_k,n_k} \in C_{p^r,j} \setminus D_{p^r,j}$ and $c_1, \ldots, c_k \in K \setminus \{0\}$ such that

$$c_1 x_{\alpha_1,n_1} + c_2 x_{\alpha_2,n_2} + \cdots + c_k x_{\alpha_k,n_k} = 0$$

in $H^1(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C})$. Suppose $n_1 < n_2 < \cdots < n_k$. Then we obtain $x_{\alpha_1,n_1} \in D_{p^r,j}$. It is a contradiction. □

Remark 4.2. Remark ${}^{\#}C_{p^r,j} = 3p^r$.

Here assume ${}^{\#}D_{p^r,j} \geq p^r$. Then we know

$$\dim_K H^1(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C}) \leq 2p^r$$

by Lemma 4.1. On the other hand we know its Euler characteristic (4.4). Therefore we have ${}^{\#}D_{p^r,j} = p^r$, $\dim_K H^1(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C}) = 2p^r$ and $H^0(\mathcal{O}_Y(-\sigma j p^r C)|_{\sigma p^r C}) = 0$.

Therefore, in order to prove (4.3), it is enough to show

$$\begin{array}{c} {}^{\#}D_{p^r,j} \geq p^r \text{ for any non-negative integer } r \\ \text{and a positive integer } j \text{ such that } (j,p) = 1. \end{array} \tag{4.6}$$

In the rest of this paper, we shall prove (4.6) for each prime number p such that $p \geq 5$.

By (4.5) and (4.6) in [15], $x_{\alpha,n}$ and $z_{\alpha,n}$ are contained in $x^n F$. By definition (2.4), we know

$$x_{\alpha,n}x_{\alpha',n'} = x_{\alpha+\alpha',n+n'}$$

if either α or α' is even. In particular, we have

$$x_{\alpha,n}x^m = x_{\alpha,n}(x_{0,1})^m = x_{\alpha,n+m}.$$

By (2.6), we know

$$z_{\alpha,n}z_{\alpha',n'} = z_{\alpha+\alpha',n+n'}$$

if either $\alpha - n$ or $\alpha' - n'$ is even. We have

$$x_{\alpha,n} = z_{\alpha,n}$$

in $F(n, n+1)$ by (4.15) in [15]. We have

(4.7)

$$x_{\alpha_1,n_1}x_{\alpha_2,n_2}\cdots x_{\alpha_t,n_t} = x_{\alpha_1+\cdots+\alpha_t,n_1+\cdots+n_t} = z_{\alpha_1+\cdots+\alpha_t,n_1+\cdots+n_t} = z_{\alpha_1,n_1}z_{\alpha_2,n_2}\cdots z_{\alpha_t,n_t}$$

in $F(n_1 + \cdots + n_t, n_1 + \cdots + n_t + 1)$ by (4.8) in [15].

We have

$$z_{10k,12k} = x_{10k,12k}\xi^{-6k} \quad \text{where} \quad \xi = (1-x)^2(1-x+vx)^{-1} \tag{4.8}$$

by definition (2.6).

Suppose that k is a positive integer such that $(k, p) = 1$. We have

$$\begin{aligned} z_{10k,12k} - x_{10k,12k} &= x_{10k,12k}(\xi^{-6k} - 1) \\ &= x_{10k,12k}((1-x)^{-12k}(1-x+vx)^{6k} - 1) \\ &\equiv x_{10k,12k}(6kx + 6kvx) \\ &\equiv (6k)z_{10k,12k+1} + (6k)x_{10k+1,12k+1} \end{aligned}$$

modulo $x^{12k+2}F$.

Here suppose $jp^r \le k < (j+1)p^r$. Remark that $x_{10k,12k} \in A(12jp^r, 12k+2)$, and $z_{10k,12k}, z_{10k,12k+1} \in B(12jp^r, 12k+2)$. Recall that the characteristic of the base field K is not 2 or 3. Since $6k \neq 0$, we know that

$$x_{10k,12k}vx = x_{10k+1,12k+1} \equiv 0 \tag{4.9}$$

modulo $A(12jp^r, 12k+2) + B(12jp^r, 12k+2)$ in $F(12jp^r, 12k+2)$. Thus we know

$$\text{①}_k \text{ is in } D_{p^r,j} \text{ if } jp^r \le k < (j+1)p^r \text{ and } (k,p) = 1. \tag{4.10}$$

The following lemma will be frequently used later.

Lemma 4.3. *Let k be a positive integer.*

(1) *Assume that α is even. Then*

$$\begin{aligned}
x_{\alpha,n}w^k =& \sum_{q=0}^{k-1}\left\{\binom{k+q-1}{2q}(1-x)^{k-q}x_{\alpha+2q,n+2q}+\binom{k+q}{2q+1}(1-x)^{k-q-1}x_{\alpha+2q+1,n+2q+1}\right\}\\
=&(1-x)^k x_{\alpha,n}+\binom{k}{1}(1-x)^{k-1}x_{\alpha+1,n+1}\\
&+\binom{k}{2}(1-x)^{k-1}x_{\alpha+2,n+2}+\binom{k+1}{3}(1-x)^{k-2}x_{\alpha+3,n+3}\\
&+\binom{k+1}{4}(1-x)^{k-2}x_{\alpha+4,n+4}+\binom{k+2}{5}(1-x)^{k-3}x_{\alpha+5,n+5}\\
&+\binom{k+2}{6}(1-x)^{k-3}x_{\alpha+6,n+6}+\binom{k+3}{7}(1-x)^{k-4}x_{\alpha+7,n+7}+\cdots
\end{aligned}$$

(2) *Assume that α is odd. Then*

$$\begin{aligned}
x_{\alpha,n}w^k =&(1-x)^k x_{\alpha,n}\\
&+\sum_{q=0}^{k-1}\left\{\binom{k+q}{2q+1}(1-x)^{k-q}x_{\alpha+2q+1,n+2q+1}+\binom{k+q+1}{2q+2}(1-x)^{k-q-1}x_{\alpha+2q+2,n+2q+2}\right\}\\
=&(1-x)^k x_{\alpha,n}\\
&+\binom{k}{1}(1-x)^{k}x_{\alpha+1,n+1}+\binom{k+1}{2}(1-x)^{k-1}x_{\alpha+2,n+2}\\
&+\binom{k+1}{3}(1-x)^{k-1}x_{\alpha+3,n+3}+\binom{k+2}{4}(1-x)^{k-2}x_{\alpha+4,n+4}\\
&+\binom{k+2}{5}(1-x)^{k-2}x_{\alpha+5,n+5}+\binom{k+3}{6}(1-x)^{k-3}x_{\alpha+6,n+6}+\cdots
\end{aligned}$$

Proof. First assume that α is even. By definition (2.4), we have

$$x_{\alpha+2q,n+2q}=x_{\alpha,n}(vx)^{2q}w^{-q},\quad x_{\alpha+2q+1,n+2q+1}=x_{\alpha,n}(vx)^{2q+1}w^{-q}.$$

Put

$$\begin{aligned}
A_q &= \sum_{i=2q}^{k+q-1}\binom{k+q-1}{i}(1-x)^{k-i+q-1}(vx)^i w^{-(q-1)},\\
B_q &= \binom{k+q-1}{2q}(1-x)^{k-q}(vx)^{2q}w^{-q}+\binom{k+q}{2q+1}(1-x)^{k-q-1}(vx)^{2q+1}w^{-q}.
\end{aligned}$$

It is enough to show

$$w^k=\sum_{q=0}^{k-1}B_q. \tag{4.11}$$

We have

$$
\begin{aligned}
A_q = & \sum_{i=2q}^{k+q-1} \binom{k+q-1}{i} (1-x)^{k-i+q-1}(vx)^i w^{-q}(1-x+vx) \\
= & \sum_{i=2q}^{k+q-1} \binom{k+q-1}{i} (1-x)^{k-i+q}(vx)^i w^{-q} + \sum_{j=2q+1}^{k+q} \binom{k+q-1}{j-1} (1-x)^{k-j+q}(vx)^j w^{-q} \\
= & \binom{k+q-1}{2q} (1-x)^{k-q}(vx)^{2q} w^{-q} + \sum_{i=2q+1}^{k+q} \binom{k+q}{i} (1-x)^{k-i+q}(vx)^i w^{-q} \\
= & \binom{k+q-1}{2q} (1-x)^{k-q}(vx)^{2q} w^{-q} + \binom{k+q}{2q+1} (1-x)^{k-q-1}(vx)^{2q+1} w^{-q} \\
& + \sum_{i=2(q+1)}^{k+q} \binom{k+q}{i} (1-x)^{k-i+q}(vx)^i w^{-q}. \\
= & B_q + A_{q+1}
\end{aligned}
$$

Using this formula several times, we have

$$
\begin{aligned}
w^k = & (1-x+vx)^k = \sum_{i=0}^{k} \binom{k}{i} (1-x)^{k-i}(vx)^i = B_0 + A_1 = B_0 + B_1 + A_2 = \cdots \\
= & \sum_{q=0}^{k-2} B_q + A_{k-1} = \sum_{q=0}^{k-1} B_q
\end{aligned}
$$

since $A_{k-1} = B_{k-1}$. We have proved (1).

