

SOME NECESSARY AND SUFFICIENT CONDITION FOR FINITE GENERATION OF SYMBOLIC REES RINGS

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ABSTRACT. Consider the blow-up Y of a weighted projective plane at a point in the open orbit over a field of characteristic 0. We assume that there exists a curve C on Y such that $C^2 < 0$ and $C.E = 1$, where E is the exceptional curve.

In this paper we give a (very simple) necessary and sufficient condition for finite generation of the Cox ring of Y (Theorem 1.2). It is an affirmative answer to a conjecture due to He and Kurano-Nishida.

1. INTRODUCTION

For pairwise coprime positive integers a, b and c , let \mathfrak{p} be the defining ideal of the space monomial curve (t^a, t^b, t^c) in K^3 , where K is a field. The ideal \mathfrak{p} is generated by at most three binomials in $P = K[x, y, z]$ (Herzog [12]). The symbolic Rees rings of space monomial primes are deeply studied by many authors. Huneke [13] and Cutkosky [1] developed criteria for finite generation of such rings. In 1994, Goto-Nishida-Watanabe [9] first found examples of infinitely generated symbolic Rees rings of space monomial primes. Recently, using toric geometry, González-Karu [4] found some sufficient conditions for the symbolic Rees rings of space monomial primes to be infinitely generated.

Cutkosky [1] found the geometric meaning of the symbolic Rees rings of space monomial primes. Let $\mathbb{P}(a, b, c)$ be the weighted projective surface with degree a, b, c . Let Y be the blow-up at a point in the open orbit of the toric variety $\mathbb{P}(a, b, c)$. Then the Cox ring of Y is isomorphic to the extended symbolic Rees ring of the space monomial prime \mathfrak{p} . Therefore, the symbolic Rees ring of the space monomial prime \mathfrak{p} is finitely generated if and only if the Cox ring of Y is finitely generated, that is, Y is a Mori dream space. A curve C on Y is called a negative curve if $C^2 < 0$ and C is different from the exceptional curve E . Here suppose $\sqrt{abc} \notin \mathbb{Q}$. Cutkosky [1] proved that the symbolic Rees ring of the space monomial prime \mathfrak{p} is finitely generated if and only if the following two conditions are satisfied:

- (1) There exists a negative curve C .
- (2) There exists a curve D on Y such that $C \cap D = \emptyset$.

In the case of $\text{ch}(K) > 0$, Cutkosky [1] proved that the symbolic Rees ring is Noetherian if there exists a negative curve.

The first author was supported by JSPS KAKENHI Grant Number 19H00637.

The second author was supported by JSPS KAKENHI Grant Numbers 18K03226, 19H00637.

This work was partly supported by Osaka City University Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849.

The existence of negative curves is a very difficult and important problem, that is deeply related to the Nagata conjecture (Proposition 5.2 in Cutkosky-Kurano [2]) and the rationality of Seshadri constant. The existence of negative curves is studied in González-AnayaGonzález-Karu [7], [8], Kurano-Matsuoka [15] and Kurano [14].

Examples that have a negative curve C such that $C.E \geq 2$ are studied in González-AnayaGonzález-Karu [5], [6] and Kurano-Nishida [16].

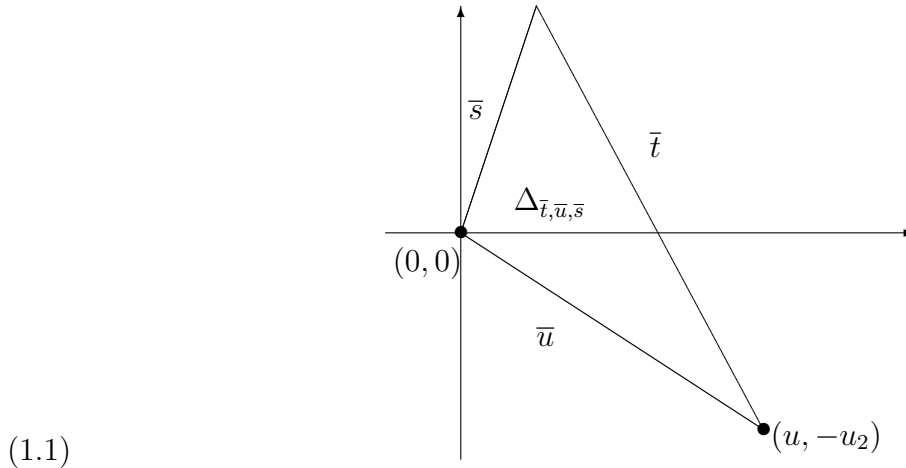
In this paper, we study the case where there exists a negative curve C such that $C.E = 1$ in the case of $\text{ch}(K) = 0$.

Now, we state the main result of this paper precisely.

Let K be a field. Suppose that a, b, c are pairwise coprime positive integers. Let \mathfrak{p} be the kernel of the K -algebra map $\varphi : P = K[x, y, z] \rightarrow K[T]$ defined by $\varphi(x) = T^a$, $\varphi(y) = T^b$, $\varphi(z) = T^c$. Assume that \mathfrak{p} is not complete intersection. Then we know

$$\mathfrak{p} = (x^s - y^{t_1} z^{u_1}, y^t - x^{s_2} z^{u_2}, z^u - x^{s_3} y^{t_3})$$

with positive integers $s_2, s_3, t_1, t_3, u_1, u_2$ such that $s = s_2 + s_3$, $t = t_1 + t_3$, $u = u_1 + u_2$. One can prove $\text{GCD}(s_2, s_3) = \text{GCD}(t_1, t_3) = \text{GCD}(u_1, u_2) = 1$ as in the proof of Proposition 4.8 in [16]. We put $\bar{t} = -t/t_3$, $\bar{u} = -u_2/u$, $\bar{s} = s_2/s_3$. Remark $\bar{t} < -1 < \bar{u} < 0 < \bar{s}$. Here consider the triangle $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ as follows:



The slopes of edges of this triangle are \bar{t} , \bar{u} , \bar{s} respectively.

Definition 1.1 (Ebina [3], Matsuura [17], Uchisawa [18]). For $i = 1, 2, \dots, u$, we put

$$\ell_i = \# \{(\alpha, \beta) \in \Delta_{\bar{t}, \bar{u}, \bar{s}} \cap \mathbb{Z}^2 \mid \alpha = i\}.$$

Note that $\ell_u = 1$ and $\ell_i \geq 1$ for all $i = 1, 2, \dots, u$. We sort the sequence $\ell_1, \ell_2, \dots, \ell_u$ into ascending order

$$\ell'_1 \leq \ell'_2 \leq \dots \leq \ell'_u.$$

We say that the condition EMU is satisfied for (a, b, c) if

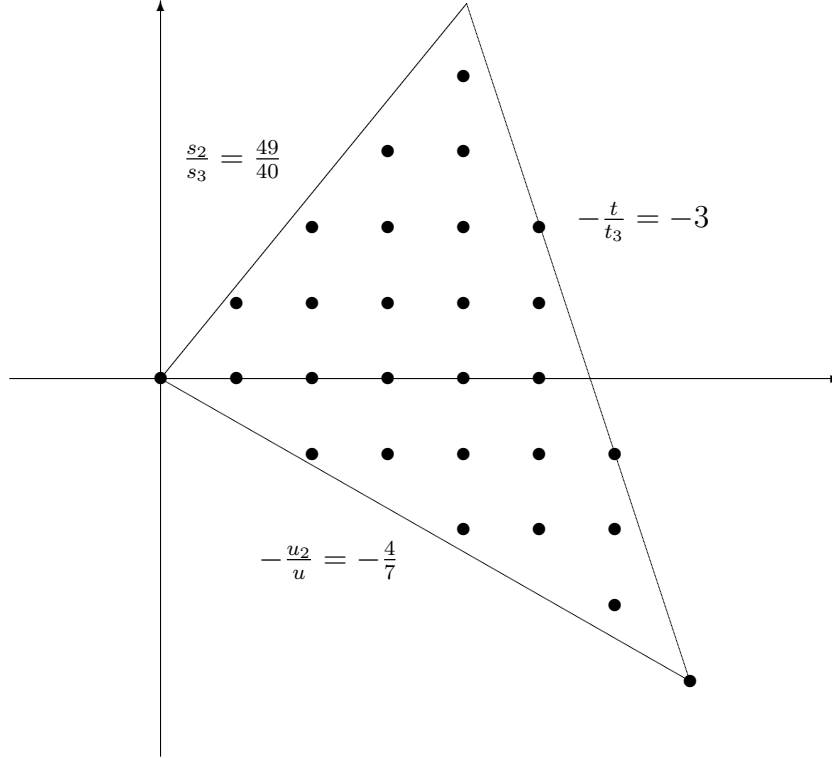
$$\ell'_i \geq i$$

for $i = 1, 2, \dots, u$.

Let me give an example. Suppose $(a, b, c) = (17, 503, 169)$. Then

$$\mathfrak{p} = (x^{89} - y^2 z^3, y^3 - x^{49} z^4, z^7 - x^{40} y)$$

and the proper transform of $z^7 - x^{40} y$ is the negative curve.



Then $u = 7$ and

$$\ell_1 = 2, \ell_2 = 4, \ell_3 = 5, \ell_4 = 7, \ell_5 = 5, \ell_6 = 3, \ell_7 = 1.$$

Therefore

$$\ell'_1 = 1, \ell'_2 = 2, \ell'_3 = 3, \ell'_4 = 4, \ell'_5 = 5, \ell'_6 = 5, \ell'_7 = 7.$$

The condition EMU is not satisfied in this case.

We put $\mathfrak{p}^{(n)} = \mathfrak{p}^n P_{\mathfrak{p}} \cap P$, and call it the n th symbolic power of \mathfrak{p} . Consider the symbolic Rees ring

$$R_s(\mathfrak{p}) := P[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3, \dots] \subset P[t].$$

Assume that $z^u - x^{s_3}y^{t_3}$ is the negative curve, i.e., $\sqrt{abc} > uc$. If the condition EMU is satisfied for (a, b, c) , the symbolic Rees ring of \mathfrak{p} is Noetherian by Proposition 4.6 in [16]. In this paper, we shall prove the converse as follows:

Theorem 1.2. *Let a, b, c be pairwise coprime positive integers. Let K be a field of characteristic 0. Assume that \mathfrak{p} is not complete intersection. Suppose that $z^u - x^{s_3}y^{t_3}$ is a negative curve, i.e., $\sqrt{abc} > uc$.*

Then $R_s(\mathfrak{p})$ is Noetherian if and only if the condition EMU is satisfied for (a, b, c) .

It is an affirmative answer to a conjecture due to He [11]. It was also implicitly conjectured in Kurano-Nishida [16].

Remark 1.3. By the same argument as in the proof of Theorem 1.2, we can prove the following:

Let K be a field of characteristic 0. Let Δ_{r_1, r_2, r_3} be the triangle defined in Section 2. We put $X_\Delta = \text{Proj}(E(\Delta_{r_1, r_2, r_3}))$, where $E(\Delta_{r_1, r_2, r_3})$ is the Ehrhart ring as in (2.3). Let Y_Δ be the blow-up of X_Δ at $(1, 1) \in (K^\times)^2 \subset X_\Delta$. Assume

$$\frac{1}{r_3 - r_2} + \frac{1}{r_2 - r_1} < 1.$$

Then the Cox ring of Y_Δ is Noetherian if and only if the condition EMU is satisfied for Δ_{r_1, r_2, r_3} .

The organization of this paper is as follows.

Section 2 is devoted to the preliminary. We give a new criterion for finite generation in the case where there exists a negative curve C such that $C.E = 1$ in section 3. We give an algebraic description of this new criterion in section 4. We shall give another proof to the finite generation in the case of $\text{ch}(K) > 0$ (Cutkosky [1]) and a result in the case of $\text{ch}(K) = 0$ (Kurano-Nishida [16]) in section 5. We classify (a, b, c) 's for which the condition EMU is not satisfied in section 6. We prove Theorem 1.2 in section 7.

2. PRELIMINARY

In this paper we assume that rings are commutative with 1. Let \mathbb{N} (resp. \mathbb{N}_0 , \mathbb{Z} , \mathbb{R} , $\mathbb{R}_{\geq 0}$) be the set of positive integers (resp. non-negative integers, integers, real numbers, non-negative real numbers).

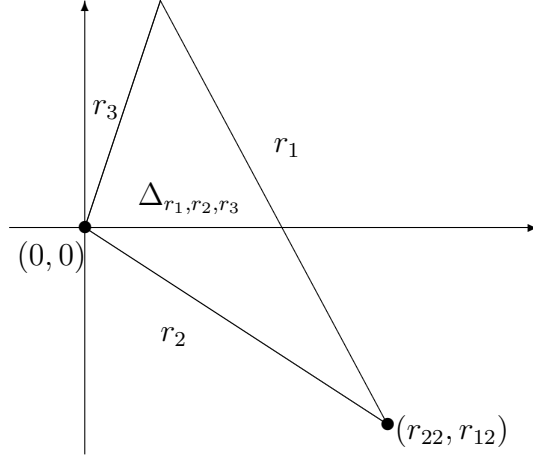
Let r_1, r_2, r_3 be rational numbers such that

$$(2.1) \quad r_1 < -1 < r_2 < 0 < r_3.$$

Let r_{ij} be integers such that

$$(2.2) \quad r_i = r_{i1}/r_{i2} \text{ with } \text{GCD}(r_{i1}, r_{i2}) = 1 \text{ and } r_{i2} > 0 \text{ for } i = 1, 2, 3.$$

Let Δ_{r_1, r_2, r_3} be the rational triangle with slopes r_1, r_2, r_3 such that its lower edge is the line segment connecting the end points $(0, 0)$ and (r_{22}, r_{12}) as below.



We define the Ehrhart ring of Δ_{r_1, r_2, r_3} over a field K by

$$(2.3) \quad E(\Delta_{r_1, r_2, r_3}) = \bigoplus_{(\alpha, \beta) \in m\Delta_{r_1, r_2, r_3} \cap \mathbb{Z}^2} K v^\alpha w^\beta t^m \subset K[v^{\pm 1}, w^{\pm 1}, t],$$

where v, w, t are algebraically independent over K .

Let K be a field. Suppose that a, b, c are pairwise coprime positive integers. Let \mathfrak{p} be the kernel of the K -algebra map $\varphi : P = K[x, y, z] \rightarrow K[T]$ defined by $\varphi(x) = T^a, \varphi(y) = T^b, \varphi(z) = T^c$. In this paper we always assume that

\mathfrak{p} is not complete intersection.

Then we know

$$\mathfrak{p} = I_2 \left(\begin{array}{ccc} x^{s_2} & y^{t_3} & z^{u_1} \\ y^{t_1} & z^{u_2} & x^{s_3} \end{array} \right) = (x^s - y^{t_1} z^{u_1}, y^t - x^{s_2} z^{u_2}, z^u - x^{s_3} y^{t_3})$$

with positive integers $s_2, s_3, t_1, t_3, u_1, u_2$, where $I_2(\cdot)$ is the ideal generated by 2×2 minors of this given 2×3 -matrix, and $s = s_2 + s_3, t = t_1 + t_3, u = u_1 + u_2$ (Herzog [12]). We put, $\bar{t} = -t/t_3, \bar{u} = -u_2/u, \bar{s} = s_2/s_3$. We think that $P = K[x, y, z]$ is a graded polynomial ring with $\deg(x) = a, \deg(y) = b, \deg(z) = c$. Then the Veronesean subring $S^{(ab)}$ is isomorphic to $E(\Delta_{\bar{t}, \bar{u}, \bar{s}})$ as in section 4 in [16].

We have the following proposition:

Proposition 2.1. *Let r_i and r_{ij} be numbers satisfying (2.1) and (2.2). Then the following three conditions are equivalent:*

- (1) *The class group of $\text{Proj } E(\Delta_{r_1, r_2, r_3})$ is torion-free.*
- (2) $\mathbb{Z} \begin{pmatrix} r_{11} \\ r_{12} \end{pmatrix} + \mathbb{Z} \begin{pmatrix} r_{21} \\ r_{22} \end{pmatrix} + \mathbb{Z} \begin{pmatrix} r_{31} \\ r_{32} \end{pmatrix} = \mathbb{Z}^2$.
- (3) *There exist pairwise coprime positive integers a, b, c such that $\bar{t} = r_1, \bar{u} = r_2$ and $\bar{s} = r_3$.*

We omit a proof. By this proposition, we immediately obtain the following corollary.

Corollary 2.2. *Let r_i and r_{ij} be numbers satisfying (2.1) and (2.2). Let ϵ be any positive number. Then there exists a positive number r'_3 with $r_3 - \epsilon < r'_3 < r_3$ such that there exist pairwise coprime positive integers a, b, c such that $\bar{t} = r_1$, $\bar{u} = r_2$ and $\bar{s} = r'_3$.*

Remark 2.3. There exists a one to one correspondence between

$$A := \left\{ (a, b, c) \mid \begin{array}{l} a, b, c \text{ are pairwise coprime positive integers,} \\ \mathfrak{p} \text{ is not complete intersection} \end{array} \right\}$$

and

$$B := \left\{ \Delta_{r_1, r_2, r_3} \mid \begin{array}{l} r_1, r_2, r_3 \text{ are rational numbers,} \\ r_1 < -1 < r_2 < 0 < r_3, \\ \mathbb{Z} \begin{pmatrix} r_{11} \\ r_{12} \end{pmatrix} + \mathbb{Z} \begin{pmatrix} r_{21} \\ r_{22} \end{pmatrix} + \mathbb{Z} \begin{pmatrix} r_{31} \\ r_{32} \end{pmatrix} = \mathbb{Z}^2 \end{array} \right\}.$$

We define a map $\Psi : A \rightarrow B$ to be $\Psi((a, b, c)) = \Delta_{\bar{t}, \bar{u}, \bar{s}}$.

We define a map $\Phi : B \rightarrow A$ to be $\Phi(\Delta_{r_1, r_2, r_3}) = (a, b, c)$, where a, b, c are pairwise coprime positive integers satisfying

$$b \begin{pmatrix} r_{11} \\ r_{12} \end{pmatrix} - c \begin{pmatrix} r_{21} \\ r_{22} \end{pmatrix} + a \begin{pmatrix} r_{31} \\ r_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By Proposition 2.1, we can prove that Φ is the inverse mapping of Ψ .

