INFINITELY GENERATED SYMBOLIC REES RINGS OF SPACE MONOMIAL CURVES HAVING NEGATIVE CURVES

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Dedicated to Professor Shiro Goto on the occasion of his 72nd birthday.

Abstract. In this paper, we shall study finite generation of symbolic Rees rings of the defining ideal $\mathfrak{p}$ of the space monomial curve (t^a, t^b, t^c) for pairwise coprime integers a, b, c. Suppose that the base field is of characteristic 0 and the above ideal $\mathfrak{p}$ is minimally generated by three polynomials. In Theorem 1.1, under the assumption that the homogeneous element ξ of the minimal degree in $\mathfrak{p}$ is a negative curve, we determine the minimal degree of an element η such that the pair $\{\xi, \eta\}$ satisfies Huneke's criterion in the case where the symbolic Rees ring is Noetherian. By this result, we can decide whether the symbolic Rees ring $\mathcal{R}_s(\mathfrak{p})$ is Notherian using computers. We give a necessary and sufficient conditions for finite generation of the symbolic Rees ring of $\mathfrak{p}$ in Proposition 4.10 under some assumptions. We give an example of an infinitely generated symbolic Rees ring of $\mathfrak{p}$ in which the homogeneous element of the minimal degree in $\mathfrak{p}^{(2)}$ is a negative curve in Example 5.7. We give a simple proof to (generalized) Huneke's criterion.

1. Introduction

Let $\mathfrak{p}_K(a, b, c)$ be the defining ideal of the space monomial curve (t^a, t^b, t^c) in K^3, where K is a field. The ideal $\mathfrak{p}_K(a, b, c)$ is generated by at most three binomials in $K[x, y, z]$ (Herzog [9]). The symbolic Rees rings of space monomial primes are deeply studied by many authors. Huneke [10] and Cutkosky [2] developed criteria for finite generation of such rings. In 1994, Goto-Nishida-Watanabe [7] first found examples of infinitely generated symbolic Rees rings of space monomial primes. Recently, using toric geometry, González-Karu [5] found some sufficient conditions for the symbolic Rees rings of space monomial primes to be infinitely generated.

Cutkosky [2] found the geometric meaning of the symbolic Rees rings of space monomial primes. Let $\mathbb{P}_K(a, b, c)$ be the weighted projective surface with degree a, b, c. Let $X_K(a, b, c)$ be the blow-up at a point in the open orbit of the toric variety $\mathbb{P}_K(a, b, c)$. Then the Cox ring of $X_K(a, b, c)$ is isomorphic to the extended symbolic Rees ring of the space monomial prime $\mathfrak{p}_K(a, b, c)$. Therefore, the symbolic Rees ring of the space monomial prime $\mathfrak{p}_K(a, b, c)$ is finitely generated if and only if the Cox ring of $X_K(a, b, c)$ is finitely generated, that is, $X_K(a, b, c)$ is a Mori dream space. A curve C on $X_K(a, b, c)$ is called a negative curve if $C^2 < 0$ and C is different from the exceptional curve E. Here suppose $\sqrt{abc} \notin \mathbb{Q}$. Cutkosky [2] proved that the symbolic Rees ring of the space monomial prime $\mathfrak{p}_K(a, b, c)$ is finitely generated if and only if the following two conditions are satisfied:

(1) There exists a negative curve C.
(2) There exists a curve D on $X_K(a,b,c)$ such that $C \cap D = \emptyset$.

All known examples ([7], [5]) of the infinitely generated symbolic Rees rings of $\mathfrak{p}_K(a,b,c)$ satisfy the following conditions:

(I) there exists a negative curve C such that $C.E = 1$.
(II) the characteristic of K is 0.

In this paper, we give an example of an infinitely generated symbolic Rees ring such that there exists a negative curve C with $C.E = 2$. Furthermore, in the case where both (I) and (II) as above are satisfied, we determine the minimal value of the degree of the curve D which satisfies the condition (2) as above in the case where the symbolic Rees ring is finitely generated.

The existence of negative curves is a very difficult problem, that is deeply related to the Nagata conjecture (Proposition 5.2 in Cutkosky-Kurano [3]).