Next assume that α is odd. By definition (2.4), we have

$$
x_{\alpha+2q,n+2q} = x_{\alpha,n}(vx)^{2q} w^{-q}, \quad x_{\alpha+2q+1,n+2q+1} = x_{\alpha,n}(vx)^{2q+1} w^{-q-1}.
$$

It is enough to show

$$
\begin{aligned}
w^k = & (1-x)^k \\
& + \sum_{q=0}^{k-1} \left\{ \binom{k+q}{2q+1} (1-x)^{k-q}(vx)^{2q+1} w^{-q-1} + \binom{k+q+1}{2q+2} (1-x)^{k-q-1}(vx)^{2q+2} w^{-q-1} \right\}
\end{aligned}
$$

By (4.11), we obtain

$$\begin{aligned}
w^k =&\sum_{q=0}^{k-1}\binom{k+q-1}{2q}(1-x)^{k-q}(vx)^{2q}w^{-q} + \sum_{q=0}^{k-1}\binom{k+q}{2q+1}(1-x)^{k-q-1}(vx)^{2q+1}w^{-q} \\
=&(1-x)^k + \sum_{q'=0}^{k-2}\binom{k+q'}{2q'+2}(1-x)^{k-q'-1}(vx)^{2q'+2}w^{-q'-1} \\
&+\sum_{q=0}^{k-1}\binom{k+q}{2q+1}(1-x)^{k-q}(vx)^{2q+1}w^{-q-1} + \sum_{q=0}^{k-1}\binom{k+q}{2q+1}(1-x)^{k-q-1}(vx)^{2q+2}w^{-q-1} \\
=&(1-x)^k \\
&+\sum_{q=0}^{k-1}\binom{k+q}{2q+1}(1-x)^{k-q}(vx)^{2q+1}w^{-q-1} + \sum_{q=0}^{k-1}\binom{k+q+1}{2q+2}(1-x)^{k-q-1}(vx)^{2q+2}w^{-q-1}.
\end{aligned}$$

□

Remark that, by this lemma, we can rewite $x_{\alpha,n}w^k$ into a K-linear combination of $x_{\beta,m}$'s since $x_{\beta,m}x^\ell = x_{\beta,m+\ell}$.

In the rest, we shall prove Example 1.4 (2) by dividing into some cases.

[III-1] Assume that p is 5. Taking pth power of the equation (4.9), we obtain

$$(x_{10k,12k}vx)^p \equiv 0$$

modulo $A(12kp, 12kp+2p) + B(12kp, 12kp+2p)$ in $F(12kp, 12kp+2p)$. We have

$$\begin{aligned}
&(x_{10k,12k}vx)^p = x_{10kp,12kp}(vx)^5 = x_{10kp,12kp}v^5w^{-2}x^5w^2 \\
=&x_{10kp+5,12kp+5}(1-x+vx)^2 = x_{10kp+5,12kp+5}(1-2x+2vx+x^2-2vx^2+v^2x^2) \\
\equiv&x_{10kp+5,12kp+5} - 2x_{10kp+5,12kp+6} + 2x_{10kp+6,12kp+6}
\end{aligned}$$

modulo $x^{12kp+7}F$. Since $x_{10kp+5,12kp+5}, x_{10kp+6,12kp+6} \in A(12kp, 12kp+7)$, we have

$$x_{10kp+5,12kp+6} \equiv 0$$

modulo $A(12kp, 12kp+7) + B(12kp, 12kp+7)$ in $F(12kp, 12kp+7)$. Taking p^{h-1}th power of it for $h > 0$, we have

$$x_{10kp^h+5p^{h-1},12kp^h+6p^{h-1}} \equiv 0$$

modulo $A(12kp^h, 12kp^h+6p^{h-1}+1)+B(12kp^h, 12kp^h+6p^{h-1}+1)$ in $F(12kp^h, 12kp^h+6p^{h-1}+1)$ by (4.7). Here remark

$$\begin{aligned}
10kp^h + 5p^{h-1} &= 10\left(kp^h + \frac{p-1}{2}(p^{h-2}+p^{h-3}+\cdots+p+1)\right)+5, \\
12kp^h + 6p^{h-1} &= 12\left(kp^h + \frac{p-1}{2}(p^{h-2}+p^{h-3}+\cdots+p+1)\right)+6.
\end{aligned}$$

Therefore we know
(4.12)
$②_{kp^h+2(p^{h-2}+p^{h-3}+\cdots+p+1)}$ is in $D_{p^r,j}$ if $r \geq h > 0$, $jp^r \leq kp^h < (j+1)p^r$ and $(k,p)=1$.

Then $D_{p^r,j}$ contains $p^r - p^{r-1}$ elements of the form $①_d$ as in (4.10) and p^{r-1} elements of the form $②_d$ as in (4.12). Therefore $D_{p^r,j}$ contains at least p^r elements. We know that $\mathrm{Cox}(Y)$ is not Noetherian by (4.6).

[III-2] Assume that $p = 10f+3$, where $f > 0$. let k be a positive integer such that $(k,p)=1$. Put $e = kp+f$. First we shall prove

(4.13) $$x_{10e+5,12e+6} \equiv 0$$

modulo $A(12kp, 12e+7) + B(12kp, 12e+7)$ in $F(12kp, 12e+7)$.

Taking the pth power of (4.9), we obtain

(4.14) $$x_{10kp,12kp}(vx)^p \equiv 0$$

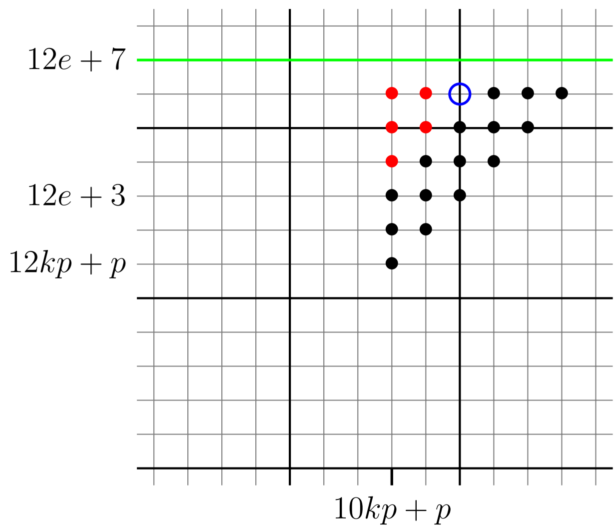
modulo $A(12kp, 12e+7) + B(12kp, 12e+7)$ in $F(12kp, 12e+7)$ since $12kp + 2p > 12(kp+f)+7 = 12e+7$. We have

(4.15) $$\begin{aligned} & x_{10kp,12kp}(vx)^p = x_{10kp,12kp}x_{p,p}w^{5f+1} \\ =&(1-x)^{5f+1}x_{10kp+p,12kp+p} + (5f+1)(1-x)^{5f+1}x_{10kp+p+1,12kp+p+1} \\ &+ \binom{5f+2}{2}(1-x)^{5f}x_{10kp+p+2,12kp+p+2} + \cdots \end{aligned}$$

by Lemma 4.3 (2). Here recall that $x_{\alpha,n}x^q = x_{\alpha,n+q}$. If $x_{\alpha,n}$ appears in (4.15), (α,n) safisfies

(4.16) $$\begin{cases} \alpha \geq 10kp+p \\ n \geq \alpha + 2kp \end{cases}$$

Remark that, if $n \geq 10e+7$, then $x_{\alpha,n} = 0$ in $F(12kp, 12e+7)$.