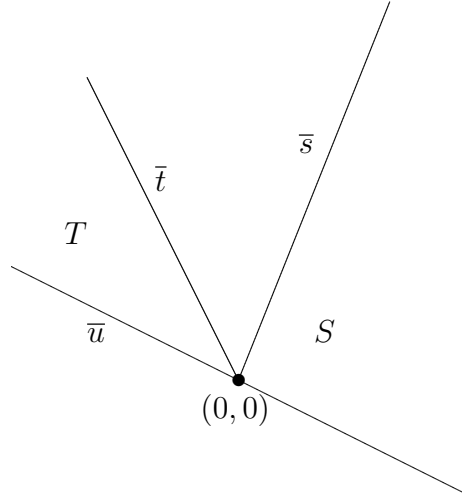
3. A NEW CRITERION FOR FINITE GENERATION OF SYMBOLIC REES RINGS

Let $K, a, b, c, P = K[x, y, z], \mathfrak{p}, \Delta_{\bar{t}, \bar{u}, \bar{s}}$ be as in section 2. In the rest of this paper, we always assume that

- (3.1)
 - \mathfrak{p} is not complete intersection, and
 - $z^u - x^{s_3}y^{t_3}$ is the negative curve, i.e., $uc < \sqrt{abc}$.

The second condition above is equivalent to that the area of $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is bigger than $u^2/2$. It is also equivalent to

(3.2)
$$\frac{1}{\bar{s} - \bar{u}} + \frac{1}{\bar{u} - \bar{t}} < 1.$$



Let S and T be the cone in \mathbb{R}^2 defined by

$$\begin{aligned} S &= \mathbb{R}_{\geq 0}(u, -u_2) + \mathbb{R}_{\geq 0}(s_3, s_2), \\ T &= \mathbb{R}_{\geq 0}(-u, u_2) + \mathbb{R}_{\geq 0}(-t_3, t). \end{aligned}$$

We put

$$\begin{aligned} K[S] &= \bigoplus_{(\alpha, \beta) \in S \cap \mathbb{Z}^2} K v^\alpha w^\beta \subset K[v^{\pm 1}, w^{\pm 1}], \\ K[T] &= \bigoplus_{(\alpha, \beta) \in T \cap \mathbb{Z}^2} K v^\alpha w^\beta \subset K[v^{\pm 1}, w^{\pm 1}]. \end{aligned}$$

Let $\pi : Y \rightarrow X = \text{Proj } P$ be the blow-up at $V_+(\mathfrak{p})$. Let C be the proper transform of $V_+(z^u - x^{s_3}y^{t_3})$. Then we have $\pi(C) = V_+(z^u - x^{s_3}y^{t_3}) \simeq \mathbb{P}_K^1$. Put $U_1 = \text{Spec } K[S]$ and $U_2 = \text{Spec } K[T]$. Then U_i 's are affine open sets of X such that $U_1 \cup U_2 \supset \pi(C)$. Since C is isomorphic to $\pi(C)$,

$$C = (\pi^{-1}(U_1) \cap C) \cup (\pi^{-1}(U_2) \cap C)$$

is an affine open covering of C . For a positive integer ℓ , ℓC is the closed subscheme of Y defined by the ideal sheaf $\mathcal{O}_Y(-\ell C)$. In general, for a scheme W , W is affine if and only if so is W_{red} (e.g., Exercise 3.1 in section III in [10]). Therefore

$$(3.3) \quad \ell C = (\pi^{-1}(U_1) \cap \ell C) \cup (\pi^{-1}(U_2) \cap \ell C)$$

is an affine open covering of ℓC .

Let H be a Weil divisor on Y satisfying $\mathcal{O}_Y(H) = \pi^* \mathcal{O}_X(1)$. (Here remark that $V_+(\mathfrak{p})$ is a non-singular point of X . Therefore $\pi^* \mathcal{O}_X(1)$ is a reflexive sheaf on Y .) Let E be the exceptional divisor of the blow-up $\pi : Y \rightarrow X$. Then we have

$$\text{Cl}(Y) = \mathbb{Z}H + \mathbb{Z}E \simeq \mathbb{Z}^2$$

and

$$H^2 = \frac{1}{abc}, \quad E^2 = -1, \quad H.E = 0.$$

Then we have the following criterion for finite generation of $R_s(\mathfrak{p})$.

Theorem 3.1 (Cutkosky). *The following conditions are equivalent:*

- (1) $R_s(\mathfrak{p})$ is finitely generated over P (equivalently, $R_s(\mathfrak{p})$ is Noetherian).
- (2) The Cox ring $\text{Cox}(Y)$ of Y is finitely generated over K (equivalently, $\text{Cox}(Y)$ is Noetherian).
- (3) There exists a curve D on Y satisfying $C \cap D = \emptyset$.

Since $C \sim ucH - E$, we know $C \cdot (abH - uE) = 0$. One can show that, if there exists a curve D satisfying the condition (3) in Theorem 3.1, there exists a positive integer m such that $D \sim m(abH - uE)$. (Since D does not contain the points corresponding to the end points of the lower edge of $\Delta_{\bar{i}, \bar{u}, \bar{s}}$, we know that the degree of $\pi(D)$ must be a multiple of ab .) The triangle $\Delta_{\bar{i}, \bar{u}, \bar{s}}$ is corresponding to the toric divisor $\mathcal{O}_X(ab)$. We know that $abH - uE$ is a Cartier divisor at each point in Y except for the point corresponding to the top vertex of $\Delta_{\bar{i}, \bar{u}, \bar{s}}$. On the other hand, the curve C does not pass through this point. Ultimately, $\mathcal{O}_Y(m(abH - uE))|_{\ell C}$ is an invertible sheaf on ℓC for any m and ℓ .

Here we have the following theorem:

Theorem 3.2. *The following conditions are equivalent:*

- (1) $R_s(\mathfrak{p})$ is finitely generated over P .
- (2) There exists a positive integer m such that $\mathcal{O}_Y(m(abH - uE))|_{\ell C} \simeq \mathcal{O}_{\ell C}$ for any positive integer ℓ .
- (3) There exists a positive integer m such that $\mathcal{O}_Y(m(abH - uE))|_{muC} \simeq \mathcal{O}_{muC}$.

Proof. First we shall show (1) \implies (2). Since $R_s(\mathfrak{p})$ is finitely generated over P , there exists a curve D on Y satisfying $C \cap D = \emptyset$ as in Theorem 3.1. Then $D \sim m(abH - uE)$ for some positive integer m . Since $C \cap D = \emptyset$, the section in $H^0(Y, \mathcal{O}_Y(m(abH - uE)))$ corresponding to D does not vanish at any point in C . Hence it does not vanish at any point in ℓC for any positive integer ℓ . Therefore $\mathcal{O}_Y(m(abH - uE))|_{\ell C}$ has a global section nowhere vanishes. Thus we have obtained $\mathcal{O}_Y(m(abH - uE))|_{\ell C} \simeq \mathcal{O}_{\ell C}$.

The implication (2) \implies (3) is trivial.

Next we shall prove (3) \implies (1). Consider the tensor product

$$\mathcal{O}_Y(m(abH - uE)) \otimes_{\mathcal{O}_Y} (0 \rightarrow \mathcal{O}_Y(-muC) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{muC} \rightarrow 0).$$

Here remark that $\mathcal{O}_Y(m(abH - uE))$ is invertible at any point in muC . Therefore we have the following exact sequence

$$0 \rightarrow \mathcal{O}_Y(m(abH - uE - uC)) \rightarrow \mathcal{O}_Y(m(abH - uE)) \rightarrow \mathcal{O}_Y(m(abH - uE))|_{muC} \rightarrow 0.$$

Taking a long exact sequence on cohomologies, we obtain the following exact sequence:

$$(3.4) \quad H^0(Y, \mathcal{O}_Y(m(abH - uE))) \xrightarrow{\psi} H^0(muC, \mathcal{O}_Y(m(abH - uE))|_{muC}) \rightarrow H^1(Y, \mathcal{O}_Y(m(abH - uE - uC))).$$

Since $C \sim ucH - E$, we know

$$H^1(Y, \mathcal{O}_Y(m(abH - uE - uC))) = H^1(Y, \mathcal{O}_Y(m(ab - u^2c)H)).$$

Since

$$R^i \pi_* (\mathcal{O}_Y(m(ab - u^2c)H)) = \begin{cases} \mathcal{O}_X(m(ab - u^2c)) & (i = 0) \\ 0 & (i > 0) \end{cases},$$

we have

$$H^1(Y, \mathcal{O}_Y(m(ab - u^2c)H)) \simeq H^1(X, \mathcal{O}_X(m(ab - u^2c)))$$

by the Leray spectral sequence. Then we have

$$H^1(X, \mathcal{O}_X(m(ab - u^2c))) \simeq H^2_{(x,y,z)}(P)_{m(ab-u^2c)} = 0.$$

Ultimately, the map ψ in the exact sequence (3.4) is surjective. Since $\mathcal{O}_Y(m(abH - uE))|_{muC} \simeq \mathcal{O}_{muC}$, there exists $f \in H^0(Y, \mathcal{O}_Y(m(abH - uE)))$ such that $\psi(f)$ does not vanish at any point in muC . Then any irreducible component corresponding to the effective divisor $\text{div}(f) + m(abH - uE)$ does not meet C . Therefore $R_s(\mathfrak{p})$ is Noetherian by Theorem 3.1. \square

Corollary 3.3. *Assume $\mathcal{O}_Y(m(abH - uE))|_{muC} \simeq \mathcal{O}_{muC}$ for some $m > 0$. Then we have $\mathcal{O}_Y(mm'(abH - uE))|_{mm'uC} \simeq \mathcal{O}_{mm'uC}$ for any $m' > 0$.*

Proof. By Theorem 3.2, we have $\mathcal{O}_Y(m(abH - uE))|_{mm'uC} \simeq \mathcal{O}_{mm'uC}$ for any $m' > 0$. The assertion immediately follows from it. \square

Applying this theorem, we can give another proof to

- $\mathcal{O}_Y(p^e(abH - uE))|_{p^e uC} \simeq \mathcal{O}_{p^e uC}$ for $e \gg 0$, in particular $R_s(\mathfrak{p})$ is always Noetherian in the case of $\text{ch}(K) = p > 0$ (Cutkosky [1]), and
- that the finite generation of $R_s(\mathfrak{p})$ is equivalent to $\mathcal{O}_Y(abH - uE)|_{uC} \simeq \mathcal{O}_{uC}$ in the case of $\text{ch}(K) = 0$ (Kurano-Nishida [16])

in Proposition 5.7 and 5.9.

4. ALGEBRAIC DESCRIPTION OF THEOREM 3.2

In this section we shall describe Theorem 3.2 using algebraic method.

Consider the affine openset $U_1 = \text{Spec } K[S]$ of X . Then

$$M_1 = (v - 1) + (\{v^\alpha w^\beta (w - 1) \mid (\alpha, \beta), (\alpha, \beta + 1) \in S\})$$

is the maximal ideal of $K[S]$ corresponding to $V_+(\mathfrak{p})$. Consider the following affine openset of the blow-up at M_1 :

$$A := K[S] \left[\left\{ \frac{v^\alpha w^\beta (w - 1)}{v - 1} \mid (\alpha, \beta), (\alpha, \beta + 1) \in S \right\} \right] \subset K[v^{\pm 1}, w^{\pm 1}, \frac{w - 1}{v - 1}].$$

Then $\text{Spec } A$ is an affine open subset of Y . The defining ideal of C is

$$\left(\left\{ \frac{v^\alpha w^\beta (w - 1)}{v - 1} \mid (\alpha, \beta), (\alpha, \beta + 1) \in S \right\} \right) A.$$

Consider the affine openset $U_2 = \text{Spec } K[T]$ of X . Then

$$M_2 = (v^{-1}w - 1) + (\{v^\alpha w^\beta (w - 1) \mid (\alpha, \beta), (\alpha, \beta + 1) \in T\})$$

is the maximal ideal of $K[T]$ corresponding to $V_+(\mathfrak{p})$. Here remark $\bar{t} < -1 < \bar{u} < 0$ and $v^{-1}w - 1 \in K[T]$. Consider the following affine openset of the blow-up at M_2 :

$$B := K[T] \left[\left\{ \frac{v^\alpha w^\beta (w-1)}{v^{-1}w-1} \mid (\alpha, \beta), (\alpha, \beta+1) \in T \right\} \right] \subset K[v^{\pm 1}, w^{\pm 1}, \frac{w-1}{v^{-1}w-1}].$$

Then $\text{Spec } B$ is an affine open subset of Y . The defining ideal of C is

$$\left(\left\{ \frac{v^\alpha w^\beta (w-1)}{v^{-1}w-1} \mid (\alpha, \beta), (\alpha, \beta+1) \in T \right\} \right) B.$$

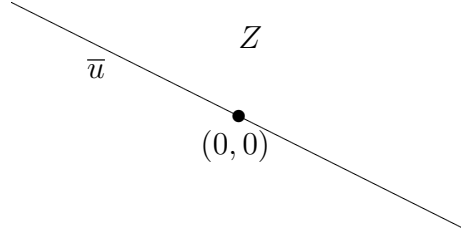
Here it is easy to see

$$\pi^{-1}(U_1) \cap \ell C \subset \text{Spec } A, \quad \pi^{-1}(U_2) \cap \ell C \subset \text{Spec } B$$

for any $\ell > 0$. By (3.3),

$$(4.1) \quad \ell C = (\text{Spec } A)|_{\ell C} \cup (\text{Spec } B)|_{\ell C}$$

is an affine open covering of ℓC .



Let Z be the cone $\mathbb{R}(u, -u_2) + \mathbb{R}_{\geq 0}(0, 1)$ as above. Put

$$x = \frac{w-1}{v-1}, \quad y = \frac{w-1}{v^{-1}w-1}.$$

Put

$$K[Z] = \bigoplus_{(\alpha, \beta) \in Z \cap \mathbb{Z}^2} K v^\alpha w^\beta \subset K[v^{\pm 1}, w^{\pm 1}],$$

$$L = K[Z] \left[x, y, \frac{v-1}{v^{-1}w-1}, \frac{v^{-1}w-1}{v-1} \right] \subset K[v^{\pm 1}, w^{\pm 1}] \left[\frac{1}{v-1}, \frac{1}{v^{-1}w-1} \right],$$

$$F = K[Z][x] \subset L,$$

$$G = K[Z][y] \subset L.$$

Here remark that $v, w, v^{-1}w$ are contained in $K[Z]$ since $-1 < \bar{u} < 0$. Since

$$(4.2) \quad \frac{v^{-1}w-1}{v-1} = -v^{-1}w + x \in F,$$

the map $F \hookrightarrow L$ is a localization. Since

$$\frac{v-1}{v^{-1}w-1} = -v + y \in G,$$

the map $G \hookrightarrow L$ is a localization.

Let ℓ be a positive integer. We know that w is a unit of $F/x^\ell F$ since $F/xF = K[Z]/(w-1)$. We know that $v^{-1}w$ is a unit of $F/x^\ell F$ since $w = (v^{-1}w)v$. Since x is a nilpotent in $F/x^\ell F$, $\frac{v^{-1}w-1}{v-1}$ is a unit in $F/x^\ell F$ by (4.2). We can show that $\frac{v-1}{v^{-1}w-1}$ is a unit in $G/y^\ell G$ in the same way. Thus we obtain isomorphisms

$$(4.3) \quad F/x^\ell F \simeq L/x^\ell L = L/y^\ell L \simeq G/y^\ell G$$

for any $\ell > 0$.

For $\alpha \in \mathbb{Z}$ and $n \geq 0$, we define

$$x_{\alpha,n} := v^\alpha w^{\lceil \alpha \bar{u} \rceil} x^n \in F,$$

where $\lceil \alpha \bar{u} \rceil$ is the least integer such that $\lceil \alpha \bar{u} \rceil \geq \alpha \bar{u}$.

Proposition 4.1. *We have*

$$(4.4) \quad F = \bigoplus_{\alpha \in \mathbb{Z}} \bigoplus_{n \geq 0} Kx_{\alpha,n}$$

$$(4.5) \quad x^\ell F = \bigoplus_{\alpha \in \mathbb{Z}} \bigoplus_{n \geq \ell} Kx_{\alpha,n}$$

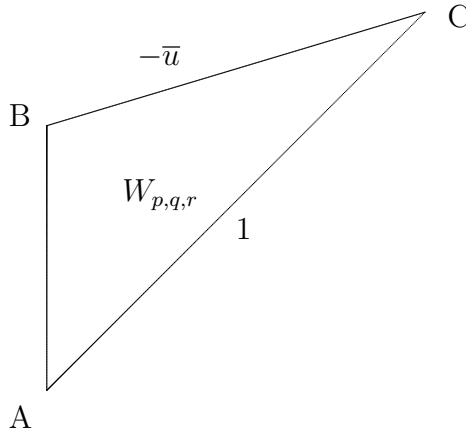
for any positive integer ℓ .

Proof. We have

$$F = K[Z][x] = \sum_{(\alpha,\beta) \in \mathbb{Z} \cap \mathbb{Z}^2} \sum_{n \geq 0} K v^\alpha w^\beta x^n \supset \sum_{\alpha \in \mathbb{Z}} \sum_{n \geq 0} Kx_{\alpha,n}$$

by definition.

We shall prove the opposite containment. For rational numbers p, q, r such that $q < r$, let $W_{p,q,r}$ be the triangle with vertices $A(p, q)$, $B(p, r)$, $C(\frac{r+p-q+p\bar{u}}{1+\bar{u}}, \frac{r+q\bar{u}}{1+\bar{u}})$.



The slope of the lower edge of $W_{p,q,r}$ is 1, and that of the upper edge is $-\bar{u}$. We shall prove the following claim:

Claim 4.2. *Let α, β, n be integers such that $\beta \geq \alpha\bar{u}$ and $n \geq 0$. Then we have¹*

$$v^\alpha w^\beta x^n \in \sum_{(\gamma, m) \in W_{\alpha, n, \beta - \alpha\bar{u} + n} \cap \mathbb{Z}^2} Kx_{\gamma, m},$$

where the coefficient of $x_{\alpha, n}$ in $v^\alpha w^\beta x^n$ is 1.

First suppose $0 \leq \beta - \alpha\bar{u} < 1$. Then $v^\alpha w^\beta x^n$ is equal to $x_{\alpha, n}$ since $\beta = \lceil \alpha\bar{u} \rceil$.