(4.17)

Remark $10kp + p = 10e + 3$. If (α, n) in the area (4.16) satisfies $n \le \alpha + 2e$, then $x_{\alpha,n} \in A(12kp, 12e + 7)$. We know that $x_{10e+3,12e+5}$, $x_{10e+3,12e+6}$, $x_{10e+4,12e+6}$ are equivalent to 0 modulo $B(12kp, 12e + 7)$ in $F(12kp, 12e + 7)$ by (4.15) in [15]. Since $(12e + 4) - (12kp + p) = 2f + 1$, we have

(4.18)

$$\begin{aligned}
&x_{10kp,12kp}(vx)^p \\
&\equiv -\binom{5f+1}{2f+1}x_{10e+3,12e+4} - (5f+1)\binom{5f+1}{2f+1}x_{10e+4,12e+5} - \binom{5f+2}{2}\binom{5f}{2f+1}x_{10e+5,12e+6} \\
&= -\binom{5f+1}{2f+1}\left(x_{10e+3,12e+4} + (5f+1)x_{10e+4,12e+5} + \frac{(5f+2)3f}{2}x_{10e+5,12e+6}\right)
\end{aligned}$$

modulo $A(12kp, 12e + 7) + B(12kp, 12e + 7)$ in $F(12kp, 12e + 7)$. Here $-\binom{5f+1}{2f+1} \neq 0$ in K. By (4.14), (4.18), we know

$$x_{10e+3,12e+4} + (5f+1)x_{10e+4,12e+5} + \frac{(5f+2)3f}{2}x_{10e+5,12e+6} \equiv 0 \tag{4.19}$$

modulo $A(12kp, 12e + 7) + B(12kp, 12e + 7)$ in $F(12kp, 12e + 7)$. Furthermore we have

$$\begin{aligned}
0 \equiv\, &z_{10e+3,12e+4} = v^{10e+3}w^{e+1}x^{12e+4}(1 + x + x^2 + \cdots)^{12e+4} \\
\equiv\, &x_{10e+3,12e+4}w^{6e+2} \\
\equiv\, &x_{10e+3,12e+4} + (6e+2)x_{10e+4,12e+5} + \binom{6e+3}{2}x_{10e+5,12e+6}
\end{aligned} \tag{4.20}$$

modulo $A(12kp, 12e + 7) + B(12kp, 12e + 7)$ in $F(12kp, 12e + 7)$ by Lemma 4.3 (2). We have

$$\begin{aligned}
0 \equiv\, &z_{10e+4,12e+5} = v^{10e+4}w^{e+1}x^{12e+5}(1 + x + x^2 + \cdots)^{12e+5} \\
\equiv\, &x_{10e+4,12e+5}w^{6e+3} \\
\equiv\, &x_{10e+4,12e+5} + (6e+3)x_{10e+5,12e+6}
\end{aligned} \tag{4.21}$$

modulo $A(12kp, 12e + 7) + B(12kp, 12e + 7)$ in $F(12kp, 12e + 7)$ by Lemma 4.3 (1). Here remark $e = kp + f \equiv f \mod p$. By (4.19), (4.20), (4.21), we obtain

$$\frac{3f(3f+2)}{2}x_{10e+5,12e+6} \equiv 0$$

modulo $A(12kp, 12e + 7) + B(12kp, 12e + 7)$ in $F(12kp, 12e + 7)$. Since $\frac{3f(3f+2)}{2} \neq 0$ in K, we obtain (4.13).

For $h > 0$, taking the p^{h-1}th power of the above equation, we obtain

$$x_{10ep^{h-1}+5p^{h-1},12ep^{h-1}+6p^{h-1}} \equiv 0$$

modulo $A(12kp^h, 12ep^{h-1}+6p^{h-1}+1)+B(12kp^h, 12ep^{h-1}+6p^{h-1}+1)$ in $F(12kp^h, 12ep^{h-1}+6p^{h-1}+1)$ by (4.7). Here remark

$$\begin{aligned}10ep^{h-1}+5p^{h-1} =& 10kp^h+10fp^{h-1}+5p^{h-1} \\ =& 10\left(kp^h+fp^{h-1}+\frac{p-1}{2}(p^{h-2}+p^{h-3}+\cdots+p+1)\right)+5, \\ 12ep^{h-1}+6p^{h-1} =& 12kp^h+12fp^{h-1}+6p^{h-1} \\ =& 12\left(kp^h+fp^{h-1}+\frac{p-1}{2}(p^{h-2}+p^{h-3}+\cdots+p+1)\right)+6.\end{aligned}$$

Therefore we know

(4.22)

$\textcircled{2}_{kp^h+fp^{h-1}+(5f+1)(p^{h-2}+p^{h-3}+\cdots+p+1)}$ is in $D_{p^r,j}$ if $r \ge h > 0$, $jp^r \le kp^h < (j+1)p^r$ and $(k,p)=1$.

Then $D_{p^r,j}$ contains $p^r - p^{r-1}$ elements of the form $\textcircled{1}_d$ as in (4.10) and p^{r-1} elements of the form $\textcircled{2}_d$ as in (4.22). Therefore $D_{p^r,j}$ contains at least p^r elements. We know that $\mathrm{Cox}(Y)$ is not Noetherian by (4.6).

[III-3] Assume that $p = 10f+1$, where $f > 0$.

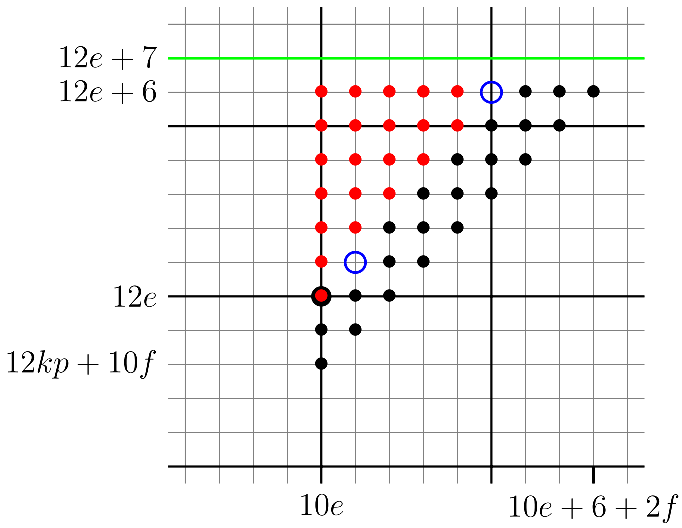
Let k be a positive integer such that $(k,p)=1$. Put $e = kp+f$.

First we shall prove that

(4.23) $$x_{10e+5,12e+6} \equiv 0$$

modulo $A(12kp, 12e+7)+B(12kp, 12e+7)$ in $F(12kp, 12e+7)$.

Consider the triangle T with three vertices $(10e, 12kp+10f)$, $(10e, 12e+6)$, $(10e+6+2f, 12e+6)$.



(4.24)

For $c_0, c_1, \ldots, c_6 \in K$, we put

$$\begin{aligned}[c_0, c_1, \ldots, c_6] =& c_0x_{10e+1,12e+1}+c_1x_{10e,12e+1}+c_2x_{10e+1,12e+2}+c_3x_{10e+2,12e+3} \\ &+c_4x_{10e+3,12e+4}+c_5x_{10e+4,12e+5}+c_6x_{10e+5,12e+6}.\end{aligned}$$

If (α, n) is in the triangle T such that $n \le \alpha+2e$, then $x_{\alpha,n}$ is in $A(12kp+10f, 12e+7)$ except for $x_{10e+1,12e+1}$. If (α, n) is in the triangle T such that $n \ge \alpha+2e+2$, then $x_{\alpha,n}$ is in $B(12kp+10f, 12e+7)$. Therefore any K-linear combination of $x_{\alpha,n}$'s in the triangle T is equivalent to some $[c_0, c_1, \dots, c_6]$ modulo $A(12kp+10f, 12e+7) + B(12kp+10f, 12e+7)$ in $F(12kp+10f, 12e+7)$.

Taking the pth power of (4.9), we obtain

$$x_{10kp,12kp}(vx)^p \equiv 0$$

modulo $A(12kp, 12kp+7)+B(12kp, 12kp+7)$ in $F(12kp, 12kp+7)$ since $12kp+2p = 12kp+20f+2 > 12kp+12f+7 = 12e+7$. Here remark that $x_{10kp,12kp}(vx)^p \bmod x^{12e+7}F$ is a K-linear combination of $x_{\alpha,n}$'s in the triangle T since $(10kp+p, 12kp+p) = (10e+1, 12kp+10f+1) \in T$. We have

(4.25)

$$\begin{aligned}
&x_{10kp,12kp}(vx)^p = x_{10kp+p,12kp+p}w^{5f} = x_{10e+1,12e+1-2f}w^{5f} \\
=&(1-x)^{5f}x_{10e+1,12e+1-2f} + \binom{5f}{1}(1-x)^{5f}x_{10e+2,12e+2-2f} + \binom{5f+1}{2}(1-x)^{5f-1}x_{10e+3,12e+3-2f} \\
&+ \binom{5f+1}{3}(1-x)^{5f-1}x_{10e+4,12e+4-2f} + \binom{5f+2}{4}(1-x)^{5f-2}x_{10e+5,12e+5-2f} + \cdots \\
\equiv& \left[\tbinom{5f}{2f}, 0, -\tbinom{5f}{2f+1}, -5f\tbinom{5f}{2f+1}, -\tbinom{5f+1}{2}\tbinom{5f-1}{2f+1}, -\tbinom{5f+1}{3}\tbinom{5f-1}{2f+1}, -\tbinom{5f+2}{4}\tbinom{5f-2}{2f+1}\right]
\end{aligned}$$

modulo $A(12kp+10f, 12e+7) + B(12kp+10f, 12e+7)$ in $F(12kp+10f, 12e+7)$ by Lemma 4.3 (2).

By (4.8) we obtain

(4.26)

$$\begin{aligned}
0 \equiv& z_{10e,12e} = x_{10e,12e}\xi^{-6e} = (1-x)^{-12e}x_{10e,12e}w^{6e} \\
&=(1-x)^{-6e}x_{10e,12e} + \binom{6e}{1}(1-x)^{-6e-1}x_{10e+1,12e+1} + \binom{6e}{2}(1-x)^{-6e-1}x_{10e+2,12e+2} \\
&\quad + \binom{6e+1}{3}(1-x)^{-6e-2}x_{10e+3,12e+3} + \binom{6e+1}{4}(1-x)^{-6e-2}x_{10e+4,12e+4} \\
&\quad + \binom{6e+2}{5}(1-x)^{-6e-3}x_{10e+5,12e+5} + \cdots \\
&\equiv \left[6e, 6e, 6e(6e+1), \tbinom{6e}{2}(6e+1), \tbinom{6e+1}{3}(6e+2), \tbinom{6e+1}{4}(6e+2), \tbinom{6e+2}{5}(6e+3)\right]
\end{aligned}$$

modulo $A(12kp+10f, 12e+7) + B(12kp+10f, 12e+7)$ in $F(12kp+10f, 12e+7)$ by Lemma 4.3 (1).

We have

$$0 \equiv z_{10e,12e+1} \equiv x_{10e,12e+1}w^{6e+1} \tag{4.27}$$
$$\equiv x_{10e,12e+1} + (6e+1)x_{10e+1,12e+2} + \binom{6e+1}{2}x_{10e+2,12e+3}$$
$$+ \binom{6e+2}{3}x_{10e+3,12e+4} + \binom{6e+2}{4}x_{10e+4,12e+5}$$
$$+ \binom{6e+3}{5}x_{10e+5,12e+6} + \cdots$$
$$\equiv \left[0, 1, 6e+1, \tbinom{6e+1}{2}, \tbinom{6e+2}{3}, \tbinom{6e+2}{4}, \tbinom{6e+3}{5}\right]$$

modulo $A(12kp+10f, 12e+7) + B(12kp+10f, 12e+7)$ in $F(12kp+10f, 12e+7)$ by Lemma 4.3 (1).

We have

$$0 \equiv z_{10e+1,12e+2} \equiv x_{10e+1,12e+2}w^{6e+1} \tag{4.28}$$
$$\equiv x_{10e+1,12e+2} + (6e+1)x_{10e+2,12e+3} + \binom{6e+2}{2}x_{10e+3,12e+4}$$
$$+ \binom{6e+2}{3}x_{10e+4,12e+5} + \binom{6e+3}{4}x_{10e+5,12e+6} + \cdots$$
$$\equiv \left[0, 0, 1, 6e+1, \tbinom{6e+2}{2}, \tbinom{6e+2}{3}, \tbinom{6e+3}{4}\right]$$

modulo $A(12kp+10f, 12e+7) + B(12kp+10f, 12e+7)$ in $F(12kp+10f, 12e+7)$ by Lemma 4.3 (2).

We have

(4.29)

$$0 \equiv z_{10e+2,12e+3} \equiv x_{10e+2,12e+3}w^{6e+2}$$
$$\equiv x_{10e+2,12e+3} + (6e+2)x_{10e+3,12e+4} + \binom{6e+2}{2}x_{10e+4,12e+5} + \binom{6e+3}{3}x_{10e+5,12e+6} + \cdots$$
$$\equiv \left[0, 0, 0, 1, 6e+2, \tbinom{6e+2}{2}, \tbinom{6e+3}{3}\right]$$

modulo $A(12kp+10f, 12e+7) + B(12kp+10f, 12e+7)$ in $F(12kp+10f, 12e+7)$ by Lemma 4.3 (1).

We have

$$0 \equiv z_{10e+3,12e+4} \equiv x_{10e+3,12e+4}w^{6e+2} \tag{4.30}$$
$$\equiv x_{10e+3,12e+4} + (6e+2)x_{10e+4,12e+5} + \binom{6e+3}{2}x_{10e+5,12e+6} + \cdots$$
$$\equiv \left[0, 0, 0, 0, 1, 6e+2, \tbinom{6e+3}{2}\right]$$

modulo $A(12kp+10f, 12e+7) + B(12kp+10f, 12e+7)$ in $F(12kp+10f, 12e+7)$ by Lemma 4.3 (2).

We have

$$
\begin{aligned}
(4.31) \qquad 0 \equiv & z_{10e+4,12e+5} \equiv x_{10e+4,12e+5} w^{6e+3} \\
\equiv & x_{10e+4,12e+5} + (6e+3) x_{10e+5,12e+6} + \cdots \\
\equiv & \, [0,0,0,0,0,1,6e+3]
\end{aligned}
$$

modulo $A(12kp+10f, 12e+7) + B(12kp+10f, 12e+7)$ in $F(12kp+10f, 12e+7)$ by Lemma 4.3 (1).

Here remark $e \equiv f \mod p$.

By (4.25), (4.26), (4.27), (4.28), (4.29), (4.30), (4.31), we obtain

$$
\begin{aligned}
0 \equiv & \frac{(2f+1)!(3f)!}{(5f)!}(4.25) - \frac{2f+1}{6f}(4.26) + (2f+1)(4.27) + (3f)(4.28) - \frac{1+14f+18f^2}{2}(4.29) \\
& + \frac{4+53f+108f^2+63f^3}{6}(4.30) - \frac{(1+12f+15f^2)(4+5f+3f^2)}{6}(4.31) \\
= & \left[0,0,0,0,0,0, \frac{-24-308f-600f^2-205f^3+114f^4-117f^5}{40}\right] \\
= & \frac{-(10f+1)(2501757+5782430f+2175700f^2-1257000f^3+1170000f^4)+101757}{4 \times 10^6} x_{10e+5,12e+6}
\end{aligned}
$$

modulo $A(12kp, 12e+7) + B(12kp, 12e+7)$ in $F(12kp, 12e+7)$. Since $101757 = 3 \times 107 \times 317$, it is not equivalent to 0 modulo p. (Recall that $p = 10f+1 \equiv 1 \mod 10$.) Thus we obtain (4.23).

For $h > 0$, taking the p^{h-1}th power of the above equation, we obtain

$$
x_{10ep^{h-1}+5p^{h-1}, 12ep^{h-1}+6p^{h-1}} \equiv 0
$$

modulo $A(12kp^h, 12ep^{h-1}+6p^{h-1}+1) + B(12kp^h, 12ep^{h-1}+6p^{h-1}+1)$ in $F(12kp^h, 12ep^{h-1}+6p^{h-1}+1)$ by (4.7). Here remark

$$
\begin{aligned}
10ep^{h-1} + 5p^{h-1} = & 10kp^h + 10fp^{h-1} + 5p^{h-1} \\
= & 10\left(kp^h + fp^{h-1} + \frac{p-1}{2}(p^{h-2}+p^{h-3}+\cdots+p+1)\right) + 5, \\
12ep^{h-1} + 6p^{h-1} = & 12kp^h + 12fp^{h-1} + 6p^{h-1} \\
= & 12\left(kp^h + fp^{h-1} + \frac{p-1}{2}(p^{h-2}+p^{h-3}+\cdots+p+1)\right) + 6.
\end{aligned}
$$

Therefore we know

(4.32)

$ⓑ_{kp^h+fp^{h-1}+5f(p^{h-2}+p^{h-3}+\cdots+p+1)}$ is in $D_{p^r,j}$ if $r \geq h > 0$, $jp^r \leq kp^h < (j+1)p^r$ and $(k,p) = 1$.

Then $D_{p^r,j}$ contains $p^r - p^{r-1}$ elements of the form $①_d$ as in (4.10) and p^{r-1} elements of the form $②_d$ as in (4.32). Therefore $D_{p^r,j}$ contains at least p^r elements. We know that $\mathrm{Cox}(Y)$ is not Noetherian by (4.6).

[III-4] Assume that $p = 10f - 1$, where $f > 1$.