Next suppose $\beta - \alpha\bar{u} \geq 1$. Since $w^\beta = w^{\beta-1} + w^{\beta-1}(w-1)$, we have

$$(4.6) \quad v^\alpha w^\beta x^n = v^\alpha w^{\beta-1} x^n + v^\alpha w^{\beta-1} (v-1) x^{n+1} = v^\alpha w^{\beta-1} x^n + v^{\alpha+1} w^{\beta-1} x^{n+1} - v^\alpha w^{\beta-1} x^{n+1}.$$

We assume that this claim is true if $\beta - \alpha\bar{u}$ is smaller. Then we know

$$\begin{aligned} v^\alpha w^{\beta-1} x^n &\in \sum_{(\gamma, m) \in W_{\alpha, n, \beta-1-\alpha\bar{u}+n} \cap \mathbb{Z}^2} Kx_{\gamma, m}, \\ v^{\alpha+1} w^{\beta-1} x^{n+1} &\in \sum_{(\gamma, m) \in W_{\alpha+1, n+1, \beta-1-(\alpha+1)\bar{u}+n+1} \cap \mathbb{Z}^2} Kx_{\gamma, m}, \\ v^\alpha w^{\beta-1} x^{n+1} &\in \sum_{(\gamma, m) \in W_{\alpha, n+1, \beta-1-\alpha\bar{u}+n+1} \cap \mathbb{Z}^2} Kx_{\gamma, m}. \end{aligned}$$

It is easy to see that triangles $W_{\alpha, n, \beta-1-\alpha\bar{u}+n}$, $W_{\alpha+1, n+1, \beta-1-(\alpha+1)\bar{u}+n+1}$ and $W_{\alpha, n+1, \beta-1-\alpha\bar{u}+n+1}$ are contained in $W_{\alpha, n, \beta-\alpha\bar{u}+n}$. Thus we have proved the claim by (4.6).

In the rest of this proof, we shall show that $x_{\alpha, n}$'s are linearly independent over K . Suppose

$$(4.7) \quad \sum_{\alpha, n} c_{\alpha, n} x_{\alpha, n} = 0 \quad (c_{\alpha, n} \in K).$$

Let γ be the minimal number which satisfies $c_{\gamma, m} \neq 0$ for some $m \geq 0$. We think

$$x_{\alpha, n} \in F \subset K[[v, w]][v^{-1}, w^{-1}].$$

Taking the coefficient of v^γ of (4.7), we have

$$\sum_{n \geq 0} c_{\gamma, n} w^{\lceil \gamma\bar{u} \rceil} (1-w)^n = 0.$$

It implies $c_{\gamma, n} = 0$ for any $n \geq 0$. It is a contradiction. We have proved (4.4) in Proposition 4.1.

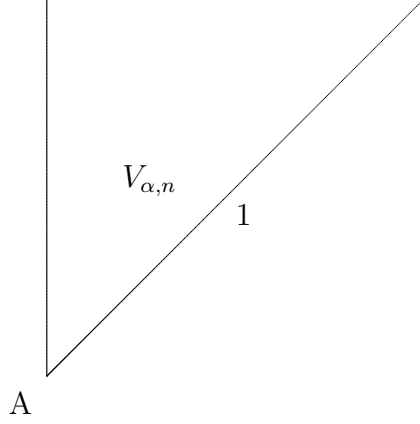
We have

$$x^\ell F = \sum_{(\alpha, \beta) \in Z \cap \mathbb{Z}^2} \sum_{n \geq \ell} K v^\alpha w^\beta x^n \supset \sum_{\alpha \in \mathbb{Z}} \sum_{n \geq \ell} K x_{\alpha, n} = \bigoplus_{\alpha \in \mathbb{Z}} \bigoplus_{n \geq \ell} K x_{\alpha, n}.$$

Using Claim 4.2, we can prove the opposite containment. □

¹If the characteristic of K is 0, we can prove $v^\alpha w^\beta x^n \in \sum_{(\gamma, m) \in W_{\alpha, n, \beta - \alpha\bar{u} + n} \cap \mathbb{Z}^2} \mathbb{Z} x_{\gamma, m}$.

Remark 4.3. For integers α, n , let $V_{\alpha,n}$ be the cone with vertex $A(\alpha, n)$ as below:



By Claim 4.2, we have²

$$(4.8) \quad x_{\alpha,n}x_{\alpha',n'} = v^{\alpha+\alpha'}w^{[\alpha\bar{u}]+[\alpha'\bar{u}]}x^{n+n'} \in \sum_{(\gamma,m) \in V_{\alpha+\alpha',n+n'} \cap \mathbb{Z}^2} Kx_{\gamma,m},$$

where the coefficient of $x_{\alpha+\alpha',n+n'}$ is 1. Here remark $[(\alpha + \alpha')\bar{u}] \leq [\alpha\bar{u}] + [\alpha'\bar{u}]$.

Proposition 4.4. *We have the following equalities:*

$$(4.9) \quad A = \sum_{\substack{n \geq 0 \\ (\alpha, \beta), (\alpha, \beta + n) \in S}} K v^{\alpha} w^{\beta} x^n = \bigoplus_{\alpha \geq 0} \bigoplus_{\substack{n \geq 0 \\ (\alpha, [\alpha\bar{u}] + n) \in S}} K x_{\alpha,n}$$

$$(4.10) \quad x^{\ell} F \cap A = \sum_{\substack{n \geq \ell \\ (\alpha, \beta), (\alpha, \beta + n) \in S}} K v^{\alpha} w^{\beta} x^n = \bigoplus_{\alpha \geq 0} \bigoplus_{\substack{n \geq \ell \\ (\alpha, [\alpha\bar{u}] + n) \in S}} K x_{\alpha,n}$$

for any positive integer ℓ .

Proof. First we shall prove

$$A = \sum_{\substack{n \geq 0 \\ (\alpha, \beta), (\alpha, \beta + n) \in S}} K v^{\alpha} w^{\beta} x^n.$$

It is easy to see that the right hand side is a subring of $K[v^{\pm 1}, w^{\pm 1}, \frac{1}{v-1}]$. Therefore A is included in the right hand side.

Suppose $n \geq 0$ and $(\alpha, \beta), (\alpha, \beta + n) \in S$. We shall prove $v^{\alpha} w^{\beta} x^n \in A$ by induction on n . It follows by definition when $n = 0, 1$. Assume $n \geq 2$. Then, by the induction hypothesis,

$$v^{\alpha+1} w^{\beta} x^n - v^{\alpha} w^{\beta} x^n = v^{\alpha}(v-1)w^{\beta} x^n = v^{\alpha} w^{\beta} (w-1)x^{n-1} = v^{\alpha} w^{\beta+1} x^{n-1} - v^{\alpha} w^{\beta} x^{n-1} \in A$$

²If the characteristic of K is 0, we can prove $x_{\alpha,n}x_{\alpha',n'} = v^{\alpha+\alpha'}w^{[\alpha\bar{u}]+[\alpha'\bar{u}]}x^{n+n'} \in \sum_{(\gamma,m) \in V_{\alpha+\alpha',n+n'} \cap \mathbb{Z}^2} \mathbb{Z}x_{\gamma,m}$.

since $(\alpha, \beta + 1), (\alpha, \beta + n), (\alpha, \beta), (\alpha, \beta + n - 1) \in S$. Therefore it is enough to show $v^{\alpha+1}w^\beta x^n \in A$. Here remark $(\alpha + 1, \beta), (\alpha + 1, \beta + n) \in S$. By this argument, it is enough to show $v^{\alpha+q}w^\beta x^n \in A$ for $q \gg 0$. Suppose $(q, 1) \in S$. Then $v^q x \in A$ since $(q, 0), (q, 1) \in S$. We have

$$v^{\alpha+q}w^\beta x^n = (v^q x)(v^\alpha w^\beta x^{n-1}) \in A.$$

Next we shall prove

$$\sum_{\substack{n \geq 0 \\ (\alpha, \beta), (\alpha, \beta + n) \in S}} K v^\alpha w^\beta x^n = \bigoplus_{\alpha \geq 0} \bigoplus_{\substack{n \geq 0 \\ (\alpha, \lceil \alpha \bar{u} \rceil + n) \in S}} K x_{\alpha, n}.$$

Here remark that $x_{\alpha, n}$'s are linearly independent by Proposition 4.1. If $\alpha \geq 0$, we have $(\alpha, \lceil \alpha \bar{u} \rceil) \in S$. Therefore the right hand side is included in the left one. Next we shall prove the opposite containment. Suppose $n \geq 0$ and $(\alpha, \beta), (\alpha, \beta + n) \in S$. We shall show that $v^\alpha w^\beta x^n$ is in the right hand side. By Claim 4.2, we have

$$v^\alpha w^\beta x^n \in \sum_{(\gamma, m) \in W_{\alpha, n, \beta - \alpha \bar{u} + n} \cap \mathbb{Z}^2} K x_{\gamma, m}.$$

It is enough to show $\gamma \geq 0$, $m \geq 0$ and $(\gamma, \lceil \gamma \bar{u} \rceil + m) \in S$ for any $(\gamma, m) \in W_{\alpha, n, \beta - \alpha \bar{u} + n} \cap \mathbb{Z}^2$. Remember $\gamma \geq \alpha \geq 0$ and $m \geq n \geq 0$. Put $q = \gamma - \alpha \geq 0$. Since $(\gamma, m) \in W_{\alpha, n, \beta - \alpha \bar{u} + n}$, we have

$$0 \leq m \leq \beta - \alpha \bar{u} + n - q \bar{u} = \beta - \gamma \bar{u} + n.$$

Adding $\lceil \gamma \bar{u} \rceil$, we have

$$\lceil \gamma \bar{u} \rceil \leq \lceil \gamma \bar{u} \rceil + m \leq \beta + n + (\lceil \gamma \bar{u} \rceil - \gamma \bar{u}).$$

Since $0 \leq \lceil \gamma \bar{u} \rceil - \gamma \bar{u} < 1$, we have

$$\lceil \gamma \bar{u} \rceil \leq \lceil \gamma \bar{u} \rceil + m \leq \beta + n.$$

Since $(\alpha, \beta + n) \in S$, we know $(\gamma, \beta + n) \in S$. Therefore $(\gamma, \lceil \gamma \bar{u} \rceil + m) \in S$. We have completed the proof of (4.9).

(4.10) follows from (4.4), (4.5) and (4.9). \square

For $\alpha \in \mathbb{Z}$ and $n \geq 0$, we define

$$y_{\alpha, n} := v^\alpha w^{\lceil \alpha \bar{u} \rceil} y^n \in G.$$

Proposition 4.5. *We have*

$$(4.11) \quad G = \bigoplus_{\alpha \in \mathbb{Z}} \bigoplus_{n \geq 0} K y_{\alpha, n}$$

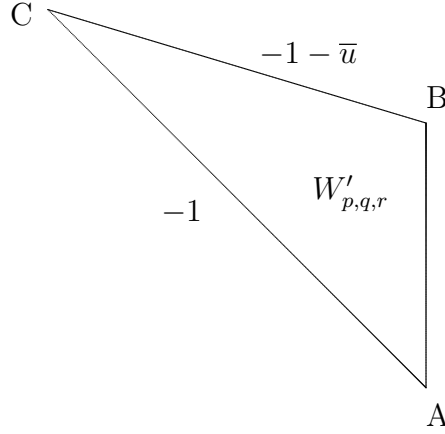
$$(4.12) \quad y^\ell G = \bigoplus_{\alpha \in \mathbb{Z}} \bigoplus_{n \geq \ell} K y_{\alpha, n}$$

for any positive integer ℓ .

Proof. We have

$$G = K[Z][y] = \sum_{(\alpha,\beta) \in \mathbb{Z} \cap \mathbb{Z}^2} \sum_{n \geq 0} K v^\alpha w^\beta y^n \supset \sum_{\alpha \in \mathbb{Z}} \sum_{n \geq 0} K y_{\alpha,n}.$$

We shall prove the opposite containment. For rational numbers p, q, r such that $q < r$, let $W'_{p,q,r}$ be the triangle with vertices $A(p, q)$, $B(p, r)$, $C(\frac{r-q+p\bar{u}}{\bar{u}}, \frac{q-r+q\bar{u}}{\bar{u}})$.



The slope of the lower edge of $W'_{p,q,r}$ is -1 , and that of the upper edge is $-1 - \bar{u}$.

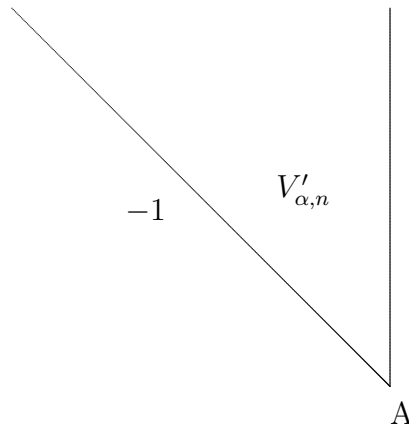
Let α, β, n be integers such that $\beta \geq \alpha\bar{u}$ and $n \geq 0$. Then we can prove³

$$v^\alpha w^\beta y^n \in \sum_{(\gamma,m) \in W'_{\alpha,n,\beta-\alpha\bar{u}+n} \cap \mathbb{Z}^2} K y_{\gamma,m}$$

in the same way as in Claim 4.2.

Thus (4.11) and (4.12) is proved as Proposition 4.4. \square

Remark 4.6. For integers α, n , let $V'_{\alpha,n}$ be the cone with vertex $A(\alpha, n)$ as below:



³The coefficient of $y_{\alpha,n}$ is 1. If the characteristic of K is 0, we can prove $v^\alpha w^\beta y^n \in \sum_{(\gamma,m) \in W'_{\alpha,n,\beta-\alpha\bar{u}+n} \cap \mathbb{Z}^2} \mathbb{Z} y_{\gamma,m}$.

By the proof of Proposition 4.5, we have⁴

$$y_{\alpha,n}y_{\alpha',n'} = v^{\alpha+\alpha'}w^{[\alpha\bar{u}]+[\alpha'\bar{u}]}y^{n+n'} \in \sum_{(\gamma,m) \in V'_{\alpha+\alpha',n+n'} \cap \mathbb{Z}^2} Ky_{\gamma,m},$$

where the coefficient of $y_{\alpha+\alpha',n+n'}$ is 1. Here remark $[(\alpha + \alpha')\bar{u}] \leq [\alpha\bar{u}] + [\alpha'\bar{u}]$.

Proposition 4.7. *We have the following equalities:*

$$(4.13) \quad B = \sum_{\substack{n \geq 0 \\ (\alpha, \beta), (\alpha, \beta + n) \in T}} K v^\alpha w^\beta y^n = \bigoplus_{\alpha \leq 0} \bigoplus_{\substack{n \geq 0 \\ (\alpha, [\alpha\bar{u}] + n) \in T}} Ky_{\alpha,n}$$

$$(4.14) \quad x^\ell G \cap B = \sum_{\substack{n \geq \ell \\ (\alpha, \beta), (\alpha, \beta + n) \in T}} K v^\alpha w^\beta y^n = \bigoplus_{\alpha \leq 0} \bigoplus_{\substack{n \geq \ell \\ (\alpha, [\alpha\bar{u}] + n) \in T}} Ky_{\alpha,n}$$

for any positive integer ℓ .

We omit a proof since we can prove it in the same way as Proposition 4.4.

We put

$$\begin{aligned} F_\ell &= F/x^\ell F, \\ G_\ell &= G/y^\ell G, \\ A_\ell &= A/(x^\ell F \cap A), \\ B_\ell &= B/(y^\ell G \cap B). \end{aligned}$$

By (4.3), we have

$$B_\ell \subset G_\ell \simeq F_\ell \supset A_\ell.$$

Let

$$\psi : B_\ell \rightarrow F_\ell$$

be the above inclusion. Then, since $-v^{-1}(1-x) = \frac{v^{-1}w-1}{v-1}$, we know

$$\psi(y) = -\frac{vx}{1-x}$$

and

$$\psi(y_{\alpha,n}) = v^\alpha w^{[\alpha\bar{u}]} \left(-\frac{vx}{1-x} \right)^n = (-1)^n v^{\alpha+n} w^{[\alpha\bar{u}]} (x + x^2 + x^3 + \dots)^n.$$

For $\alpha \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, we put

$$z_{\alpha,n} := (-1)^n \psi(y_{\alpha-n,n}) = v^\alpha w^{[(\alpha-n)\bar{u}]} (x + x^2 + x^3 + \dots)^n.$$

Since $n \geq 0$, $[(\alpha - n)\bar{u}] \geq [\alpha\bar{u}]$. By Claim 4.2, we can describe⁵

$$(4.15) \quad z_{\alpha,n} = \sum_{(\gamma,m) \in V_{\alpha,n} \cap \mathbb{Z}^2} c_{\gamma,m} x_{\gamma,m} \quad (c_{\gamma,m} \in K)$$

⁴If the characteristic of K is 0, we can prove $y_{\alpha,n}y_{\alpha',n'} = v^{\alpha+\alpha'}w^{[\alpha\bar{u}]+[\alpha'\bar{u}]}y^{n+n'} \in \sum_{(\gamma,m) \in V'_{\alpha+\alpha',n+n'} \cap \mathbb{Z}^2} \mathbb{Z}y_{\gamma,m}$.

⁵If the characteristic of K is 0, we can prove $c_{\gamma,m} \in \mathbb{Z}$.

with $c_{\alpha,n} = 1$. Then, we have

$$(4.16) \quad F_\ell = \bigoplus_{\alpha \in \mathbb{Z}} \bigoplus_{\ell > n \geq 0} Kx_{\alpha,n},$$

$$A_\ell = \bigoplus_{\alpha \geq 0} \bigoplus_{\substack{\ell > n \geq 0 \\ (\alpha, [\alpha\bar{u}] + n) \in S}} Kx_{\alpha,n},$$

$$(4.17) \quad \psi(B_\ell) = \bigoplus_{\alpha \in \mathbb{Z}} \bigoplus_{\substack{\ell > n \geq 0 \\ (\alpha, [\alpha\bar{u}] + n) \in T}} Ky_{\alpha,n} = \bigoplus_{\alpha \in \mathbb{Z}} \bigoplus_{\substack{\ell > n \geq 0 \\ (\alpha - n, [(\alpha - n)\bar{u}] + n) \in T}} Kz_{\alpha,n}.$$

We have

$$(\text{Spec } A)|_{\ell C} = \text{Spec } A_\ell, \quad (\text{Spec } B)|_{\ell C} = \text{Spec } B_\ell$$

and

$$\ell C = \text{Spec } A_\ell \cup \text{Spec } B_\ell, \quad \text{Spec } F_\ell = \text{Spec } A_\ell \cap \text{Spec } B_\ell$$

by (4.1).