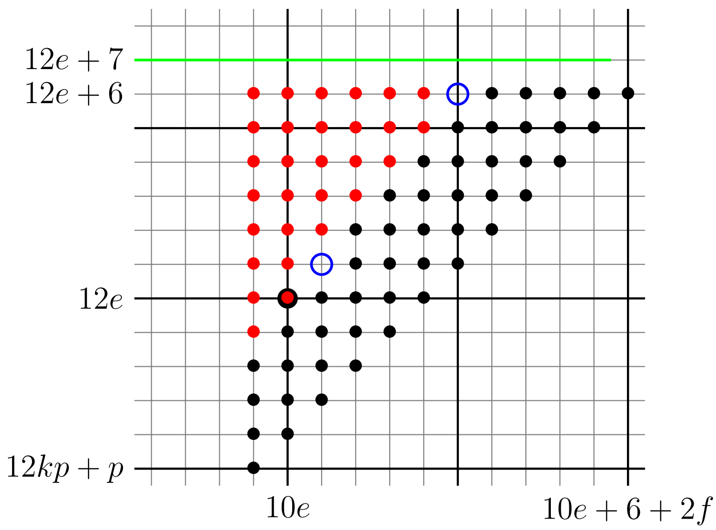
Let k be a positive integer such that $(k,p) = 1$. Put $e = kp + f$.

First we shall prove that

$$x_{10e+5,12e+6} \equiv 0 \tag{4.33}$$

modulo $A(12kp, 12e+7) + B(12kp, 12e+7)$ in $F(12kp, 12e+7)$.

Consider the triangle T' with three vertices $(10e-1, 12kp+p)$, $(10e-1, 12e+6)$, $(10e+6+2f, 12e+6)$. Here remark $10kp + p = 10e - 1$.



(4.34)

For $d_1, d_2, c_0, c_1 \ldots, c_6 \in K$, we put

$$\begin{aligned}[d_1, d_2, c_0, c_1 \ldots, c_6] =& d_1 x_{10e-1,12e-1} + d_2 x_{10e+1,12e+1} + c_0 x_{10e-1,12e} + c_1 x_{10e,12e+1} + c_2 x_{10e+1,12e+2} \\ &+ c_3 x_{10e+2,12e+3} + c_4 x_{10e+3,12e+4} + c_5 x_{10e+4,12e+5} + c_6 x_{10e+5,12e+6}\end{aligned}$$

If (α, n) is in the triangle T' such that $n \le \alpha + 2e$, then $x_{\alpha,n}$ is in $A(12kp+p, 12e+7)$ except for $x_{10e-1,12e-1}$ and $x_{10e+1,12e+1}$. If (α, n) is in the triangle T' such that $n \ge \alpha + 2e + 2$, then $x_{\alpha,n}$ is in $B(12kp + p, 12e + 7)$. Therefore any K-linear combination of $x_{\alpha,n}$'s in the triangle T' is equivalent to some $[d_1, d_2, c_0, c_1, \ldots, c_6]$ modulo $A(12kp + p, 12e + 7) + B(12kp + p, 12e + 7)$ in $F(12kp + p, 12e + 7)$.

Taking the pth power of (4.9), we obtain

$$x_{10kp,12kp}(vx)^p \equiv 0$$

modulo $A(12kp, 12e + 7) + B(12kp, 12e + 7)$ in $F(12kp, 12e + 7)$. Here remark $12kp + 2p = 12kp + 20f - 2 > 12kp + 12f + 7 = 12e + 7$ because $f > 1$. We know that $x_{10kp,12kp}(vx)^p \mod x^{12e+7}F$ is a K-linear combination of $x_{\alpha,n}$'s in the triangle

T'. We have

(4.35)

$$
\begin{aligned}
& x_{10kp,12kp}(vx)^p = x_{10kp+p,12kp+p}w^{5f-1} = x_{10e-1,12e-1-2f}w^{5f-1} \\
=& (1-x)^{5f-1}x_{10e-1,12e-1-2f} + (5f-1)(1-x)^{5f-1}x_{10e,12e-2f} + \binom{5f}{2}(1-x)^{5f-2}x_{10e+1,12e+1-2f} \\
& + \binom{5f}{3}(1-x)^{5f-2}x_{10e+2,12e+2-2f} + \binom{5f+1}{4}(1-x)^{5f-3}x_{10e+3,12e+3-2f} + \\
& + \binom{5f+1}{5}(1-x)^{5f-3}x_{10e+4,12e+4-2f} + \binom{5f+2}{6}(1-x)^{5f-4}x_{10e+5,12e+5-2f} + \cdots \\
\equiv & \Big[\tbinom{5f-1}{2f}, \tbinom{5f}{2}\tbinom{5f-2}{2f}, -\tbinom{5f-1}{2f+1}, -(5f-1)\tbinom{5f-1}{2f+1}, -\tbinom{5f}{2}\tbinom{5f-2}{2f+1}, -\tbinom{5f}{3}\tbinom{5f-2}{2f+1}, \\
& -\tbinom{5f+1}{4}\tbinom{5f-3}{2f+1}, -\tbinom{5f+1}{5}\tbinom{5f-3}{2f+1}, -\tbinom{5f+2}{6}\tbinom{5f-4}{2f+1}\Big]
\end{aligned}
$$

modulo $A(12kp+p, 12e+7) + B(12kp+p, 12e+7)$ in $F(12kp+p, 12e+7)$ by Lemma 4.3 (2).

We have

(4.36)

$$
\begin{aligned}
0 \equiv & z_{10e-1,12e-1} \equiv x_{10e-1,12e-1}w^{6e-1}(1+x+x^2+\cdots)^{12e-1} \\
& \equiv \Big(1+(12e-1)x\Big)\Big((1-x)^{6e-1}x_{10e-1,12e-1} + (6e-1)(1-x)^{6e-1}x_{10e,12e} \\
& \quad + \binom{6e}{2}(1-x)^{6e-2}x_{10e+1,12e+1} + \binom{6e}{3}(1-x)^{6e-2}x_{10e+2,12e+2} \\
& \quad + \binom{6e+1}{4}(1-x)^{6e-3}x_{10e+3,12e+3} + \binom{6e+1}{5}(1-x)^{6e-3}x_{10e+4,12e+4} \\
& \quad + \binom{6e+2}{6}(1-x)^{6e-4}x_{10e+5,12e+5} + \cdots\Big) \\
& \equiv \big[1, \tbinom{6e}{2}, 6e, (6e-1)6e, \tbinom{6e}{2}(6e+1), \tbinom{6e}{3}(6e+1), \tbinom{6e+1}{4}(6e+2), \tbinom{6e+1}{5}(6e+2), \tbinom{6e+2}{6}(6e+3)\big]
\end{aligned}
$$

modulo $A(12kp+p, 12e+7) + B(12kp+p, 12e+7)$ in $F(12kp+p, 12e+7)$ by Lemma 4.3 (2).

We have

(4.37)

$$
\begin{aligned}
0 \equiv & z_{10e,12e} = x_{10e,12e} w^{6e} (1 + x + x^2 + \cdots)^{12e} \\
= & (1+12ex)\left((1-x)^{6e} x_{10e,12e} + \binom{6e}{1}(1-x)^{6e-1} x_{10e+1,12e+1} + \binom{6e}{2}(1-x)^{6e-1} x_{10e+2,12e+2} \right. \\
& + \binom{6e+1}{3}(1-x)^{6e-2} x_{10e+3,12e+3} + \binom{6e+1}{4}(1-x)^{6e-2} x_{10e+4,12e+4} \\
& \left. + \binom{6e+2}{5}(1-x)^{6e-3} x_{10e+5,12e+5} + \cdots \right) \\
\equiv & \left[0, 6e, 0, 6e, 6e(6e+1), \tbinom{6e}{2}(6e+1), \tbinom{6e+1}{3}(6e+2), \tbinom{6e+1}{4}(6e+2), \tbinom{6e+2}{5}(6e+3)\right]
\end{aligned}
$$

modulo $A(12kp+p, 12e+7) + B(12kp+p, 12e+7)$ in $F(12kp+p, 12e+7)$ by Lemma 4.3 (1).

We have

(4.38)

$$
\begin{aligned}
0 \equiv & z_{10e-1,12e} \equiv x_{10e-1,12e} w^{6e} \\
= & x_{10e-1,12e} + \binom{6e}{1} x_{10e,12e+1} + \binom{6e+1}{2} x_{10e+1,12e+2} + \binom{6e+1}{3} x_{10e+2,12e+3} \\
& + \binom{6e+2}{4} x_{10e+3,12e+4} + \binom{6e+2}{5} x_{10e+4,12e+5} + \binom{6e+3}{6} x_{10e+5,12e+6} + \cdots \\
\equiv & \left[0, 0, 1, 6e, \tbinom{6e+1}{2}, \tbinom{6e+1}{3}, \tbinom{6e+2}{4}, \tbinom{6e+2}{5}, \tbinom{6e+3}{6}\right]
\end{aligned}
$$

modulo $A(12kp+p, 12e+7) + B(12kp+p, 12e+7)$ in $F(12kp+p, 12e+7)$ by Lemma 4.3 (2).

We have

(4.39)

$$
\begin{aligned}
0 \equiv & z_{10e,12e+1} \equiv x_{10e,12e+1} w^{6e+1} \\
= & x_{10e,12e+1} + \binom{6e+1}{1} x_{10e+1,12e+2} + \binom{6e+1}{2} x_{10e+2,12e+3} + \binom{6e+2}{3} x_{10e+3,12e+4} \\
& + \binom{6e+2}{4} x_{10e+4,12e+5} + \binom{6e+3}{5} x_{10e+5,12e+6} + \cdots \\
\equiv & \left[0, 0, 0, 1, 6e+1, \tbinom{6e+1}{2}, \tbinom{6e+2}{3}, \tbinom{6e+2}{4}, \tbinom{6e+3}{5}\right]
\end{aligned}
$$

modulo $A(12kp+p, 12e+7) + B(12kp+p, 12e+7)$ in $F(12kp+p, 12e+7)$ by Lemma 4.3 (1).

We have

(4.40)

$$\begin{aligned}
0 \equiv & z_{10e+1,12e+2} \equiv x_{10e+1,12e+2}w^{6e+1} \\
= & x_{10e+1,12e+2} + \binom{6e+1}{1}x_{10e+2,12e+3} + \binom{6e+2}{2}x_{10e+3,12e+4} + \binom{6e+2}{3}x_{10e+4,12e+5} \\
& + \binom{6e+3}{4}x_{10e+5,12e+6} + \cdots \\
\equiv & \left[0,0,0,0,1,6e+1,\tbinom{6e+2}{2},\tbinom{6e+2}{3},\tbinom{6e+3}{4}\right]
\end{aligned}$$

modulo $A(12kp+p,12e+7)+B(12kp+p,12e+7)$ in $F(12kp+p,12e+7)$ by Lemma 4.3 (2).

We have

(4.41)

$$\begin{aligned}
0 \equiv & z_{10e+2,12e+3} \equiv x_{10e+2,12e+3}w^{6e+2} \\
= & x_{10e+2,12e+3} + \binom{6e+2}{1}x_{10e+3,12e+4} + \binom{6e+2}{2}x_{10e+4,12e+5} + \binom{6e+3}{3}x_{10e+5,12e+6} + \cdots \\
\equiv & \left[0,0,0,0,0,1,6e+2,\tbinom{6e+2}{2},\tbinom{6e+3}{3}\right]
\end{aligned}$$

modulo $A(12kp+p,12e+7)+B(12kp+p,12e+7)$ in $F(12kp+p,12e+7)$ by Lemma 4.3 (1).

We have

$$\begin{aligned}
0 \equiv & z_{10e+3,12e+4} \equiv x_{10e+3,12e+4}w^{6e+2} \\
= & x_{10e+3,12e+4} + \binom{6e+2}{1}x_{10e+4,12e+5} + \binom{6e+3}{2}x_{10e+5,12e+6} + \cdots \\
\equiv & \left[0,0,0,0,0,0,1,6e+2,\tbinom{6e+3}{2}\right]
\end{aligned} \tag{4.42}$$

modulo $A(12kp+p,12e+7)+B(12kp+p,12e+7)$ in $F(12kp+p,12e+7)$ by Lemma 4.3 (2).

We have

$$\begin{aligned}
0 \equiv & z_{10e+4,12e+5} \equiv x_{10e+4,12e+5}w^{6e+3} \\
= & x_{10e+4,12e+5} + \binom{6e+3}{1}x_{10e+5,12e+6} + \cdots \\
\equiv & [0,0,0,0,0,0,0,1,6e+3]
\end{aligned} \tag{4.43}$$

modulo $A(12kp+p,12e+7)+B(12kp+p,12e+7)$ in $F(12kp+p,12e+7)$ by Lemma 4.3 (1).

Here remark $e \equiv f \mod p$.

By (4.35), (4.36), (4.37), (4.38), (4.39), (4.40), (4.41), (4.42), (4.43), we obtain

$$
\begin{aligned}
0 \equiv & \frac{(2f+1)!(3f-1!)}{(5f-1)!}(4.35) - (1+2f)(4.36) + \frac{(1+2f)(-1+21f)}{12}(4.37) - (1-9f-12f^2)(4.38) \\
& - \frac{-2+15f+49f^2+42f^3}{2}(4.39) - \frac{2-14f-51f^2-45f^3}{2}(4.40) \\
& - \frac{-12+79f+335f^2+432f^3+198f^4}{12}(4.41) + \frac{(1+f)^2(-8+66f+177f^2+81f^3)}{8}(4.42) \\
& - \frac{-120+718f+3821f^2+6630f^3+3480f^4-5508f^5-7101f^6}{120}(4.43) \\
= & \left[0,0,0,0,0,0,0,0,\frac{3(2+3f)(120-872f-2670f^2-3213f^3+4907f^4+22509f^5+24147f^6)}{720}\right]
\end{aligned}
$$

modulo $A(12kp, 12e+7) + B(12kp, 12e+7)$ in $F(12kp, 12e+7)$. We have

(4.44)

$$
\begin{aligned}
& 10^7 \cdot 3(2+3f)(120-872f-2670f^2-3213f^3+4907f^4+22509f^5+24147f^6) \\
= & 3(20+3\cdot 10f)(12\cdot 10^7 - 872\cdot 10^5(10f) - 267\cdot 10^5(10f)^2 - 3213\cdot 10^3(10f)^3 \\
& + 4907\cdot 10^2(10f)^4 + 22509\cdot 10(10f)^5 + 24147\cdot(10f)^6) \\
\equiv & 3\cdot 23\cdot 3626937 = 3^5\cdot 23\cdot 44777 \mod (10f-1).
\end{aligned}
$$

Therefore it is not equivalent to 0 modulo p. (Recall that $p = 10f-1 \equiv 9 \mod 10$.) Thus we obtain (4.33).

For $h > 0$, taking the p^{h-1}th power of (4.33), we obtain

$$x_{10ep^{h-1}+5p^{h-1},12ep^{h-1}+6p^{h-1}} \equiv 0$$

modulo $A(12kp^h, 12ep^{h-1}+6p^{h-1}+1)+B(12kp^h, 12ep^{h-1}+6p^{h-1}+1)$ in $F(12kp^h, 12ep^{h-1}+6p^{h-1}+1)$ by (4.7). Here remark

$$
\begin{aligned}
10ep^{h-1}+5p^{h-1} = & 10kp^h + 10fp^{h-1} + 5p^{h-1} \\
= & 10\left(kp^h + fp^{h-1} + \frac{p-1}{2}(p^{h-2}+p^{h-3}+\cdots+p+1)\right) + 5, \\
12ep^{h-1}+6p^{h-1} = & 12kp^h + 12fp^{h-1} + 6p^{h-1} \\
= & 12\left(kp^h + fp^{h-1} + \frac{p-1}{2}(p^{h-2}+p^{h-3}+\cdots+p+1)\right) + 6.
\end{aligned}
$$

Therefore we know

(4.45)

$â‘¡_{kp^h+fp^{h-1}+(5f-1)(p^{h-2}+p^{h-3}+\cdots+p+1)}$ is in $D_{p^r,j}$ if $r \geq h > 0$, $jp^r \leq kp^h < (j+1)p^r$ and $(k,p)=1$.

Then $D_{p^r,j}$ contains $p^r - p^{r-1}$ elements of the form $â‘ _d$ as in (4.10) and p^{r-1} elements of the form $â‘¡_d$ as in (4.45). Therefore $D_{p^r,j}$ contains at least p^r elements. We know that $\mathrm{Cox}(Y)$ is not Noetherian by (4.6).

[III-5] Assume that $p = 10f+7$, where $f \geq 0$.