By Theorem 3.2, it is very important whether $\mathcal{O}_Y(m(abH - uE))|_{\ell C}$ is a free sheaf or not. We denote $\mathcal{O}_Y(m(abH - uE))|_{\ell C}$ by $\mathcal{L}_{m,\ell}$. As we have seen just before Theorem 3.2, it is a locally free sheaf on ℓC . Here put

$$\xi := v^u w^{-u_2} \left(\frac{v^{-1}w - 1}{1 - v} \right)^u = \left(\frac{w - v}{1 - v} \right)^u w^{-u_2} = (1-x)^u (1-x+vx)^{-u_2} = (1-x_{0,1})^u (1-x_{0,1}+x_{1,1})^{-u_2}.$$

Remark that the constant term of ξ is 1. We know that

$$(4.18) \quad \begin{aligned} & \bullet \mathcal{L}_{m,\ell}|_{\text{Spec } A_\ell} \simeq \mathcal{O}_{\text{Spec } A_\ell}, \\ & \bullet \mathcal{L}_{m,\ell}|_{\text{Spec } B_\ell} \simeq \mathcal{O}_{\text{Spec } B_\ell}, \\ & \bullet \text{ the transition function of } \mathcal{L}_{m,\ell} \text{ is } \xi^m \in F_\ell^\times. \end{aligned}$$

Here $()^\times$ is the set of unit elements in the given ring. Thus we obtain the following:

Proposition 4.8. $\mathcal{L}_{m,\ell} \simeq \mathcal{O}_{\ell C} \iff \exists \xi_A \in A_\ell^\times, \exists \xi_B \in B_\ell^\times$ such that $\xi^m = \xi_A \xi_B \in F_\ell^\times$

Remark 4.9. Let m be a positive integer. Then the following conditions are equivalent:

- (1) $\mathcal{O}_Y(m(abH - uE))|_{muC} \simeq \mathcal{O}_{muC}$.
- (2) $\exists \xi_A \in A_{mu}^\times, \exists \xi_B \in B_{mu}^\times$ such that $\xi^m = \xi_A \xi_B \in F_{mu}^\times$.
- (3) There exists $g \in [\mathfrak{p}^{(mu)}]_{mab}$ such that $z^u - x^{s_3}y^{t_3}$, g satisfying Huneke's criterion [13], that is, $\ell_P(P/(z^u - x^{s_3}y^{t_3}, g, x)) = mua$ holds.
- (4) There exists an effective Weil divisor D such that $D \sim m(abH - uE)$ and $C \cap D = \emptyset$.

5. STRATEGY TO PROVE FINITE/INFINITE GENERATION

For $i = 1, 2, \dots, mu$, we put

$$q_i = \# \{(\alpha, \beta) \in m\Delta_{\bar{i}, \bar{u}, \bar{s}} \cap \mathbb{Z}^2 \mid \alpha = i\}.$$

Note that $q_{mu} = 1$ and $q_i \geq 1$ for all $i = 1, 2, \dots, mu$. We sort the sequence q_1, q_2, \dots, q_{mu} into ascending order

$$(5.1) \quad q'_1 \leq q'_2 \leq \dots \leq q'_{mu}.$$

We say that the condition EMU is satisfied for $m\Delta_{\bar{t}, \bar{u}, \bar{s}}$ if

$$q'_i \geq i$$

for $i = 1, 2, \dots, mu$. If the condition EMU for $m\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is satisfied for some $m > 0$, then $R_s(\mathfrak{p})$ is Noetherian. One can prove it in the same way as the proof of Proposition 4.6 in [16].

By definition, the condition EMU is satisfied for (a, b, c) iff the condition EMU is satisfied for $\Delta_{\bar{t}, \bar{u}, \bar{s}}$.

Let (a, b, c) be $(53, 48, 529)$. It satisfies (3.1). It is not difficult to prove that the condition EMU is satisfied for $(53, 48, 529)$. Therefore $R_s(\mathfrak{p})$ is Noetherian. However the condition EMU for $m\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is not satisfied for any $m \geq 2$.

The following lemma is proved by Zhuang He. We give a proof of it for a reader below.

Lemma 5.1 (He [11]). *Assume that the condition EMU for $m\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is not satisfied. Then there exists an integer d satisfying $1 \leq d < mu$ such that the sequence $q'_1, q'_2, \dots, q'_{mu}$ defined in (5.1) coincides with*

$$1, 2, 3, \dots, d-1, d, d, \dots$$

Proof. For $i \in \mathbb{Z}$, we put

$$(5.2) \quad \begin{aligned} a_i &= \#\{(\alpha, \beta) \in S \cap \mathbb{Z}^2 \mid \alpha = i\} \\ b_i &= \#\{(\alpha, \beta) \in T \cap \mathbb{Z}^2 \mid \alpha = i\}. \end{aligned}$$

By definition, we have

$$\dots \geq b_{-3} \geq b_{-2} \geq b_{-1} \geq 1 \leq a_1 \leq a_2 \leq a_3 \leq \dots$$

and

$$\lim_{i \rightarrow \infty} a_i = \lim_{i \rightarrow \infty} b_{-i} = \infty.$$

Here we remark $a_1 > 0$ and $b_{-1} > 0$ since $\bar{t} < -1 < \bar{u} < 0 < \bar{s}$. We sort the sequence $1, a_1, b_{-1}, a_2, b_{-2}, a_3, b_{-3}, a_4, b_{-4}, \dots$ into ascending order

$$c_1 \leq c_2 \leq \dots \leq c_i \leq \dots$$

For $i > 0$, let e_i be the positive integer such that $c_{e_i} \leq i < c_{e_i+1}$. Then it is easy to see the following:

$$(5.3) \quad c_i = q'_i \text{ for } i = 1, 2, \dots, mu, \text{ where } q'_i \text{ is defined in (5.1).}$$

$$(5.4) \quad \text{The condition EMU for } m\Delta_{\bar{t}, \bar{u}, \bar{s}} \text{ is not satisfied if and only if there exists } i \text{ satisfying } 0 < i < mu \text{ and } e_i > i.$$

By definition, we have

$$\begin{aligned} a_i &= [i\bar{s}] - [i\bar{u}] + 1 \\ b_{-i} &= [-i\bar{t}] - [-i\bar{u}] + 1 \end{aligned}$$

for $i > 0$, where $[i\bar{s}]$ is the maximal integer such that $i\bar{s} \geq [i\bar{s}]$. Since

$$\begin{aligned} [(i+j)\bar{s}] &\geq [i\bar{s}] + [j\bar{s}] \\ [(i+j)\bar{u}] &\leq [i\bar{u}] + [j\bar{u}], \end{aligned}$$

we obtain

$$(5.5) \quad \begin{aligned} a_{i+j} &\geq a_i + a_j - 1 \\ b_{-(i+j)} &\geq b_{-i} + b_{-j} - 1 \end{aligned}$$

for positive integers i and j .

If $a_1 = 1$ or $b_{-1} = 1$, then $q'_1 = q'_2 = 1$. Here remark $u \geq 2$ and (5.3).

Next assume $a_1 \geq 2$ and $b_{-1} \geq 2$. By (5.5), we have

$$1 < a_1 < a_2 < a_3 < \cdots$$

and

$$1 < b_{-1} < b_{-2} < b_{-3} < \cdots .$$

Since the condition EMU for $m\Delta_{\bar{i}, \bar{u}, \bar{s}}$ is not satisfied, the set

$$\{a_1, a_2, a_3, \dots\} \cap \{b_{-1}, b_{-2}, b_{-3}, \dots\}$$

is not empty. Let d be the minimal number of the above set. Since the condition EMU for $m\Delta_{\bar{i}, \bar{u}, \bar{s}}$ is not satisfied, we know $d < mu$. Suppose $a_f = b_{-f'} = d$. Then

$$(5.6) \quad \{a_1, a_2, a_3, \dots, a_{f-1}\} \coprod \{b_{-1}, b_{-2}, b_{-3}, \dots, b_{-(f'-1)}\} \subset \{2, 3, \dots, d-1\}.$$

By (5.3), it is enough to show that the above is the equality, that is, $f + f' - 2 = d - 2$. Assume the contrary, i.e. $d \geq f + f' + 1$.

By (5.5), we have $a_{nf} \geq nd - (n-1)$ and $b_{-nf'} \geq nd - (n-1)$. Since $d \geq f + f' + 1$, we have $nd - (n-1) \geq n(f + f') + 1$. Therefore we know

$$(5.7) \quad a_{nf} \geq n(f + f') + 1, \quad b_{-nf'} \geq n(f + f') + 1.$$

Hence we have

$$e_{n(f+f')+1} \leq n(f + f') + 1, \quad e_{n(f+f')} \leq n(f + f') - 1$$

for any $n \geq 1$ by (5.3). By (5.5) and (5.7), we have

$$a_{nf+1} \geq n(f + f') + a_1, \quad a_{nf+2} \geq n(f + f') + a_2, \quad \dots, \quad a_{nf+(f-1)} \geq n(f + f') + a_{f-1}$$

and

$$b_{-(nf'+1)} \geq n(f + f') + b_{-1}, \quad b_{-(nf'+2)} \geq n(f + f') + b_{-2}, \quad \dots, \quad b_{-(nf'+(f'-1))} \geq n(f + f') + b_{-(f'-1)}.$$

Then, by (5.6), we know

$$e_{n(f+f')+2} \leq n(f + f') + 2, \quad e_{n(f+f')+3} \leq n(f + f') + 3, \quad \dots, \quad e_{(n+1)(f+f')-1} \leq (n+1)(f + f') - 1.$$

It contradicts to (5.4). \square

Definition 5.2. Assume that the condition EMU for $m\Delta_{\bar{t},\bar{u},\bar{s}}$ is not satisfied. The integer d as in Lemma 5.1 is called the minimal degree of $m\Delta_{\bar{t},\bar{u},\bar{s}}$.

The above definition is due to He [11].

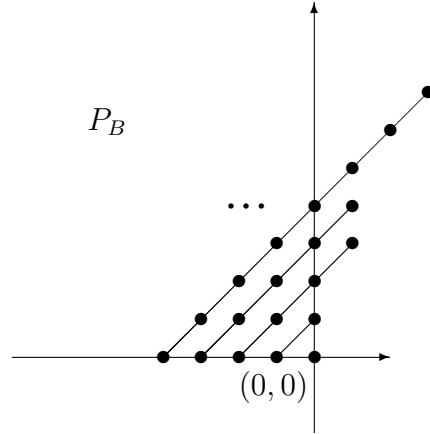
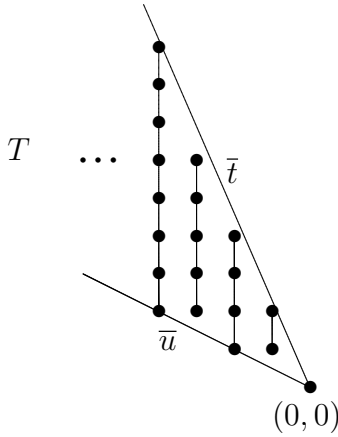
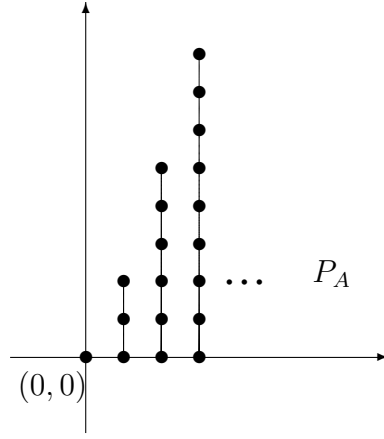
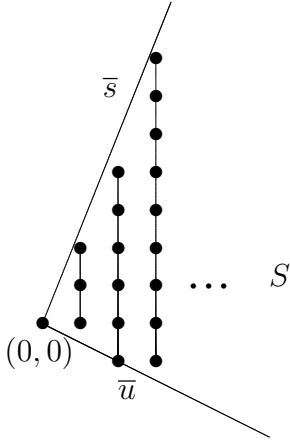
For integers p, q , we put

$$[p, q]_{\mathbb{Z}} := \{n \in \mathbb{Z} \mid p \leq n \leq q\}, \quad [p, q)_{\mathbb{Z}} := \{n \in \mathbb{Z} \mid p \leq n < q\}.$$

We put

$$P_A := \left\{ (\alpha, n) \in \mathbb{Z}^2 \mid \begin{array}{l} \alpha \geq 0, n \geq 0, \\ (\alpha, \lceil \alpha \bar{u} \rceil + n) \in S \end{array} \right\} = \left\{ (\alpha, n) \in \mathbb{Z}^2 \mid \begin{array}{l} \alpha \geq 0, n \geq 0, \\ a_\alpha \geq n + 1 \end{array} \right\},$$

$$P_B := \left\{ (\alpha, n) \in \mathbb{Z}^2 \mid \begin{array}{l} \alpha \in \mathbb{Z}, n \geq 0, \\ (\alpha - n, \lceil (\alpha - n) \bar{u} \rceil + n) \in T \end{array} \right\} = \left\{ (\alpha, n) \in \mathbb{Z}^2 \mid \begin{array}{l} \alpha \in \mathbb{Z}, n \geq 0, \\ b_{\alpha-n} \geq n + 1 \end{array} \right\}.$$



Lemma 5.3. (1) Let m be a positive integer. The following two conditions are equivalent:

- (a) The condition EMU is satisfied for $m\Delta_{\bar{t},\bar{u},\bar{s}}$.
- (b) $\mathbb{Z} \times [0, mu)_{\mathbb{Z}} \subset P_A \cup P_B$.

- (2) The set $\mathbb{N}_0^2 \setminus (P_A \cup P_B)$ is finite.
- (3) Assume that the condition EMU for $m\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is not satisfied. Then there exist integers d and f such that
- $mu > d \geq f \geq 0$,
 - $(f, d) \notin P_A \cup P_B$
 - $P_A \cup P_B \cup \{(f, d)\} \supset [0, f]_{\mathbb{Z}} \times [0, d]_{\mathbb{Z}}$,
 - $P_A \cap P_B \cap ([0, f]_{\mathbb{Z}} \times [0, d]_{\mathbb{Z}}) = \{(0, 0)\}$.

Proof. First we shall prove (1). Remember $u \geq 2$ since \mathfrak{p} is not complete intersection. Therefore the condition (a) implies $a_1 \geq 2$ and $b_{-1} \geq 2$. On the other hand, assume the condition (b). Since $(0, 1) \notin P_A$, we have $(0, 1) \in P_B$ and $b_{-1} \geq 2$. Since $(1, 1) \notin P_B$, we have $(1, 1) \in P_A$ and $a_1 \geq 2$.

From now on, we assume $a_1 \geq 2$ and $b_{-1} \geq 2$. Then we know

$$(5.8) \quad \begin{aligned} & 2 \leq a_1 < a_2 < a_3 < \cdots, \\ & 2 \leq b_{-1} < b_{-2} < b_{-3} < \cdots, \\ & \text{if } (\alpha, n) \in P_A, \text{ then } (\alpha + 1, n), (\alpha + 1, n + 1) \in P_A, \\ & \text{if } (\alpha, n) \in P_B, \text{ then } (\alpha - 1, n), (\alpha, n + 1) \in P_B \end{aligned}$$

by (5.5). We choose q such that $a_q < mu \leq a_{q+1}$. Then, by (5.8), the condition (b) is equivalent to

$$(0, a_0), (1, a_1), (2, a_2), \dots, (q, a_q) \in P_B,$$

where we put $a_0 = b_0 = 1$. It is also equivalent to

$$b_{i-a_i} \geq a_i + 1$$

for $i = 0, 1, 2, \dots, q$. Then, the condition (b) is equivalent to

- $b_{i-a_i} \geq a_i + 1$, $b_{i-a_{i-1}} \geq a_i + 2$, \dots , $b_{(i+1)-a_{i+1}+1} \geq a_{i+1} - 1$ for $i = 0, 1, 2, \dots, q-1$, and
- $b_{q-a_q} \geq a_q + 1$, $b_{q-a_{q-1}} \geq a_q + 2$, \dots , $b_{q-mu+2} \geq mu - 1$.

It is equivalent to the condition (a).

Next we shall prove (2). For $\alpha \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, we know

$$(\alpha, n) \in P_A \iff (\alpha, \lceil \alpha \bar{u} \rceil + n) \in S \iff \begin{cases} \alpha \geq 0 \\ 0 \leq n \leq \alpha \bar{s} - \lceil \alpha \bar{u} \rceil. \end{cases}$$

Since $-\alpha \bar{u} \geq -\lceil \alpha \bar{u} \rceil > -\alpha \bar{u} - 1$, we have

$$(5.9) \quad \left\{ (\alpha, n) \mid \begin{array}{l} \alpha \geq 0 \\ 0 \leq n \leq \alpha(\bar{s} - \bar{u}) - 1 \end{array} \right\} \subset P_A \subset \left\{ (\alpha, n) \mid \begin{array}{l} \alpha \geq 0 \\ 0 \leq n \leq \alpha(\bar{s} - \bar{u}) \end{array} \right\}.$$

Furthermore we have

$$(\alpha, n) \in P_B \iff \begin{cases} n \geq 0 \\ (\alpha - n, \lceil (\alpha - n) \bar{u} \rceil + n) \in T \end{cases} \iff \begin{cases} n \geq \alpha \\ 0 \leq n \leq (\alpha - n) \bar{t} - \lceil (\alpha - n) \bar{u} \rceil. \end{cases}$$

By (3.2), we know $\bar{u} - \bar{t} > 1$. By $-(\alpha - n) \bar{u} \geq -\lceil (\alpha - n) \bar{u} \rceil > -(\alpha - n) \bar{u} - 1$, we have

$$(5.10) \quad \left\{ (\alpha, n) \mid \begin{array}{l} n \geq \alpha, \quad n \geq 0 \\ n \geq \frac{\bar{u} - \bar{t}}{\bar{u} - \bar{t} - 1} \alpha + \frac{1}{\bar{u} - \bar{t} - 1} \end{array} \right\} \subset P_B \subset \left\{ (\alpha, n) \mid \begin{array}{l} n \geq \alpha, \quad n \geq 0 \\ n \geq \frac{\bar{u} - \bar{t}}{\bar{u} - \bar{t} - 1} \alpha \end{array} \right\}.$$

By (3.2), we have

$$1 < \frac{\bar{u} - \bar{t}}{\bar{u} - \bar{t} - 1} < \bar{s} - \bar{u}.$$

Then (2) immediately follows from (5.9) and (5.10).

Next we shall prove (3). If $b_{-1} = 1$, then $(f, d) = (0, 1)$ satisfies our requirements. If $b_{-1} \geq 2$ and $a_1 = 1$, then $(f, d) = (1, 1)$ satisfies our requirements. From now on, we assume $a_1 \geq 2$ and $b_{-1} \geq 2$. Then (5.8) is satisfied. Let d be the minimal degree defined in Definition 5.2. Suppose $a_f = b_{-f'} = d$. Then $d = f + f'$ by Lemma 5.1. We have $(0, 0) \in P_A \cap P_B$ and $(f, d) \notin P_A \cup P_B$ since $a_f = d$ and $b_{f-d} = b_{-f'} = d$.

Since $(0, 1) \in P_B \setminus P_A$, we have $(0, q) \in P_B \setminus P_A$ for any $q > 0$. Since $(f, d) \notin P_B$, we have $(f, q) \in P_A \setminus P_B$ for any $0 \leq q < d$.

Suppose $0 < p < f$. We have $(p, a_p - 1) \in P_A$ and $(p, a_p) \notin P_A$. Since

$$\{b_{-1}, b_{-2}, \dots, b_{-(f'-1)}\} = \{2, 3, \dots, d-1\} \setminus \{a_1, a_2, \dots, a_{f-1}\},$$

we have $a_p - 1 \geq b_{p-a_p+1}$ and $a_p + 1 \leq b_{p-a_p}$. Therefore we have $(p, a_p - 1) \notin P_B$ and $(p, a_p) \in P_B$. Ultimately, we obtain $(p, q) \in P_B \setminus P_A$ if $q \geq a_p$, and $(p, q) \in P_A \setminus P_B$ if $0 \leq q < a_p$. \square

We define

$$F'_\ell := \bigoplus_{\substack{\ell > n \geq 0 \\ (\alpha, n) \in V_{0,0} \cap \mathbb{Z}^2}} Kx_{\alpha, n} \subset F_\ell = F/x^\ell F = \bigoplus_{\alpha \in \mathbb{Z}} \bigoplus_{\ell > n \geq 0} Kx_{\alpha, n}.$$

By (4.8), F'_ℓ is a subring of F_ℓ . Recall (4.16). We define

$$A'_\ell := A_\ell \cap F'_\ell = \bigoplus_{\substack{\ell > n \geq 0 \\ (\alpha, n) \in V_{0,0} \cap \mathbb{Z}^2 \\ (\alpha, \lceil \alpha \bar{u} \rceil + n) \in S}} Kx_{\alpha, n}.$$

Recall (4.17). We define

$$B'_\ell := \psi(B_\ell) \cap F'_\ell = \sum_{\substack{\ell > n \geq 0 \\ (\alpha, n) \in V_{0,0} \cap \mathbb{Z}^2 \\ (\alpha - n, \lceil (\alpha - n) \bar{u} \rceil + n) \in T}} Kz_{\alpha, n}.$$

The above equality follows from (4.15) and (4.17).

Lemma 5.4. *Let ℓ be a positive integer. Suppose $\eta = \eta_A \eta_B$ for $\eta \in F'_\ell^\times$, $\eta_A \in A'_\ell^\times$ and $\eta_B \in B'_\ell^\times$.*

Then we have $\eta_A \in A'_\ell^\times$ and $\eta_B \in B'_\ell^\times$.

Proof. It is easy to see that the constant terms (the coefficient of $x_{0,0}$) of η , η_A and η_B are not 0 by (4.8), (5.9) and (5.10).

Since $\eta_A \eta_B \in F'_\ell$, it follows that η_A and η_B are in F'_ℓ by (4.8), (4.15), (5.9) and (5.10). \square

Definition 5.5. For $0 \leq \alpha \leq n$, we define

$$U_{\alpha,n} = \{(\gamma, m) \in V_{0,0} \cap \mathbb{Z}^2 \mid 0 < m < n\} \cup \{(0, n), (1, n), \dots, (\alpha - 1, n)\}.$$

We say that

$$(5.11) \quad \zeta = \sum_{(\gamma,m) \in V_{0,0} \cap \mathbb{Z}^2} h_{\gamma,m} x_{\gamma,m} \text{ is 0 in } U_{\alpha,n}$$

if $h_{\gamma,m} = 0$ for $(\gamma, m) \in U_{\alpha,n}$.

Remark 5.6. Suppose that the constant term of $\zeta \in F'_\ell$ is 1. Take $(\alpha, n) \in V_{0,0} \setminus \{(0, 0)\}$. We assume that $(\alpha, n) \in P_A \cup P_B$. We do the following procedure:

- (5.12)
 - Let c be the coefficient of $x_{\alpha,n}$ in ζ .
 - If $(\alpha, n) \in P_A$, then we multiply $1 - cx_{\alpha,n}$ to ζ .
 - If $(\alpha, n) \in P_B \setminus P_A$, then we multiply $1 - cz_{\alpha,n}$ to ζ .

Then we know that the element ζ' obtained after the above procedure satisfies

- $\zeta - \zeta'$ is 0 in $U_{\alpha,n}$,
- the coefficient of $x_{\alpha,n}$ in ζ' is 0.

Then we know the following:

- (1) Suppose that $\eta \in F'_\ell^\times$ is 0 in $U_{\alpha,n}$. If $U_{\gamma,m} \setminus U_{\alpha,n} \subset P_A \cup P_B$, there exists $\eta_A \in A'_\ell^\times$ and $\eta_B \in B'_\ell^\times$ such that $\eta\eta_A\eta_B$ is 0 in $U_{\gamma,m}$.
- (2) Suppose that $\eta \in F'_\ell^\times$ is 0 in $U_{\alpha,n}$. Assume that $(\alpha', n) \in P_A$ for $\alpha < \alpha' \leq n$. We know that there exists $\eta_A \in A'_\ell^\times$ such that $\eta\eta_A = 1 + ex_{\alpha,n}$ in $F'_{n+1}{}^\times$ for some $e \in K$.

Remember that ξ is the transition function of $\mathcal{L}_{1,u}$ as in (4.18). The following is a special case of Cutkosky's theorem [1].

Proposition 5.7. Assume $\text{ch}(K) = p > 0$.

Then there exist $\xi_A \in A'_{p^e u}{}^\times$ and $\xi_B \in B'_{p^e u}{}^\times$ such that $\xi^{p^e} = \xi_A \xi_B$ in $F'_{p^e u}{}^\times$ for $e \gg 0$.

In particular $R_s(\mathfrak{p})$ is Noetherian.

Proof. By Lemma 5.3 (2), we may assume that the second coordinate of each points in $\mathbb{N}_0^2 \setminus (P_A \cup P_B)$ is less than p^e . Then ξ^{p^e} is 0 in U_{0,p^e} . By Remark 5.6 (1), we know that $\xi^{p^e} \eta_A \eta_B$ is 0 in $U_{0,p^e u}$ for some $\eta_A \in A'_{p^e u}{}^\times$ and $\eta_B \in B'_{p^e u}{}^\times$. Then we have $\xi^{p^e} = c \eta_A^{-1} \eta_B^{-1}$ in $F'_{p^e u}{}^\times$ for $c \in K^\times$, $\eta_A^{-1} \in A'_{p^e u}{}^\times$ and $\eta_B^{-1} \in B'_{p^e u}{}^\times$. \square

Lemma 5.8. Let K be a field of characteristic 0. Let C be a local K -algebra with $\dim_K C < \infty$. Let \mathfrak{m} be the maximal ideal of C .

Let $\varphi \in C$ is an element such that $\varphi \equiv 1 \pmod{\mathfrak{m}}$. Let n be a positive integer.

Then there exists the unique element $\varphi' \in C$ such that $\varphi' \equiv 1 \pmod{\mathfrak{m}}$ and $\varphi'^n = \varphi$.

We omit a proof. Here we give another proof to Theorem 1.1 in [16].

Proposition 5.9. Assume $\text{ch}(K) = 0$. Let m be a positive integer. Assume that there exist $\xi_A \in A'_{mu}{}^\times$ and $\xi_B \in B'_{mu}{}^\times$ such that $\xi^m = \xi_A \xi_B$ in $F'_{mu}{}^\times$.

Then there exist $\xi_{A,1} \in A'_u{}^\times$ and $\xi_{B,1} \in B'_u{}^\times$ such that $\xi = \xi_{A,1} \xi_{B,1}$ in $F'_u{}^\times$.

Proof. We may assume that the constant terms of ξ_A, ξ_B are 1. By Lemma 5.8, there exist $\xi'_A \in A'_{mu}{}^\times$ and $\xi'_B \in B'_{mu}{}^\times$ such that their constant terms are 1 and $\xi_A{}^m = \xi'_A, \xi_B{}^m = \xi'_B$. Then we have $\xi = \xi'_A \xi'_B$ in $F'_u{}^\times$ by the uniqueness. \square

Lemma 5.10. *Assume that the condition EMU for $m\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is not satisfied. Choose d and f satisfying Lemma 5.3 (3). Suppose that there exist $\eta_A \in A'_{d+1}{}^\times$ and $\eta_B \in B'_{d+1}{}^\times$ such that $\xi^m \eta_A \eta_B = 1 + cx_{f,d}$ in $F'_{d+1}{}^\times$ with $c \in K^\times$.*

Then there do not exist $\xi_A \in A'_{mu}{}^\times$ and $\xi_B \in B'_{mu}{}^\times$ satisfying $\xi^m = \xi_A \xi_B$ in $F'_{mu}{}^\times$.

Proof. Assume the contrary. Then we have

$$\xi_A \xi_B \eta_A \eta_B = (\xi_A \eta_A)(\xi_B \eta_B) = 1 + cx_{f,d}$$

in $F'_{d+1}{}^\times$. Let $U_{\alpha,n}$ be the set defined in Definition 5.5. Assume that both $\xi_A \eta_A$ and $\xi_B \eta_B$ are 0 in $U_{\alpha,n}$, and the coefficient of $x_{\alpha,n}$ in either $\xi_A \eta_A$ or $\xi_B \eta_B$ is not zero for some α and n . Therefore we have $(\alpha, n) \in P_A \cup P_B$ and $(\alpha, n) \neq (f, d)$. Since $1 + cx_{f,d}$ is 0 in $U_{\alpha,n}$, we know that one of the following is satisfied:

- $n < d$
- $\alpha < f$ and $n = d$

Therefore we know that only one of P_A and P_B contains (α, n) . If the coefficient of $x_{\alpha,n}$ in $\xi_A \eta_A$ is not 0, then $(\alpha, n) \in P_A$. Then $(\alpha, n) \notin P_B$ and the coefficient of $x_{\alpha,n}$ in $(\xi_A \eta_A)(\xi_B \eta_B)$ is not 0. It is a contradiction. If the coefficient of $x_{\alpha,n}$ in $\xi_B \eta_B$ is not 0, then $(\alpha, n) \in P_B$. Then $(\alpha, n) \notin P_A$ and the coefficient of $x_{\alpha,n}$ in $(\xi_A \eta_A)(\xi_B \eta_B)$ is not 0. It is a contradiction. \square

Lemma 5.11. *Assume that the condition EMU for $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is not satisfied. Choose d and f satisfying Lemma 5.3 (3) with $m = 1$. Furthermore assume*

$$(5.13) \quad (\mathbb{Z} \times [0, u]_{\mathbb{Z}}) \setminus (P_A \cup P_B) = \{(f, d)\}.$$

Then the following conditions are equivalent:

- (1) *If $\xi \eta_A \eta_B = 1 + cx_{f,d}$ in $F'_{d+1}{}^\times$ is satisfied for some $c \in K$, $\eta_A \in A'_{d+1}{}^\times$ and $\eta_B \in B'_{d+1}{}^\times$, then $c \neq 0$.*
- (2) *There do not exist $\xi_A \in A'_u{}^\times$ and $\xi_B \in B'_u{}^\times$ satisfying $\xi = \xi_A \xi_B$ in $F'_u{}^\times$.*

Proof. By Remark 5.6, we can prove that there exist $c \in K$, $\eta_A \in A'_{d+1}{}^\times$ and $\eta_B \in B'_{d+1}{}^\times$ satisfying $\xi \eta_A \eta_B = 1 + cx_{f,d}$ in $F'_{d+1}{}^\times$. Then, by Lemma 5.10, we know (1) \Rightarrow (2).

Next we shall prove the converse. Assume that (1) is not satisfied. Then there exist $\eta_A \in A'_u{}^\times$ and $\eta_B \in B'_u{}^\times$ such that $\xi \eta_A \eta_B = 1$ in $F'_{d+1}{}^\times$. By Remark 5.6 (1), we know that there exist $\eta'_A \in A'_u{}^\times$ and $\eta'_B \in B'_u{}^\times$ such that $\xi \eta_A \eta_B \eta'_A \eta'_B = 1$ in $F'_u{}^\times$ by the assumption (5.13). Therefore we have $\xi = (\eta_A \eta'_A)^{-1} (\eta_B \eta'_B)^{-1}$ in $F'_u{}^\times$. \square

6. CLASSIFICATION OF (a, b, c) 's WHICH DO NOT SATISFY THE CONDITION EMU

We want to classify (a, b, c) 's which do not satisfy the condition EMU.

Remark 6.1. If the condition EMU for $m\Delta_{\bar{i},\bar{u},\bar{s}}$ is satisfied for some $m > 0$, then $R_s(\mathfrak{p})$ is Noetherian. One can prove it in the same way as the proof of Proposition 4.6 in [16].

Remark that we may assume $\ell_1 \geq \ell_{u-1}$ by exchanging x for y if necessary.

If $\ell_1 = 1$ or $\ell_{u-1} = 1$, the condition EMU for $\Delta_{\bar{i},\bar{u},\bar{s}}$ is not satisfied. In this case, $R_s(\mathfrak{p})$ is not Noetherian by Theorem 1.2 in González-Karu [4].

If $\ell_1 = \ell_{u-1} = 2$ and $u \geq 3$, then the condition EMU for $\Delta_{\bar{i},\bar{u},\bar{s}}$ is not satisfied. In this case, $R_s(\mathfrak{p})$ is not Noetherian by Theorem 1.2 in González-Karu [4].

If $\ell_1 \geq 3$ and $\ell_{u-1} \geq 3$, then the condition EMU for $\Delta_{\bar{i},\bar{u},\bar{s}}$ is satisfied by Lemma 5.1.

The remaining case is $\ell_1 = n \geq 3$ and $\ell_{u-1} = 2$.

In the rest of this section, we suppose that n is a positive integer such that

$$n \geq 3.$$

We want to classify (a, b, c) 's which do not satisfy the condition EMU for (a, b, c) such that $a_1 = n$ and $b_{-1} = 2$.

First we define the sequence of integers

$$\{f_i, g_i \mid i = 0, 1, \dots\}$$

as

$$\begin{pmatrix} f_0 \\ g_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} f_i \\ g_i \end{pmatrix} = \begin{pmatrix} n-2 & 1 \\ n-3 & 1 \end{pmatrix}^i \begin{pmatrix} f_0 \\ g_0 \end{pmatrix}.$$

We know

$$(f_0, g_0) = (1, 1), \quad (f_1, g_1) = (n-1, n-2), \quad (f_2, g_2) = (n^2 - 2n, n^2 - 3n + 1), \quad \dots$$

Here remark that

$$1 = f_0 < f_1 < f_2 < \dots$$

and

$$1 = g_0 \leq g_1 \leq g_2 \leq \dots$$

For $i > 0$, we have

$$\begin{pmatrix} f_i & f_{i-1} \\ g_i & g_{i-1} \end{pmatrix} = \begin{pmatrix} n-2 & 1 \\ n-3 & 1 \end{pmatrix}^{i-1} \begin{pmatrix} f_1 & f_0 \\ g_1 & g_0 \end{pmatrix} = \begin{pmatrix} n-2 & 1 \\ n-3 & 1 \end{pmatrix}^{i-1} \begin{pmatrix} n-1 & 1 \\ n-2 & 1 \end{pmatrix}$$

and

$$(6.1) \quad \det \begin{pmatrix} f_i & f_{i-1} \\ g_i & g_{i-1} \end{pmatrix} = 1.$$

In particular, we have $\text{GCD}(f_i, f_{i-1}) = \text{GCD}(f_i, g_i) = 1$.

Since $f_i = (n-2)f_{i-1} + g_{i-1}$ and $g_i = (n-3)f_{i-1} + g_{i-1}$, we know

$$(6.2) \quad f_i = g_i + f_{i-1}$$

for $i > 0$.

For $i \geq 2$, we know

$$(6.3) \quad f_i = (n-2)f_{i-1} + g_{i-1} = (n-2)f_{i-1} + f_{i-1} - f_{i-2} = (n-1)f_{i-1} - f_{i-2}.$$

We define a_i and b_i as in (5.2). By Lemma 5.1 and (5.3), the condition EMU is not satisfied for $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ if and only if there exist positive integers f, f', d such that

- $a_f = b_{-f'} = d$,
- $\{a_1, a_2, \dots, a_{f-1}\} \coprod \{b_{-1}, b_{-2}, \dots, b_{-(f'-1)}\} = \{2, 3, \dots, d-1\}$,
- $f + f' = d < u$.

Theorem 6.2. *Let a, b, c be pairwise coprime positive integers. Assume that \mathfrak{p} is not complete intersection. Let n be an integer such that $n \geq 3$. Then the following two conditions are equivalent:*

- (1) *The following conditions are satisfied:*
 - (a) $z^u - x^{s_3}y^{t_3}$ is the negative curve, i.e., $uc < \sqrt{abc}$,
 - (b) the condition EMU is not satisfied for $\Delta_{\bar{t}, \bar{u}, \bar{s}}$,
 - (c) $\ell_1 = n$ and $\ell_{u-1} = 2$.
- (2) *There exist $\lambda \in \mathbb{N}_0$ and $\gamma, \delta \in \mathbb{N}$ with $(\gamma, \delta) \neq (1, 1)$ and $\text{GCD}(\gamma, \delta) = 1$ satisfying the following conditions⁶:*
 - (a) $u = \gamma f_\lambda + \delta f_{\lambda+1}$, $u_2 = \gamma g_\lambda + \delta g_{\lambda+1}$,
 - (b) $n-1 \leq \bar{s} < \frac{(n-1)f_{\lambda+1}}{f_\lambda}$,
 - (c) $2 \leq -\bar{t} < \frac{2f_{\lambda+1}+1}{f_{\lambda+1}}$,
 - (d) $\frac{1}{\bar{s}-\bar{u}} + \frac{1}{\bar{u}-\bar{t}} < 1$.

We shall prove this theorem in this section.