Let k be a positive integer such that $(k,p)=1$. Put $e=kp+f$.
First we shall prove

$$x_{10e+7,12e+8} \equiv 0 \tag{4.46}$$

modulo $A(12kp,12e+9)+B(12kp,12e+9)$ in $F(12kp,12e+9)$. Taking the pth power of (4.9), we obtain

$$x_{10kp,12kp}(vx)^p \equiv 0 \tag{4.47}$$

modulo $A(12kp,12e+9)+B(12kp,12e+9)$ in $F(12kp,12e+9)$. (Remark $12kp+2p = 12kp+20f+14 > 12kp+12f+9 = 12e+9$.) Then we have

$$\begin{aligned} &x_{10kp,12kp}(vx)^p = x_{10e+7,12e+7-2f}w^{5f+3} \\ =&(1-x)^{5f+3}x_{10e+7,12e+7-2f} + \binom{5f+3}{1}(1-x)^{5f+3}x_{10e+8,12e+8-2f} + \cdots \end{aligned}$$

by Lemma 4.3 (2). The coefficient of $x_{10e+7,12e+8}$ is $-\binom{5f+3}{2f+1}$. Thus (4.46) follows from this. Therefore we know

$$ⓢ_{kp+f} \text{ is in } D_{p^r,j} \text{ if } (k,p)=1 \text{ and } jp^r \le kp < (j+1)p^r. \tag{4.48}$$

Let m be the integer satisfying

$$(10e+7)p+1 = 10m, \tag{4.49}$$

that is,

$$m = ep+7f+5 = kp^2+fp+7f+5.$$

Next we shall prove

(1) if $p \neq 44777$, then

$$x_{10m+5,12m+6} \equiv 0 \tag{4.50}$$

modulo $A(12kp^2,12m+7)+B(12kp^2,12m+7)$ in $F(12kp^2,12m+7)$,

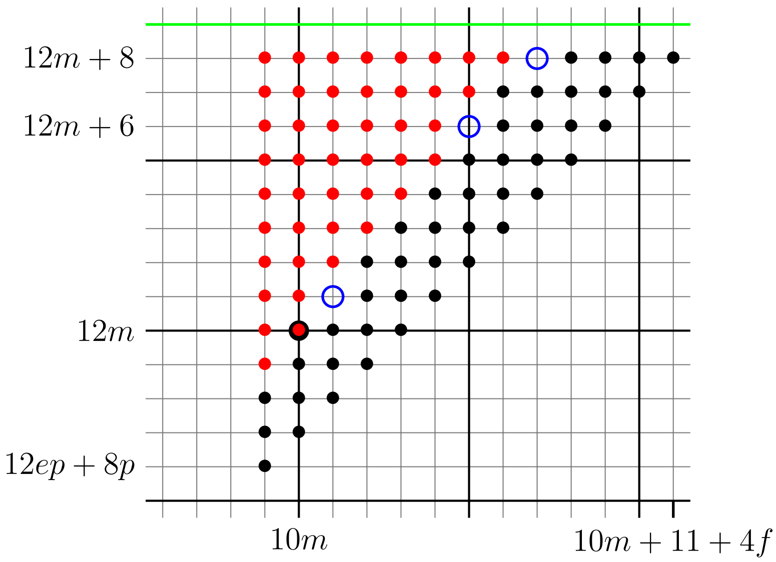
(2) if $p = 44777$, then

$$x_{10m+7,12m+8} \equiv 0 \tag{4.51}$$

modulo $A(12kp^2,12m+9)+B(12kp^2,12m+9)$ in $F(12kp^2,12m+9)$.

Consider the triangle T'' with three vertices $(10m-1, 12ep+8p)$, $(10m-1, 12m+8)$, $(10m+11+4f, 12m+8)$.

(4.52)



For $d_1, d_2, c_0, c_1 \ldots, c_7 \in K$, we put

$$
\begin{aligned}
&[d_1, d_2, c_0, c_1 \ldots, c_7] \\
=&d_1 x_{10m-1,12m-1} + d_2 x_{10m+1,12m+1} + c_0 x_{10m-1,12m} + c_1 x_{10m,12m+1} + c_2 x_{10m+1,12m+2} \\
&+ c_3 x_{10m+2,12m+3} + c_4 x_{10m+3,12m+4} + c_5 x_{10m+4,12m+5} + c_6 x_{10m+5,12m+6} + c_7 x_{10m+7,12m+8}
\end{aligned}
$$

If (α, n) is in the triangle T'' such that $n \leq \alpha + 2m$, then $x_{\alpha,n}$ is in $A(12ep+8p, 12m+9)$ except for $x_{10m-1,12m-1}$ and $x_{10m+1,12m+1}$. If (α, n) is in the triangle T'' such that $n \geq \alpha + 2m + 2$, then $x_{\alpha,n}$ is in $B(12ep + 8p, 12m + 9)$. Therefore any K-linear combination of $x_{\alpha,n}$'s in the triangle T'' is equivalent to some $[d_1, d_2, c_0, c_1, \ldots, c_7]$ modulo $A(12ep + 8p, 12m + 9) + B(12ep + 8p, 12m + 9)$ in $F(12ep + 8p, 12m + 9)$.

Since $F(12e+9, 12e+10) = A(12e+9, 12e+10) + B(12e+9, 12e+10)$, we btain

$$x_{10e+7,12e+8} \equiv 0 \tag{4.53}$$

modulo $A(12kp^2, 12e+10) + B(12kp^2, 12e+10)$ in $F(12kp^2, 12e+10)$ by (4.46). Taking the pth power of (4.53), we obtain

(4.54)

$$
\begin{aligned}
0 \equiv & (x_{10e+7,12e+8})^p = (v^{10e+7} w^{\lceil -\frac{10e+7}{2} \rceil} x^{12e+8})^p = x_{10m-1,12ep+8p} w^{5f+3} \\
= & (1-x)^{5f+3} x_{10m-1,12ep+8p} + (5f+3)(1-x)^{5f+3} x_{10m,12ep+8p+1} \\
& + \binom{5f+4}{2}(1-x)^{5f+2} x_{10m+1,12ep+8p+2} + \binom{5f+4}{3}(1-x)^{5f+2} x_{10m+2,12ep+8p+3} \\
& + \binom{5f+5}{4}(1-x)^{5f+1} x_{10m+3,12ep+8p+4} + \binom{5f+5}{5}(1-x)^{5f+1} x_{10m+4,12ep+8p+5} \\
& + \binom{5f+6}{6}(1-x)^{5f} x_{10m+5,12ep+8p+6} + \binom{5f+7}{8}(1-x)^{5f-1} x_{10m+7,12ep+8p+8} + \cdots \\
\equiv & \Big[-\tbinom{5f+3}{4f+3}, -\tbinom{5f+4}{2}\tbinom{5f+2}{4f+3}, \tbinom{5f+3}{4f+4}, (5f+3)\tbinom{5f+3}{4f+4}, \tbinom{5f+4}{2}\tbinom{5f+2}{4f+4}, \tbinom{5f+4}{3}\tbinom{5f+2}{4f+4}, \\
& \tbinom{5f+5}{4}\tbinom{5f+1}{4f+4}, \tbinom{5f+5}{5}\tbinom{5f+1}{4f+4}, \tbinom{5f+6}{6}\tbinom{5f}{4f+4}, \tbinom{5f+7}{8}\tbinom{5f-1}{4f+4} \Big]
\end{aligned}
$$

modulo $A(12ep+8p, 12m+9) + B(12ep+8p, 12m+9)$ in $F(12ep+8p, 12m+9)$ since $12ep + 10p = 12ep + 100f + 70 \geq 12ep + 84f + 69 = 12m + 9$. Here remark that (4.54) is $[-1, 0, \cdots, 0]$ if $f = 0$.

Replacing e by m and adding the last component[4] to (4.36), (4.37), (4.38), (4.39), (4.40), (4.41), (4.42), (4.43), we obtain

(4.55)
$$0 \equiv \big[1, \tbinom{6m}{2}, 6m, (6m-1)6m, \tbinom{6m}{2}(6m+1), \tbinom{6m}{3}(6m+1), \tbinom{6m+1}{4}(6m+2), \tbinom{6m+1}{5}(6m+2), \tbinom{6m+2}{6}(6m+3), \tbinom{6m+3}{8}(6m+4)\big]$$
(4.56)
$$0 \equiv \big[0, 6m, 0, 6m, 6m(6m+1), \tbinom{6m}{2}(6m+1), \tbinom{6m+1}{3}(6m+2), \tbinom{6m+1}{4}(6m+2), \tbinom{6m+2}{5}(6m+3), \tbinom{6m+3}{7}(6m+4)\big]$$
(4.57)
$$0 \equiv \big[0, 0, 1, 6m, \tbinom{6m+1}{2}, \tbinom{6m+1}{3}, \tbinom{6m+2}{4}, \tbinom{6m+2}{5}, \tbinom{6m+3}{6}, \tbinom{6m+4}{8}\big]$$
(4.58)
$$0 \equiv \big[0, 0, 0, 1, 6m+1, \tbinom{6m+1}{2}, \tbinom{6m+2}{3}, \tbinom{6m+2}{4}, \tbinom{6m+3}{5}, \tbinom{6m+4}{7}\big]$$
(4.59)
$$0 \equiv \big[0, 0, 0, 0, 1, 6m+1, \tbinom{6m+2}{2}, \tbinom{6m+2}{3}, \tbinom{6m+3}{4}, \tbinom{6m+4}{6}\big]$$
(4.60)
$$0 \equiv \big[0, 0, 0, 0, 0, 1, 6m+2, \tbinom{6m+2}{2}, \tbinom{6m+3}{3}, \tbinom{6m+4}{5}\big]$$
(4.61)
$$0 \equiv \big[0, 0, 0, 0, 0, 0, 1, 6m+2, \tbinom{6m+3}{2}, \tbinom{6m+4}{4}\big]$$
(4.62)
$$0 \equiv \big[0, 0, 0, 0, 0, 0, 0, 1, 6m+3, \tbinom{6m+4}{3}\big]$$