Definition 6.3. Let $f, p_1, p_2, f', q_1, q_2$ be rational numbers such that $f > 0$ and $f' > 0$. Let $\Gamma(-f', p_1, p_2; f, q_1, q_2)$ the set of lattice points consisting of

$$\{(\alpha, \beta) \in \mathbb{Z}^2 \mid \alpha \leq 0, \frac{-p_2\alpha}{f'} \leq \beta \leq \frac{-p_1\alpha}{f'}\}$$

and

$$\{(\alpha, \beta) \in \mathbb{Z}^2 \mid 0 < \alpha, \frac{q_2\alpha}{f} < \beta \leq \frac{q_1\alpha}{f}\}.$$

Lemma 6.4. *Let f and f' be positive integers. Let p_1, p_2, q_1, q_2 be rational numbers. For $i \in \mathbb{Z}$, we define*

$$e_i = \#\{(i, \beta) \in \mathbb{Z}^2 \mid (i, \beta) \in \Gamma(-f', p_1, p_2; f, q_1, q_2)\}.$$

Assume that if we sort the sequence $e_{-f'}, e_{-f'+1}, \dots, e_f$ into ascending order, we obtain

$$1, 2, \dots, f + f', f + f' + 1$$

with $e_f = f + f'$ and $e_{-f'} = f + f' + 1$. Let c_1, c_2, c_3, \dots be the sequence given by sorting the sequence $\{e_i \mid i \in \mathbb{Z}\}$ into ascending order.

- (1) *Assume that p_1, p_2, q_1, q_2 are integers. Then we have $c_i = i$ for any $i > 0$.*
- (2) *Assume that p_1, q_1, q_2 are integers and $p_2 \notin \mathbb{Z}$. Then there exists $i > 0$ such that*

$$c_1 = 1, c_2 = 2, \dots, c_{i-1} = i-1, c_i > i.$$

⁶The minimal degree d defined in Definition 5.2 is $f_\lambda + f_{\lambda+1}$ in this case.

Proof. First we prove (1). We have $e_{f+i} = e_f + e_i = f + f' + e_i$ for $i > 0$ and $e_{-(f'+i)} = e_{-f'} - 1 + e_{-i} = f + f' + e_{-i}$ for $i \geq 0$. Then we have $e_{2f} = 2(f + f')$ and $e_{-2f'} = 2(f + f') + 1$. Recall

$$\{e_{-f'}, e_{-f'+1}, \dots, e_f\} = \{1, 2, \dots, f + f' + 1\}.$$

Therefore, if we sort the sequence $e_{-2f'}, e_{-2f'+1}, \dots, e_{2f}$ into ascending order, we obtain

$$1, 2, \dots, 2(f + f'), 2(f + f') + 1.$$

Repeating this process, we shall obtain the assertion.

Next we shall prove (2). We have $e_{f+i} = e_f + e_i = f + f' + e_i$ for $i = 1, 2, \dots, f$ and

$$(6.4) \quad e_{-(f'+i)} \geq e_{-f'} - 1 + e_{-i} = f + f' + e_{-i}$$

for $i = 1, 2, \dots, f'$. If there exists i such that $e_{-(f'+i)} > f + f' + e_{-i}$, the assertion follows immediately. If (6.4) are the equalities for $i = 1, 2, \dots, f'$, we have $e_{2f} = 2(f + f')$ and $e_{-2f'} = 2(f + f') + 1$. If we sort the sequence $e_{-2f'}, e_{-2f'+1}, \dots, e_{2f}$ into ascending order, we obtain

$$1, 2, \dots, 2(f + f'), 2(f + f') + 1.$$

We repeat this process. The equalities will not hold in the future. \square

Lemma 6.5. *Let λ be a non-negative integer. We put $f = f_\lambda$, $f' = f_{\lambda+1}$, $p_1 = 2f_{\lambda+1}$, $p_2 = g_{\lambda+1}$, $q_1 = (n-1)f_\lambda$, $q_2 = -g_\lambda$ and $\Gamma = \Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_\lambda, (n-1)f_\lambda, -g_\lambda)$.*

- (1) *The integers f , f' , p_1 , p_2 , q_1 , q_2 satisfy the assumption of Lemma 6.4.*
- (2) *The set of lattice points in Γ with the first component f_λ is*

$$\{(f_\lambda, -g_\lambda + 1), (f_\lambda, -g_\lambda + 2), \dots, (f_\lambda, (n-1)f_\lambda)\}.$$

The number of this set is $f_\lambda + f_{\lambda+1}$.

- (3) *The set of lattice points in Γ with the first component $-f_{\lambda+1}$ is*

$$\{(-f_{\lambda+1}, g_{\lambda+1}), (-f_{\lambda+1}, g_{\lambda+1} + 1), \dots, (-f_{\lambda+1}, 2f_{\lambda+1})\}.$$

The number of this set is $f_\lambda + f_{\lambda+1} + 1$.

- (4) *The set of lattice points in Γ with the first component $f_{\lambda+1}$ is*

$$\{(f_{\lambda+1}, -g_{\lambda+1}), (f_{\lambda+1}, -g_{\lambda+1} + 1), \dots, (f_{\lambda+1}, (n-1)f_{\lambda+1})\}.$$

The number of this set is $f_{\lambda+1} + f_{\lambda+2} + 1$.

- (5) *The set of lattice points in Γ with the first component $-f_{\lambda+2}$ is*

$$\{(-f_{\lambda+2}, g_{\lambda+2} + 1), (-f_{\lambda+2}, g_{\lambda+2} + 2), \dots, (-f_{\lambda+2}, 2f_{\lambda+2})\}.$$

The number of this set is $f_{\lambda+1} + f_{\lambda+2}$.

Proof. (2) and (3) follow from definition immediately.

We shall prove (1) by induction on $\lambda \geq 0$.

Assume $\lambda = 0$. We have $f' = n-1$, $p_1 = 2(n-1)$, $p_2 = n-2$, $f = 1$, $q_1 = n-1$ and $q_2 = -1$. Let e_i be the number of lattice points in $\Gamma(-f', p_1, p_2; f, q_1, q_2)$ such that the first component is i . Then we obtain $e_{-n+1} = n+1$, $e_{-n+2} = n-1$,

$e_{-n+3} = n - 2, \dots, e_{-1} = 2, e_0 = 1, e_1 = n$. Thus the assumption in Lemma 6.4 is satisfied.

Assume that the assumption in Lemma 6.4 is satisfied for some $\lambda \geq 0$. We define

$$\begin{aligned} e'_i &= \#\{(i, \beta) \in \mathbb{Z}^2 \mid (i, \beta) \in \Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_\lambda, (n-1)f_\lambda, -g_\lambda)\}, \\ e''_i &= \#\{(i, \beta) \in \mathbb{Z}^2 \mid (i, \beta) \in \Gamma(-f_{\lambda+2}, 2f_{\lambda+2}, g_{\lambda+2}; f_{\lambda+1}, (n-1)f_{\lambda+1}, -g_{\lambda+1})\}. \end{aligned}$$

Remark

$$1 \leq e'_1 \leq e'_2 \leq \dots$$

and

$$1 \leq e'_{-1} \leq e'_{-2} \leq \dots$$

By Lemma 6.4 (1), if we sort the sequence $\{e'_i \mid i \in \mathbb{Z}\}$ into ascending order, we obtain

$$1, 2, 3, \dots$$

Here we have

$$(6.5) \quad e''_{f_{\lambda+1}} = (n-1)f_{\lambda+1} + g_{\lambda+1} = (n-1)f_{\lambda+1} + (f_{\lambda+1} - f_\lambda) = f_{\lambda+1} + f_{\lambda+2}$$

by (6.2) and (6.3). Furthermore, we have

$$(6.6) \quad e''_{-f_{\lambda+2}} = 2f_{\lambda+2} - g_{\lambda+2} + 1 = 2f_{\lambda+2} - (f_{\lambda+2} - f_{\lambda+1}) + 1 = f_{\lambda+1} + f_{\lambda+2} + 1.$$

By (6.1), we have

$$1 = \det \begin{pmatrix} f_{\lambda+1} & f_\lambda \\ g_{\lambda+1} & g_\lambda \end{pmatrix} = \det \begin{pmatrix} f_\lambda & f_{\lambda+1} \\ -g_\lambda & -g_{\lambda+1} \end{pmatrix}.$$

Therefore we know

$$e'_{f_{\lambda+1}} = e''_{f_{\lambda+1}} + 1 = f_{\lambda+1} + f_{\lambda+2} + 1$$

and $e''_i = e'_i$ for $0 < i < f_{\lambda+1}$. Thus (4) follows from the above equality. By (6.1), we have

$$1 = \det \begin{pmatrix} f_{\lambda+2} & f_{\lambda+1} \\ g_{\lambda+2} & g_{\lambda+1} \end{pmatrix} = \det \begin{pmatrix} -f_{\lambda+1} & -f_{\lambda+2} \\ g_{\lambda+1} & g_{\lambda+2} \end{pmatrix}.$$

Therefore we know

$$e'_{-f_{\lambda+2}} = e''_{-f_{\lambda+2}} - 1 = f_{\lambda+1} + f_{\lambda+2}$$

and $e''_i = e'_i$ for $-f_{\lambda+2} < i < 0$. Thus (5) follows from the above equality. If we sort the sequence $e''_{-f_{\lambda+2}+1}, e''_{-f_{\lambda+2}+2}, \dots, e''_{f_{\lambda+1}-1}$ into ascending order, we have

$$1, 2, \dots, f_{\lambda+1} + f_{\lambda+2} - 1.$$

By (6.5) and (6.6), we know that (1) holds for $\lambda + 1$. □

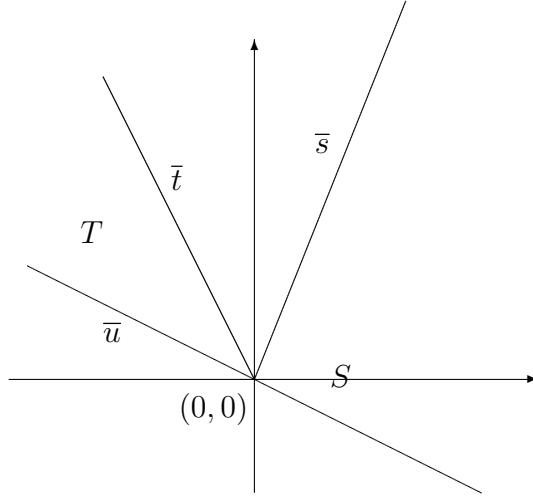
Now we start to prove (1) \Rightarrow (2) in Theorem 6.2. We put

$$f_{-1} = 0.$$

It is enough to prove the following claim:

Claim 6.6. *Assume that (1) in Theorem 6.2 is satisfied. Let λ be non-negative integer. Then we have the following:*

- (1) $_{\lambda}$ Letting d be the minimal degree of $\Delta_{\bar{t}, \bar{u}, \bar{s}}$, d does not satisfy $f_{\lambda-1} + f_{\lambda} < d < f_{\lambda} + f_{\lambda+1}$.
- (2) $_{\lambda}$ If the minimal degree of $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is $f_{\lambda} + f_{\lambda+1}$, then there exist $\gamma, \delta \in \mathbb{N}$ with $(\gamma, \delta) \neq (1, 1)$ and $\text{GCD}(\gamma, \delta) = 1$ such that u, u_2, \bar{t}, \bar{s} satisfy (a), (b), (c), (d) in (2) in Theorem 6.2.
- (3) $_{\lambda}$ If the minimal degree of $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is bigger than $f_{\lambda} + f_{\lambda+1}$, then we have
- $(S \cup T) \cap [-f_{\lambda+1}, f_{\lambda}]_{\mathbb{Z}}^1 = \Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_{\lambda}, (n-1)f_{\lambda}, -g_{\lambda}) \cap [-f_{\lambda+1}, f_{\lambda}]_{\mathbb{Z}}^1$,
 where $[-f_{\lambda+1}, f_{\lambda}]_{\mathbb{Z}}^1 := [-f_{\lambda+1}, f_{\lambda}]_{\mathbb{Z}} \times \mathbb{Z} = \{(\alpha, \beta) \in \mathbb{Z}^2 \mid -f_{\lambda+1} \leq \alpha \leq f_{\lambda}\}$.



We shall prove this claim by induction on λ .

First, assume $\lambda = 0$. Then we know $f_{-1} + f_0 = 1$ and $f_0 + f_1 = n$.

Since $a_1 = n$, the minimal degree of $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is not less than n by (5.8) and Lemma 5.1. Thus (1) $_0$ follows.

Since $(-1, 1), (-1, 2) \in T$, we know

$$(6.7) \quad (S \cup T) \cap [-(n-1), 1]_{\mathbb{Z}}^1 \supset \Gamma(-1, 2, 1; 1, n-1, -1) \cap [-(n-1), 1]_{\mathbb{Z}}^1.$$

Assume that the minimal degree of $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is n . By Lemma 5.1, we know $b_{-1} = 2$, $b_{-2} = 3, \dots, b_{-(n-1)} = n$. Thus we know that (6.7) is the equality in this case. Since $(1, -1) \notin S$ and $(-(n-1), n-2) \notin T$, we know

$$-\frac{g_0}{f_0} = -1 < \bar{u} = -\frac{u_2}{u} < -\frac{g_1}{f_1} = -\frac{n-2}{n-1}.$$

Then (a) in (2) follows since $\text{GCD}(u_2, u) = 1$ (Proposition 4.8 in [16]) and (6.1). Here we remark $(\gamma, \delta) \neq (1, 1)$ since $u > n = f_0 + f_1$. Since $a_1 = b_{-(n-1)} = n$, (b) and (c) in (2) follow. Since $z^u - x^{s_3}y^{t_3}$ is the negative curve, (3.2) is satisfied. Thus (2) $_0$ is proved.

Assume that the minimal degree is bigger than n . By (6.7) and $a_2 \geq n+2$, we know

$$b_{-(n-1)} = n+1, b_{-(n-2)} = n-1, b_{-(n-3)} = n-2, \dots, b_{-1} = 2, a_1 = n$$

by Lemma 5.1. Therefore only one of $(-(n-1), 2n-1)$ and $(-(n-1), n-2)$ belongs to T .

Here we assume $(-(n-1), 2n-1) \in T$. Consider $\Gamma(-(n-1), 2n-1, n-1; 1, n-1, -1)$. Let e_i be the number of lattice points in $\Gamma(-(n-1), 2n-1, n-1; 1, n-1, -1)$ with the first component i . Then we have

$$1 < e_1 < e_2 < \cdots, \quad 1 < e_{-1} < e_{-2} < \cdots$$

There exists $(-\alpha, \beta) \in T \cap \mathbb{Z}^2$ such that $0 < \beta < \alpha$ since $\bar{u} > -1$. We choose such $(-\alpha, \beta) \in T \cap \mathbb{Z}^2$ with α minimal. By Lemma 6.4 (1), if we sort $e_{-\alpha}, e_{-\alpha+1}, \dots, e_{-1}, e_0, e_1, \dots, e_\sigma$ ($\sigma = e_{-\alpha} - \alpha - 1$) into ascending order, we obtain

$$1, 2, 3, \dots, e_{-\alpha}.$$

If $i < -\alpha$, then $b_i \geq e_i > e_{-\alpha}$. If $\sigma < i < \alpha$, then $a_i \geq e_i > e_{-\alpha}$ by the definition of α and Lemma 6.4 (1). If $i \geq \alpha$, then $a_i > e_{-i} \geq e_{-\alpha}$. (For $i > 0$, we have

$$\begin{aligned} e_{-i} &= \lfloor \frac{2n-1}{n-1} i \rfloor - i + 1 = \lfloor \frac{n}{n-1} i \rfloor + 1 \\ a_i &\geq (n-1)i + 1. \end{aligned}$$

Therefore

$$a_i - e_{-i} \geq (n-1)i - \lfloor \frac{n}{n-1} i \rfloor \geq (n-1)i - \frac{n}{n-1}i = \frac{(n-1)^2 - n}{n-1}i > 0$$

if $n \geq 3$.) Therefore, if $i > \sigma$, then $a_i > e_{-i}$. By definition, we have

$$\Gamma(-(n-1), 2n-1, n-1; 1, n-1, -1) \cap [-\alpha, \sigma]_{\mathbb{Z}}^1 \subsetneq (S \cup T) \cap [-\alpha, \sigma]_{\mathbb{Z}}^1$$

since $b_{-\alpha} > e_{-\alpha}$. Then the condition EMU for $\Delta_{\bar{i}, \bar{u}, \bar{s}}$ is satisfied. It is a contradiction. Therefore $(-(n-1), n-2)$ belongs to T . Then we have

$$(S \cup T) \cap [-(n-1), 1]_{\mathbb{Z}}^1 = \Gamma(-(n-1), 2n-2, n-2; 1, n-1, -1) \cap [-(n-1), 1]_{\mathbb{Z}}^1.$$

Thus (3)₀ is proved.

Next, assume that (1) _{λ} , (2) _{λ} , (3) _{λ} are satisfied for some $\lambda \geq 0$. We shall prove (1) _{$\lambda+1$} , (2) _{$\lambda+1$} , (3) _{$\lambda+1$} . We may assume that the minimal degree of $\Delta_{\bar{i}, \bar{u}, \bar{s}}$ is bigger than $f_\lambda + f_{\lambda+1}$. By (3) _{λ} , we have

$$(S \cup T) \cap [-f_{\lambda+1}, f_\lambda]_{\mathbb{Z}}^1 = \Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_\lambda, (n-1)f_\lambda, -g_\lambda) \cap [-f_{\lambda+1}, f_\lambda]_{\mathbb{Z}}^1.$$

Then we know

(6.8)

$$(S \cup T) \cap [-f_{\lambda+2}, f_{\lambda+1}]_{\mathbb{Z}}^1 \supset \Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_{\lambda+1}, (n-1)f_{\lambda+1}, -g_{\lambda+1}) \cap [-f_{\lambda+2}, f_{\lambda+1}]_{\mathbb{Z}}^1.$$

By (6.1), we have

$$\begin{aligned} &\Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_{\lambda+1}, (n-1)f_{\lambda+1}, -g_{\lambda+1}) \cap (-f_{\lambda+2}, f_{\lambda+1})_{\mathbb{Z}}^1 \\ &= \Gamma(-f_{\lambda+2}, 2f_{\lambda+2}, g_{\lambda+2}; f_{\lambda+1}, (n-1)f_{\lambda+1}, -g_{\lambda+1}) \cap (-f_{\lambda+2}, f_{\lambda+1})_{\mathbb{Z}}^1. \end{aligned}$$

By Lemma 6.5 (1), if we sort the numbers of lattice points in each columns of $\Gamma(-f_{\lambda+2}, 2f_{\lambda+2}, g_{\lambda+2}; f_{\lambda+1}, (n-1)f_{\lambda+1}, -g_{\lambda+1}) \cap (-f_{\lambda+2}, f_{\lambda+1})_{\mathbb{Z}}^1$, we obtain

$$1, 2, 3, \dots, f_{\lambda+1} + f_{\lambda+2} - 1.$$

Furthermore we have

$$(6.9) \quad b_{-f_{\lambda+2}} \geq f_{\lambda+1} + f_{\lambda+2}$$

by (6.8) and Lemma 6.5 (5). Since $(-f_{\lambda+1}, g_{\lambda+1}) \in T$, we have $(f_{\lambda+1}, -g_{\lambda+1}) \notin S$. Since $(-f_{\lambda+1}, g_{\lambda+1} - 1) \notin T$, we have $(f_{\lambda+1}, -g_{\lambda+1} + 1) \in S$. Therefore

$$(6.10) \quad a_{f_{\lambda+1}} \geq (n-1)f_{\lambda+1} + g_{\lambda+1} = f_{\lambda+1} + f_{\lambda+2}$$

by (6.8) and Lemma 6.5 (4). Since the condition EMU is not satisfied for $\Delta_{\bar{t}, \bar{u}, \bar{s}}$, we have

$$(6.11) \quad (SUT) \cap (-f_{\lambda+2}, f_{\lambda+1})_{\mathbb{Z}}^1 = \Gamma(-f_{\lambda+2}, 2f_{\lambda+2}, g_{\lambda+2}; f_{\lambda+1}, (n-1)f_{\lambda+1}, -g_{\lambda+1}) \cap (-f_{\lambda+2}, f_{\lambda+1})_{\mathbb{Z}}^1.$$

Thus we know that the minimal degree of $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is bigger than or equal to $f_{\lambda+1} + f_{\lambda+2}$.

We have proved (1) $_{\lambda+1}$.

Assume that the minimal degree of $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is equal to $f_{\lambda+1} + f_{\lambda+2}$. Then (6.11) and $a_{f_{\lambda+1}} = b_{-f_{\lambda+2}} = f_{\lambda+1} + f_{\lambda+2}$ are satisfied, and (6.8) is the equality in this case. Since $(-f_{\lambda+2}, g_{\lambda+2}) \notin T$ and $(f_{\lambda+1}, -g_{\lambda+1}) \notin S$, we obtain

$$-\frac{g_{\lambda+1}}{f_{\lambda+1}} < \bar{u} = -\frac{u_2}{u} < -\frac{g_{\lambda+2}}{f_{\lambda+2}}.$$

By $\text{GCD}(u_2, u) = 1$ and (6.1), (a) in (2) follows. Here we know $(\gamma, \delta) \neq (1, 1)$ since $u > f_{\lambda+1} + f_{\lambda+2}$. Since $a_{f_{\lambda+1}} = b_{-f_{\lambda+2}} = f_{\lambda+1} + f_{\lambda+2}$, (b) and (c) in (2) follow. Since $z^u - x^{s_3}y^{t_3}$ is the negative curve, (3.2) is satisfied. Thus (2) $_{\lambda+1}$ is proved.

Assume that the minimal degree of $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is bigger than $f_{\lambda+1} + f_{\lambda+2}$. Remember that (6.8), (6.9), (6.10) and (6.11) are satisfied in this case. Only one of $a_{f_{\lambda+1}}$ and $b_{-f_{\lambda+2}}$ is equal to $f_{\lambda+1} + f_{\lambda+2}$.

First assume that $a_{f_{\lambda+1}} = f_{\lambda+1} + f_{\lambda+2}$. Since $a_{f_{\lambda+1}+1} \geq f_{\lambda+1} + f_{\lambda+2} + 2$ by (5.5), we know $b_{-f_{\lambda+2}} = f_{\lambda+1} + f_{\lambda+2} + 1$. Therefore only one of $(-f_{\lambda+2}, 2f_{\lambda+2} + 1)$ and $(-f_{\lambda+2}, g_{\lambda+2})$ belongs to T . Assume $(-f_{\lambda+2}, 2f_{\lambda+2} + 1) \in T$. Then we have

$$(SUT) \cap [-f_{\lambda+2}, f_{\lambda+1}]_{\mathbb{Z}}^1 = \Gamma(-f_{\lambda+2}, 2f_{\lambda+2} + 1, \frac{g_{\lambda+1}f_{\lambda+2}}{f_{\lambda+1}}; f_{\lambda+1}, (n-1)f_{\lambda+1}, -g_{\lambda+1}) \cap [-f_{\lambda+2}, f_{\lambda+1}]_{\mathbb{Z}}^1.$$

There exists $(-\alpha, \beta) \in T \cap \mathbb{Z}^2$ such that $0 < \beta < \frac{g_{\lambda+1}\alpha}{f_{\lambda+1}}$ since $u > f_{\lambda+1} + f_{\lambda+2}$. We choose such $(-\alpha, \beta) \in T \cap \mathbb{Z}^2$ with α minimal. Then we know $\alpha > f_{\lambda+2}$ and

$$(SUT) \cap [-\alpha, \alpha]_{\mathbb{Z}}^1 \supset \Gamma(-f_{\lambda+2}, 2f_{\lambda+2} + 1, \frac{g_{\lambda+1}f_{\lambda+2}}{f_{\lambda+1}}; f_{\lambda+1}, (n-1)f_{\lambda+1}, -g_{\lambda+1}) \cap [-\alpha, \alpha]_{\mathbb{Z}}^1.$$

Let e be the number of lattice points in $\Gamma(-f_{\lambda+2}, 2f_{\lambda+2} + 1, \frac{g_{\lambda+1}f_{\lambda+2}}{f_{\lambda+1}}; f_{\lambda+1}, (n-1)f_{\lambda+1}, -g_{\lambda+1})$ such that the first component is $-\alpha$. Then we know $b_i > b_{-\alpha} > e$ for $i < -\alpha$ since $(-\alpha, \beta) \in T$. Furthermore $a_i \geq a_{\alpha} > e$ if $i \geq \alpha$. (Since $\alpha > f_{\lambda+2}$, we know $(-f_{\lambda+1} - 1, g_{\lambda+1}) \notin T$ and $(f_{\lambda+1} + 1, -g_{\lambda+1}) \in S$. Therefore we have

$$a_{\alpha} \geq (n-1)\alpha + 2.$$

On the other hand, since

$$e = \lfloor \frac{2f_{\lambda+2} + 1}{f_{\lambda+2}} \alpha \rfloor - \lceil \frac{g_{\lambda+1}}{f_{\lambda+1}} \alpha \rceil + 1 \leq \frac{2f_{\lambda+2} + 1}{f_{\lambda+2}} \alpha - \frac{g_{\lambda+1}}{f_{\lambda+1}} \alpha + 1,$$

we know

$$e \leq \lfloor \left(\frac{2f_{\lambda+2} + 1}{f_{\lambda+2}} - \frac{g_{\lambda+1}}{f_{\lambda+1}} \right) \alpha \rfloor + 1.$$

Since

$$n - 1 \geq 2 > \frac{2f_{\lambda+2} + 1}{f_{\lambda+2}} - \frac{g_{\lambda+1}}{f_{\lambda+1}},$$

we obtain $a_\alpha > e$ immediately.) Since $b_{-\alpha} > e$, we know that the condition EMU for $\Delta_{\bar{i}, \bar{u}, \bar{s}}$ is satisfied by Lemma 6.4 (2). It is a contradiction. If $(-f_{\lambda+2}, g_{\lambda+2})$ belongs to T , (3) $_{\lambda+1}$ is satisfied.

Next we assume that $b_{-f_{\lambda+2}} = f_{\lambda+1} + f_{\lambda+2}$ and $a_{f_{\lambda+1}} \geq f_{\lambda+1} + f_{\lambda+2} + 1$. Since $(-f_{\lambda+1}, g_{\lambda+1} - 1) \notin T$ and $(-f_{\lambda+1}, g_{\lambda+1}) \in T$ by (3) $_\lambda$, we know $(f_{\lambda+1}, -g_{\lambda+1} + 1) \in S$ and $(f_{\lambda+1}, -g_{\lambda+1}) \notin S$. Then we know that S contains $(f_{\lambda+1}, (n-1)f_{\lambda+1} + 1)$. We know

$$S \cup T \supset \Gamma(-f_{\lambda+2} - 1, 2(f_{\lambda+2} + 1), \frac{g_{\lambda+1}(f_{\lambda+2} + 1)}{f_{\lambda+1}}; f_{\lambda+1}, (n-1)f_{\lambda+1} + 1, -g_{\lambda+1}) \cap [-f_{\lambda+2} - 1, f_{\lambda+1}]_{\mathbb{Z}}^1.$$

Let e'_i be the number of the lattice points in $\Gamma(-f_{\lambda+2} - 1, 2(f_{\lambda+2} + 1), \frac{g_{\lambda+1}(f_{\lambda+2} + 1)}{f_{\lambda+1}}; f_{\lambda+1}, (n-1)f_{\lambda+1} + 1, -g_{\lambda+1})$ with the first component i . Then $e'_{f_{\lambda+1}} = f_{\lambda+1} + f_{\lambda+2} + 1$, $e'_{-f_{\lambda+2}} = f_{\lambda+1} + f_{\lambda+2}$, $e'_{-f_{\lambda+2}-1} = f_{\lambda+1} + f_{\lambda+2} + 2$. The last equality follows from $(-f_{\lambda+2} - 1, g_{\lambda+2} + 1) \in \Gamma(-f_{\lambda+2} - 1, 2(f_{\lambda+2} + 1), \frac{g_{\lambda+1}(f_{\lambda+2} + 1)}{f_{\lambda+1}}; f_{\lambda+1}, (n-1)f_{\lambda+1} + 1, -g_{\lambda+1})$. (If not, the point $(-f_{\lambda+2} - 1, g_{\lambda+2} + 1)$ is in the interior of the cone spanned by $(-f_{\lambda+1}, g_{\lambda+1})$ and $(-f_{\lambda+2}, g_{\lambda+2})$. It is impossible since $f_{\lambda+1} \geq f_1 = n - 1 \geq 2$ and (6.1).) Here we remark

$$\begin{aligned} & \Gamma(-f_{\lambda+2} - 1, 2(f_{\lambda+2} + 1), \frac{g_{\lambda+1}(f_{\lambda+2} + 1)}{f_{\lambda+1}}; f_{\lambda+1}, (n-1)f_{\lambda+1} + 1, -g_{\lambda+1}) \cap (-f_{\lambda+2}, f_{\lambda+1})_{\mathbb{Z}}^1 \\ &= \Gamma(-f_{\lambda+2}, 2f_{\lambda+2}, g_{\lambda+2}; f_{\lambda+1}, (n-1)f_{\lambda+1}, -g_{\lambda+1}) \cap (-f_{\lambda+2}, f_{\lambda+1})_{\mathbb{Z}}^1. \end{aligned}$$

Therefore, if we sort $e'_{-f_{\lambda+2}+1}, e'_{-f_{\lambda+2}+2}, \dots, e'_{f_{\lambda+1}-1}$ into ascending order, we have

$$1, 2, \dots, f_{\lambda+1} + f_{\lambda+2} - 1$$

by Lemma 6.5 (1). Since the condition EMU for $\Delta_{\bar{i}, \bar{u}, \bar{s}}$ is not satisfied, we know

$$\begin{aligned} & (S \cup T) \cap [-f_{\lambda+2} - 1, f_{\lambda+1}]_{\mathbb{Z}}^1 \\ &= \Gamma(-f_{\lambda+2} - 1, 2(f_{\lambda+2} + 1), \frac{g_{\lambda+1}(f_{\lambda+2} + 1)}{f_{\lambda+1}}; f_{\lambda+1}, (n-1)f_{\lambda+1} + 1, -g_{\lambda+1}) \cap [-f_{\lambda+2} - 1, f_{\lambda+1}]_{\mathbb{Z}}^1. \end{aligned}$$

Here remark that $\Gamma(-f_{\lambda+2} - 1, 2(f_{\lambda+2} + 1), \frac{g_{\lambda+1}(f_{\lambda+2} + 1)}{f_{\lambda+1}}; f_{\lambda+1}, (n-1)f_{\lambda+1} + 1, -g_{\lambda+1})$ satisfies the assumption in Lemma 6.4. There exists $(-\alpha, \beta) \in T \cap \mathbb{Z}^2$ such that $0 < \beta < \frac{g_{\lambda+1}\alpha}{f_{\lambda+1}}$. We choose such $(-\alpha, \beta) \in T \cap \mathbb{Z}^2$ with α minimal. Then we know $\alpha > f_{\lambda+2} + 1$ and

$$(S \cup T) \cap [-\alpha, \alpha]_{\mathbb{Z}}^1 \supset \Gamma(-f_{\lambda+2} - 1, 2(f_{\lambda+2} + 1), \frac{g_{\lambda+1}(f_{\lambda+2} + 1)}{f_{\lambda+1}}; f_{\lambda+1}, (n-1)f_{\lambda+1} + 1, -g_{\lambda+1}) \cap [-\alpha, \alpha]_{\mathbb{Z}}^1.$$

Then $b_i > b_{-\alpha} > e'_{-\alpha}$ if $i < -\alpha$. Furthermore $a_i > e'_{-i} > e'_{-\alpha}$ if $i \geq \alpha$. Since $b_{-\alpha} > e'_{-\alpha}$, we know that the condition EMU for $\Delta_{\bar{i}, \bar{u}, \bar{s}}$ is satisfied by Lemma 6.4. It is a contradiction. We have completed the proof of (3) $_{\lambda+1}$.

We have completed the proof of Claim 6.6 and (1) \Rightarrow (2) in Theorem 6.2.

Assume (2) in Theorem 6.2. Then we have

$$-\frac{g_\lambda}{f_\lambda} < \bar{u} = -\frac{u_2}{u} < -\frac{g_{\lambda+1}}{f_{\lambda+1}}$$

and $u > f_\lambda + f_{\lambda+1}$. By (a), (b) and (c) in (2), we know

$$(S \cup T) \cap (-f_{\lambda+1}, f_\lambda]_{\mathbb{Z}}^1 = \Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_\lambda, (n-1)f_\lambda, -g_\lambda) \cap (-f_{\lambda+1}, f_\lambda]_{\mathbb{Z}}^1$$

and $-b_{f_{\lambda+1}} = f_\lambda + f_{\lambda+1}$. By Lemma 6.5, if we sort $1, a_1, a_2, \dots, a_{f_\lambda}, b_{-1}, b_{-2}, \dots, b_{-f_{\lambda+1}}$ into ascending order, we obtain

$$1, 2, \dots, f_\lambda + f_{\lambda+1} - 1, f_\lambda + f_{\lambda+1}, f_\lambda + f_{\lambda+1}.$$

Thus the condition EMU is not satisfied for $\Delta_{\bar{i}, \bar{u}, \bar{s}}$. (a) of (1) follows from (d) of (2). (c) of (1) is clear. We have completed the proof of Theorem 6.2. \square

Remark 6.7. Let a, b, c be pairwise coprime positive integers such that $\Delta_{\bar{i}, \bar{u}, \bar{s}}$ satisfies (2) in Theorem 6.2 with $n, \lambda, \gamma, \delta$. Let a_i and b_j be integers defined in (5.2). Let e_i be the number of lattice points in $\Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_\lambda, (n-1)f_\lambda, -g_\lambda)$ with the first component i . Then we know

- $a_i = e_i$ for $i = 1, 2, \dots, f_\lambda$,
- $b_{-i} = e_{-i}$ for $i = 1, 2, \dots, f_{\lambda+1} - 1$,
- $a_{f_\lambda} = b_{-f_{\lambda+1}} = e_{-f_{\lambda+1}} - 1 = f_\lambda + f_{\lambda+1}$.

Remark that

$$a_1, a_2, \dots, a_{f_\lambda}, b_{-1}, b_{-2}, \dots, b_{-f_{\lambda+1}}$$

are independent of γ and δ .

Remark 6.8. For given $\lambda \in \mathbb{N}_0$ and $\gamma, \delta \in \mathbb{N}$ with $(\gamma, \delta) \neq (1, 1)$ and $\text{GCD}(\gamma, \delta) = 1$, it is possible to find pairwise coprime positive integers a, b, c satisfying (a), (b), (c), (d) in (2) of Theorem 6.2 as follows.

We put $u = \gamma f_\lambda + \delta f_{\lambda+1}$, $u_2 = \gamma g_\lambda + \delta g_{\lambda+1}$, $\bar{u} = -u_2/u$, $s' = \frac{(n-1)f_\lambda + 1}{f_\lambda}$, $t' = -\frac{2f_{\lambda+1} + 1}{f_{\lambda+1}}$. Consider the triangle $\Delta_{t', \bar{u}, s'}$. Then the sequence ℓ'_1, ℓ'_2, \dots in Definition 1.1 is equal to

$$1, 2, \dots, f_\lambda + f_{\lambda+1} - 1, f_\lambda + f_{\lambda+1} + 1, f_\lambda + f_{\lambda+1} + 1, \dots$$

Then the condition EMU for $\Delta_{t', \bar{u}, s'}$ is satisfied by Lemma 5.1. Consider the convex hull P of $\Delta_{t', \bar{u}, s'} \cap \mathbb{Z}^2$. Let B be the number of lattice points in the boundary of P . Let I be the number of lattice points in the interior of P . Since the condition EMU is satisfied, we know

$$B + I \geq 1 + (1 + 2 + \dots + u) + 1 = 2 + \frac{u(u+1)}{2}$$

It is easy to see $B \leq u + 1$. Then, by Pick's theorem, we know

$$|\Delta_{t', \bar{u}, s'}| \geq |P| = \frac{B}{2} + I - 1 \geq \frac{u+1}{2} + \left\{ 2 + \frac{u(u+1)}{2} - (u+1) \right\} - 1 = \frac{u^2+1}{2} > \frac{u^2}{2}.$$

Then we have

$$\frac{1}{s' - \bar{u}} + \frac{1}{\bar{u} - t'} < 1.$$

We choose sufficiently near $\bar{s} < s'$ and $\bar{t} < t'$ (see Corollary 2.2), we can find a, b, c satisfying the requirement. (The sequence ℓ'_1, ℓ'_2, \dots of $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is equal to

$$1, 2, \dots, f_\lambda + f_{\lambda+1} - 1, f_\lambda + f_{\lambda+1}, f_\lambda + f_{\lambda+1}, \dots$$

Therefore the minimal degree is equal to $f_\lambda + f_{\lambda+1}$.)

7. A PROOF OF THEOREM 1.2

We shall prove Theorem 1.2 in this section.