modulo $A(12kp+p, 12m+7)+B(12kp+p, 12m+7)$ in $F(12kp+p, 12m+7)$. Remark that

$$m \equiv 7f + 5 \mod p. \tag{4.63}$$

[4]If $p \neq 44777$, then we do not need the last component. If $p = 44777$, then we may assume that denominators of $\binom{5f+7}{8}$, $\binom{6m+3}{8}$, $\binom{6m+3}{7}$, $\binom{6m+4}{8}$, $\binom{6m+4}{7}$ (in the last components) are units.

By (4.54), (4.55), (4.56), (4.57), (4.58), (4.59), (4.60), (4.61), (4.62), we have

$$\begin{aligned}
0 \equiv & \frac{(4f+4)!f!}{(5f+3)!}(4.54) + (4f+4)(4.55) - \frac{2(1+f)(870+2474f+1759f^2)}{6m}(4.56) \\
& - (120+289f+168f^2)(4.57) + (1860+7003f+8671f^2+3518f^3)(4.58) \\
& - \frac{3720+14708f+19303f^2+8405f^3}{2}(4.59) \\
& + \frac{65100+343772f+680605f^2+598882f^3+197669f^4}{6}(4.60) \\
& - \frac{476160+3042856f+7850330f^2+10243855f^3+6774982f^4+1819801f^5}{24}(4.61) \\
& - \frac{190940160+1516966184f+5005402706f^2+8777761165f^3+8625823355f^4+4502014011f^5+974544899f^6}{120}(4.62) \\
= & [0,0,0,0,0,0,0,0,q_1,q_2]
\end{aligned}$$

modulo $A(12ep+8p, 12m+9) + B(12ep+8p, 12m+9)$ in $F(12ep+8p, 12m+9)$, where

$$\begin{aligned}
720q_1 = & -26767572480 - 246120200736f - 967942897272f^2 - 2110412205706f^3 \\
& - 2754630615405f^4 - 2152135097539f^5 - 931716713643f^6 - 172390143619f^7
\end{aligned}$$

(4.64)

$$\begin{aligned}
40320q_2 = & -348081961328640 - 4085017940012352f - 21279829406091360f^2 \\
& - 64577996264481356f^3 - 125811467012647820f^4 - 163172345721567295f^5 \\
& - 140876419259495720f^6 - 78068028418279174f^7 - 25195471807991660f^8 \\
& - 3607880835288623f^9.
\end{aligned}$$

Here 720 is not divided by $p = 10f + 7$, and 40320 is not divided by 44777.

First assume that $p \neq 44777$. When we divide $720 \times 10^7 q_1$ by $10f + 7$ (as a polynomial of f), the reminder is -250258653. Here we have

(4.65) $$250258653 = 3^5 \times 23 \times 44777.$$

Therefore (4.50) holds. (We have to give an attention to the case $f = 0$, since some binomial coefficients are 0.)

Taking the p^{h-2}th power of (4.50) for $h \geq 2$, we obtain

$$x_{10mp^{h-2}+5p^{h-2},12mp^{h-2}+6p^{h-2}} \equiv 0$$

modulo $A(12kp^h, 12mp^{h-2}+6p^{h-2}+1)+B(12kp^h, 12mp^{h-2}+6p^{h-2}+1)$ in $F(12kp^h, 12mp^{h-2}+6p^{h-2}+1)$ by (4.7). Here remark

$$\begin{aligned}10mp^{h-2}+5p^{h-2} =&10(kp^2+fp+7f+5)p^{h-2}+5(p^{h-2}-1)+5\\ =&10\left(kp^h+fp^{h-1}+(7f+5)p^{h-2}+\frac{p-1}{2}(p^{h-3}+p^{h-4}+\cdots+p+1)\right)+5,\\ 12mp^{h-2}+6p^{h-2} =&12(kp^2+fp+7f+5)p^{h-2}+6(p^{h-2}-1)+6\\ =&12\left(kp^h+fp^{h-1}+(7f+5)p^{h-2}+\frac{p-1}{2}(p^{h-3}+p^{h-4}+\cdots+p+1)\right)+6.\end{aligned}$$

Therefore we know
(4.66)
$$⓶_{kp^h+fp^{h-1}+(7f+5)p^{h-2}+(5f+3)(p^{h-3}+p^{h-4}+\cdots+p+1)} \text{ is in } D_{p^r,j} \text{ if } r\ge h\ge 2 \text{ and } jp^r\le kp^h<(j+1)p^r.$$

Then $D_{p^r,j}$ contains p^r-p^{r-1} elements of the form ①$_d$ as in (4.10), $p^{r-1}-p^{r-2}$ elements of the form ③$_d$ as in (4.48) and p^{r-2} elements of the form ②$_d$ as in (4.66). Therefore $D_{p^r,j}$ contains at least p^r elements. We know that $\mathrm{Cox}(Y)$ is not Noetherian by (4.6).

Finally assume $p=44777$. When we divide $40320\times 10^9 q_2$ by $10f+7$ (as a polynomial of f), the reminder is -5257057765239. Here 5257057765239 is not divided by 44777. Therefore (4.51) holds. Recall that (4.46) implies (4.51) for m satisfying (4.49). Therefore, putting

$$(10m+7)p+1=10n,$$

(4.51) implies

$$x_{10n+7,12n+8}\equiv 0$$

modulo $A(12kp^3, 12n+9)+B(12kp^3, 12n+9)$ in $F(12kp^3, 12n+9)$. Here

$$n=mp+7f+5=kp^3+fp^2+(7f+5)p+(7f+5).$$

Repeating this process, we know
(4.67)
$$③_{kp^h+fp^{h-1}+(7f+5)(p^{h-2}+p^{h-3}+p^{h-4}+\cdots+p+1)} \text{ is in } D_{p^r,j} \text{ if } r\ge h\ge 1 \text{ and } jp^r\le kp^h<(j+1)p^r.$$

Then $D_{p^r,j}$ contains p^r-p^{r-1} elements of the form ①$_d$ as in (4.10), p^{r-1} elements of the form ③$_d$ as in (4.67). Therefore $D_{p^r,j}$ contains at least p^r elements. We know that $\mathrm{Cox}(Y)$ is not Noetherian by (4.6).

Remark 4.4. We put $f_1(x)=2x+1$, $f_2(x)=\frac{5x}{2}(2x+1)(3x-1)$, $f_3(x)=-(3x-1)$, $f_4(x)=-(5x-1)(3x-1)$, $f_5(x)=-\frac{5x}{2}(3x-1)(3x-2)$. $f_6(x)=-\frac{5x}{6}(5x-2)(3x-1)(3x-2)$, $f_7(x)=-\frac{(5x+1)}{24}(5x)(3x-1)(3x-2)(3x-3)$, $f_8(x)=-\frac{(5x+1)}{120}(5x)(5x-3)(3x-1)(3x-2)(3x-3)$, $f_9(x)=-\frac{(5x+2)}{720}(5x+1)(5x)(3x-1)(3x-2)(3x-3)(3x-4)$, $f_{10}(x)=-\frac{(5x+3)}{8!}(5x+2)(5x+1)(5x)(3x-1)(3x-2)(3x-3)(3x-4)(3x-5)$.

Then we have

$$\frac{(2f+1)!(3f-1!)}{(5f-1)!}(4.35) = [f_1(e), f_2(e), f_3(e), f_4(e), f_5(e), f_6(e), f_7(e), f_8(e), f_9(e)],$$

$$-\frac{(4f+4)!f!}{(5f+3)!}(4.54) = [f_1(m), f_2(m), f_3(m), f_4(m), f_5(m), f_6(m), f_7(m), f_8(m), f_9(m), f_{10}(m)].$$

Furthermore we have

$$10e \equiv 1 \mod p(= 10f-1),$$
$$10m \equiv 1 \mod p(= 10f+7).$$

Hence we obtained the same remainder in (4.44) and (4.65).

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