Lemma 7.1. *Let K be a field of characteristic 0. Let n and λ be integers such that $n \geq 3$ and $\lambda \geq 0$.*

Then there exists pairwise coprime positive integers a, b, c satisfying following conditions:

- (1) $R_s(\mathfrak{p})$ is not Noetherian.
- (2) The condition (2) in Theorem 6.2 is satisfied with $n, \lambda, \gamma = 2, \delta = 1$.
- (3) The sequence $\ell'_1, \ell'_2, \dots, \ell'_u$ in Definition 1.1 is equal to

$$(7.1) \quad 1, 2, \dots, f_\lambda + f_{\lambda+1} - 1, f_\lambda + f_{\lambda+1}, f_\lambda + f_{\lambda+1}, f_\lambda + f_{\lambda+1} + 2, f_\lambda + f_{\lambda+1} + 3, \dots, 2f_\lambda + f_{\lambda+1}$$

if $\lambda > 0$, and

$$1, 2, \dots, f_\lambda + f_{\lambda+1} - 1, f_\lambda + f_{\lambda+1}, f_\lambda + f_{\lambda+1}$$

if $\lambda = 0$.

Proof. First we assume $\lambda = 0$. By Remark 6.8, we can find pairwise coprime positive integers a, b, c satisfying (2) in Theorem 6.2 with $n, \lambda = 0, \gamma = 2, \delta = 1$. The minimal degree is equal to $f_0 + f_1 = n$. In this case, $u = 2f_0 + f_1 = n + 1$. Therefore (3) in Lemma 7.1 is satisfied by Lemma 5.1. In this case, by Remark 6.7,

$$a_1 = n, b_{-1} = 2, b_{-2} = 3, \dots, b_{n-1} = n.$$

We know that $R_s(\mathfrak{p})$ is not Noetherian by Theorem 1.2 in [4].

Next we assume $\lambda > 0$. Consider $\Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_\lambda, (n-1)f_\lambda, -g_\lambda)$. Let e_i be the number of lattice points in $\Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_\lambda, (n-1)f_\lambda, -g_\lambda)$ with the first component i . If we sort $\{e_i \mid i \in \mathbb{Z}\}$ into ascending order, we obtain

$$1, 2, 3, \dots$$

by Lemma 6.5 (1). Put $u' = -\frac{2g_\lambda + g_{\lambda+1}}{2f_\lambda + f_{\lambda+1}}$. Consider $\Delta_{-2, u', (n-1)}$. Then we have

$$(S \cup T) \cap \mathbb{Z}^2 \subset \Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_\lambda, (n-1)f_\lambda, -g_\lambda).$$

We define a_i, b_i as in (5.2).

Then we have $a_i \leq e_i$ and $b_{-i} \leq e_{-i}$ for $i > 0$. Remark

$$\cdots > b_{-(i+1)} > b_{-i} > \cdots > b_{-2} > b_{-1} > 1 < a_1 < a_2 < \cdots < a_i < a_{i+1} < \cdots$$

and remember (5.3). The sequence $\ell_1, \ell_2, \dots, \ell_{2f_\lambda+f_{\lambda+1}}$ defined in Definition 1.1 is equal to

$$a_1, a_2, \dots, a_j, b_{-k}, \dots, b_{-2}, b_{-1}, 1$$

for some j, k such that $j + k + 1 = 2f_\lambda + f_{\lambda+1}$. Since

$$a_{f_\lambda} = b_{-f_{\lambda+1}} = f_\lambda + f_{\lambda+1},$$

we know $j \geq f_\lambda$ and $k \geq f_{\lambda+1}$. We have

$$a_i = e_i \text{ for } i = 1, 2, \dots, 2f_\lambda + f_{\lambda+1} - 1$$

and

$$b_{-i} = \begin{cases} e_{-i} & i = 1, 2, \dots, f_{\lambda+1} - 1, \\ e_{-f_{\lambda+1}} - 1 & i = f_{\lambda+1}, \\ e_{-i} & i = f_{\lambda+1} + 1, \dots, f_\lambda + f_{\lambda+1} - 1. \end{cases}$$

Then we know that the sequence $\ell'_1, \ell'_2, \dots, \ell'_{2f_\lambda+f_{\lambda+1}}$ for $\Delta_{-2, u', (n-1)}$ defined in Definition 1.1 is equal to (7.1) by Lemma 6.5 (1). Here remark that this sequence does not end at $f_\lambda + f_{\lambda+1}$ by $f_\lambda > 1$ (since $\lambda > 0$). Therefore $\Delta_{-2, u', (n-1)}$ has a column with $2f_\lambda + f_{\lambda+1}$ lattice points. Then it is easy to see that there exists pairwise coprime positive integers a, b, c satisfying following conditions:

- $\Delta_{-2, u', (n-1)} \cap \mathbb{Z}^2 = \Delta_{\bar{t}, \bar{u}, \bar{s}} \cap \mathbb{Z}^2$, in particular $\bar{t} < -2$, $u' = \bar{u}$, $n - 1 < \bar{s}$.
- $|\Delta_{\bar{t}, \bar{u}, \bar{s}}| > (2f_\lambda + f_{\lambda+1})^2/2$.

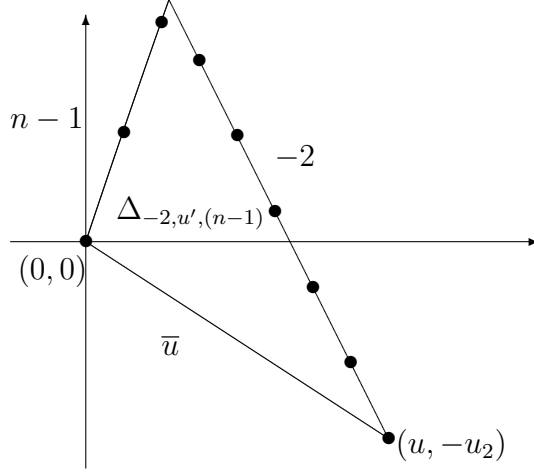
By the second condition, we know that $z^u - x^{s_3}y^{t_3}$ is a negative curve. Thus (2) in Theorem 6.2 is satisfied for $n, \lambda, \gamma = 2, \delta = 1$. Here $u = 2f_\lambda + f_{\lambda+1}$, $u_2 = 2g_\lambda + g_{\lambda+1}$.

Assume that $R_s(\mathfrak{p})$ is Noetherian. By Proposition 5.9 (Theorem 1.1 in [16]), there exists a Laurent polynomial

$$g \in \sum_{(\alpha, \beta) \in \Delta_{\bar{t}, \bar{u}, \bar{s}} \cap \mathbb{Z}^2} K v^\alpha w^\beta \cap (v - 1, w - 1)^u \subset K[v^{\pm 1}, w^{\pm 1}]$$

such that the constant term of g is not 0. We know that the coefficient of $v^u w^{-u_2}$ in g is not 0 since this curve does not meet the negative curve. Then we know that g is irreducible in $K[v^{\pm 1}, w^{\pm 1}]$ by Lemma 2.3 [6]. Consider the convex hull P of $\Delta_{\bar{t}, \bar{u}, \bar{s}} \cap \mathbb{Z}^2$. Let Q be the Newton polygon of g . Then $Q \subset P$. The number of lattice points in the boundary of P is $u + 1$ since $\Delta_{-2, u', (n-1)} \cap \mathbb{Z}^2 = \Delta_{\bar{t}, \bar{u}, \bar{s}} \cap \mathbb{Z}^2$. Here remark

that each column has just 1 point in the boundary of P as in the picture below.



The number of lattice points in P is

$$1 + (1 + 2 + \cdots + (2f_\lambda + f_{\lambda+1})) - 1 = \frac{u(u+1)}{2}$$

by (7.1). Therefore the number of lattice points in the interior of P is

$$\frac{u(u+1)}{2} - (u+1) = \frac{u(u-1)}{2} - 1.$$

Then, by Pick's theorem, we have

$$|Q| \leq |P| = \frac{u+1}{2} + \left(\frac{u(u-1)}{2} - 1 \right) - 1 = \frac{u^2 - 3}{2} < \frac{u^2}{2}.$$

Since g is irreducible, g is a u -nct in the sense of [14]. The number I of lattice points in the interior of Q is less than or equal to that of P . Therefore we have

$$I \leq \frac{u(u-1)}{2} - 1.$$

It contradicts to Lemma 4.1 in [14]. Therefore $R_s(\mathfrak{p})$ is not Noetherian.

We have completed the proof of Lemma 7.1. \square

Now we start to prove Theorem 1.2. If the condition EMU is satisfied for (a, b, c) , the symbolic Rees ring of \mathfrak{p} is Noetherian by Proposition 4.6 in [16]. Here we shall prove the converse.

Assume that the condition EMU for $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is not satisfied. Let n and λ be integers such that $n \geq 3$ and $\lambda \geq 0$. We may assume that $\ell_1 = n$ and $\ell_{u-1} = 2$ as in Remark 6.1. We shall consider pairwise coprime positive integers a, b, c satisfying (2) in Theorem 6.2 for the fixed n, λ and some γ, δ . Remark that the minimal degree is $f_\lambda + f_{\lambda+1}$. We put $d = f_\lambda + f_{\lambda+1}$. Consider P_A and P_B defined just before Lemma 5.3. Consider

$$(7.2) \quad P_A \cap (\mathbb{Z} \times [0, d]_{\mathbb{Z}})$$

and

$$(7.3) \quad P_B \cap (\mathbb{Z} \times [0, d]_{\mathbb{Z}}).$$

(7.2) and (7.3) are determined by the sequences $a_1, a_2, \dots, a_{f_\lambda} = d$ and $b_{-1}, b_{-2}, \dots, b_{-f_{\lambda+1}} = d$. Therefore (7.2) and (7.3) are independent of the choice of γ and δ (see Remark 6.7).

By Lemma 5.3 (3) and Remark 5.6 (1), (2), there exist $\eta_{A,1}, \eta_{A,2} \in A'_{d+1} \times$ and $\eta_{B,1}, \eta_{B,2} \in B'_{d+1} \times$, such that

$$(1+x)\eta_{A,1}\eta_{B,1} = 1 + q_1x_{f_\lambda, d} \text{ in } F'_{d+1} \times$$

and

$$w\eta_{A,2}\eta_{B,2} = 1 + q_2x_{f_\lambda, d} \text{ in } F'_{d+1} \times$$

for some integers q_1, q_2 .⁷ Then, for any integers h_1 and h_2 , we obtain

$$(7.4) \quad (1+x)^{h_1}w^{h_2} (\eta_{A,1}^{h_1}\eta_{A,2}^{h_2}) (\eta_{B,1}^{h_1}\eta_{B,2}^{h_2}) = (1+q_1x_{f_\lambda, d})^{h_1}(1+q_2x_{f_\lambda, d})^{h_2} = 1+(h_1q_1+h_2q_2)x_{f_\lambda, d}$$

in $F'_{d+1} \times$.

Put $d' = g_\lambda + g_{\lambda+1}$. Now consider $\Delta_{-2, -d'/d, (n-1)}$. Considering $\Gamma(-f_{\lambda+1}, 2f_{\lambda+1}, g_{\lambda+1}; f_\lambda, (n-1)f_\lambda, -g_\lambda)$, we know that the sequence $\ell'_1, \ell'_2, \dots, \ell'_u$ given in Definition 1.1 is $1, 2, \dots, d$. In particular, $\Delta_{-2, -d'/d, (n-1)}$ has a column that has d lattice points. Then it is easy to see that there exists pairwise coprime positive integers a, b, c satisfying following conditions:

- $\Delta_{-2, -d'/d, (n-1)} \cap \mathbb{Z}^2 = \Delta_{\bar{t}, \bar{u}, \bar{s}} \cap \mathbb{Z}^2$, in particular $\bar{t} < -2$, $-d'/d = \bar{u}$, $n-1 < \bar{s}$.
- $|\Delta_{\bar{t}, \bar{u}, \bar{s}}| > d^2/2$.

By the second condition, we know that $z^u - x^{s_3}y^{t_3}$ is a negative curve. Then the condition EMU for $\Delta_{\bar{t}, \bar{u}, \bar{s}}$ is satisfied and $\mathcal{O}_Y(abH - uE)|_{u_C} \simeq \mathcal{O}_{u_C}$ by Remark 4.9, Lemma 5.3 and Remark 5.6. Therefore, by Corollary 3.3, we have $\mathcal{O}_Y(m'(abH - uE))|_{m'u_C} \simeq \mathcal{O}_{m'u_C}$ for any $m' > 0$. Hence there exists $\eta_A \in A'_{m'u} \times$ and $\eta_B \in B'_{m'u} \times$ such that

$$(1+x)^{m'u}w^{-m'u_2} = \eta_A\eta_B \text{ in } F'_{m'u} \times$$

by Proposition 4.8. Here suppose $m' \geq 2$. Then we know $m'u > d+1$ and the condition EMU is not satisfied for $m'\Delta_{\bar{t}, \bar{u}, \bar{s}}$. Then we have

$$m'uq_1 + (-m'u_2)q_2 = 0$$

by Lemma 5.10 and (7.4). Thus we have $uq_1 - u_2q_2 = 0$. Since $\text{GCD}(u, u_2) = 1$, we know

$$q_1 = qu_2 = q(g_\lambda + g_{\lambda+1}), \quad q_2 = qu = q(f_\lambda + f_{\lambda+1})$$

for some integer q .

⁷The integers q_1 and q_2 depend only on n and λ . They are independent of γ and δ . Here we assume that γ and δ are positive integers. (We do not assume $(\gamma, \delta) \neq (1, 1)$.)

If $-d < \alpha < d$, then $\lceil \alpha \bar{u} \rceil$ is independent of γ and δ by (6.1), where $\bar{u} = -\frac{\gamma g_\lambda + \delta g_{\lambda+1}}{\gamma f_\lambda + \delta f_{\lambda+1}}$. By (7.2) and (7.3), we know that the rings $F'_{d+1}, A'_{d+1}, B'_{d+1}$ are independent of γ and δ .

On the other hand, consider the pairwise coprime positive integers satisfying Lemma 7.1. By the condition (3) in Lemma 7.1, we know that (5.13) is satisfied. Then, by Lemma 5.11, we know

$$q_1(2f_\lambda + f_{\lambda+1}) + q_2(-2g_\lambda - g_{\lambda+1}) \neq 0.$$

In particular, we obtain $q \neq 0$.

Here let γ and δ be positive integers such that $\text{GCD}(\gamma, \delta) = 1$ and $(\gamma, \delta) \neq (1, 1)$. Then we have

$$\begin{aligned} & q_1(\gamma f_\lambda + \delta f_{\lambda+1}) + q_2(-\gamma g_\lambda - \delta g_{\lambda+1}) \\ &= q \{(g_\lambda + g_{\lambda+1})(\gamma f_\lambda + \delta f_{\lambda+1}) + (f_\lambda + f_{\lambda+1})(-\gamma g_\lambda - \delta g_{\lambda+1})\} \\ &= q \times \det \begin{pmatrix} \gamma f_\lambda + \delta f_{\lambda+1} & f_\lambda + f_{\lambda+1} \\ \gamma g_\lambda + \delta g_{\lambda+1} & g_\lambda + g_{\lambda+1} \end{pmatrix} \\ &= q \times \det \begin{pmatrix} f_\lambda & f_{\lambda+1} \\ g_\lambda & g_{\lambda+1} \end{pmatrix} \times \det \begin{pmatrix} \gamma & 1 \\ \delta & 1 \end{pmatrix} \\ &\neq 0 \end{aligned}$$

by the choice of γ and δ , (6.1) and $q \neq 0$. Then, for pairwise coprime positive integers a, b, c satisfying (2) in Theorem 6.2 with $n, \lambda, \gamma, \delta$, the symbolic Rees ring $R_s(\mathfrak{p})$ is not Noetherian by Lemma 5.10.

We have completed the proof of Theorem 1.2. \square

REFERENCES

- [1] S. D. CUTKOSKY, *Symbolic algebras of monomial primes*, J. reine angew. Math. **416** (1991), 71–89.
- [2] S. D. CUTKOSKY AND K. KURANO, *Asymptotic regularity of powers of ideals of points in a weighted projective plane*, Kyoto J. Math. **51** (2011), 25–45.
- [3] T. EBINA, Master theses, Meiji University 2017 (Japanese).
- [4] J. L. GONZÁLEZ AND K. KARU, *Some non-finitely generated Cox rings*, Compos. Math. **152** (2016), 984–996.
- [5] J. GONZÁLEZ-ANAYA, J. L. GONZÁLEZ AND K. KARU, *Constructing non-Mori Dream Spaces from negative curves*, J. Algebra **539** (2019), 118–137.
- [6] J. GONZÁLEZ-ANAYA, J. L. GONZÁLEZ AND K. KARU, *Curves generating extremal rays in blowups of weighted projective planes*, J. of London Math. Soc. **104** (2021), 1342–1362.
- [7] J. GONZÁLEZ-ANAYA, J. L. GONZÁLEZ AND K. KARU, *The geography of negative curves*, arXiv:2104.03950, 2021.
- [8] J. GONZÁLEZ-ANAYA, J. L. GONZÁLEZ AND K. KARU, *Non-existence of negative curves*, arXiv:2110.13333, 2021.
- [9] S. GOTO, K. NISHIDA AND K.-I. WATANABE, *Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik’s question*, Proc. Amer. Math. Soc. **120** (1994), 383–392.
- [10] R. HARTSHORNE, *Algebraic Geometry*, GTM 52, Springer Verlag, 1977.
- [11] Z. HE, *New examples and non-examples of Mori dream spaces when blowing up toric surfaces*; arXiv:1703.00819, 2017.
- [12] J. HERZOG, *Generators and relations of Abelian semigroups and semigroup rings*, Manuscripta Math. **3** (1970), 175–193.
- [13] C. HUNEKE, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987), 293–318.

- [14] K. KURANO, *Equations of negative curves of blow-ups of Ehrhart rings of rational convex polygons*, J. Algebra **590** (2022), 413-438.
- [15] K. KURANO AND N. MATSUOKA, *On finite generation of symbolic Rees rings of space monomial curves and existence of negative curves*, J. Algebra **322** (2009), 3268-3290.
- [16] K. KURANO AND K. NISHIDA, *Infinitely generated symbolic Rees rings of space monomial curves having negative curves*, Michigan Math. J. **68** (2019), 405-445.
- [17] M. MATSUURA, Master theses, Meiji University 2019 (Japanese).
- [18] K. UCHISAWA, Master theses, Meiji University 2017 (Japanese).

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