In the rest of this section, we state the results of this paper precisely.

Let a, b, c be pairwise coprime integers. We regard the polynomial ring $S = K[x,y,z]$ as a $\mathbb{Z}$-graded ring by $\deg(x) = a$, $\deg(y) = b$ and $\deg(z) = c$. Let $\mathfrak{p}_K(a,b,c)$ be the kernel of the K-algebra homomorphism

$$\phi_K : S \longrightarrow K[t]$$

given by $\phi_K(x) = t^a$, $\phi_K(y) = t^b$, $\phi_K(z) = t^c$. If no confusion is possible, we simply denote $\mathfrak{p}_K(a,b,c)$ by $\mathfrak{p}$.

By a result of Herzog [9], we know that $\mathfrak{p}_K(a,b,c)$ is generated by at most three binomials. We define s, t, u to be

$$\begin{aligned} sa &= \min\{\mathbb{N}a \cap (\mathbb{N}_0 b + \mathbb{N}_0 c)\}, \\ tb &= \min\{\mathbb{N}b \cap (\mathbb{N}_0 a + \mathbb{N}_0 c)\}, \\ uc &= \min\{\mathbb{N}c \cap (\mathbb{N}_0 a + \mathbb{N}_0 b)\}, \end{aligned} \tag{1.1}$$

where $\mathbb{N}$ (resp. $\mathbb{N}_0$) denotes the set of positive integers (resp. non-negative integers). Let t_1, u_1, s_2, u_2, s_3, t_3 be non-negative integers such that $sa = t_1 b + u_1 c$, $tb = s_2 a + u_2 c$, $uc = s_3 a + t_3 b$. Then $\mathfrak{p}_K(a,b,c)$ is minimally generated by three elements if and only if s, t, $u \geq 2$. When this is the case, $\mathfrak{p}_K(a,b,c)$ is minimally generated by three elements

$$x^s - y^{t_1} z^{u_1}, \quad y^t - x^{s_2} z^{u_2}, \quad z^u - x^{s_3} y^{t_3}, \tag{1.2}$$

and t_1, u_1, s_2, u_2, s_3, t_3 must be positive integers satisfying $s = s_2 + s_3$, $t = t_1 + t_3$, $u = u_1 + u_2$.

For a prime ideal P of S, we define the symbolic Rees ring of P to be

$$\mathcal{R}_s(P) = \oplus_{n \geq 0} P^{(n)} T^n \subset S[T],$$

where $P^{(n)} = P^n S_P \cap S$ is the nth symbolic power of P and T is an indeterminate. Here, $\mathcal{R}_s(P)$ is a Noetherian ring if and only if $\mathcal{R}_s(P)$ is finitely generated over S as a ring.

In Section 2, we give a simple proof to Huneke's criterion [10]. We slightly generalize Huneke's criterion here. Furthermore, we develop the method of mod p reduction introduced in Goto-Nishida-Watanabe [7].

In Section 3, we give a proof to the following theorem:

Theorem 1.1. *Let a, b, c be pairwise coprime positive integers. Assume the following three conditions:*

(i) *K is a field of characteristic* 0,

(ii) $\mathfrak{p}_K(a,b,c)$ *is minimally generated by the three elements as in (1.2),*

(iii) $uc < \sqrt{abc}$.

Then $\mathcal{R}_s(\mathfrak{p}_K(a,b,c))$ is a Noetherian ring if and only if there exists η in $[\mathfrak{p}^{(u)}]_{ab}$ *such that $z^u - x^{s_3}y^{t_3}$ and η satisfy Huneke's condition* [10] *(see Theorem 2.5), that is,*

$$\ell_S(\, S/(x, z^u - x^{s_3}y^{t_3}, \eta)\,) = u \cdot \ell_S(\, S/(x) + \mathfrak{p}\,) \tag{1.3}$$

holds.

The condition (iii) as above implies that $z^u - x^{s_3}y^{u_3}$ is a negative curve, that is, there exists a negative curve C such that $C.E = 1$. Theorem 1.1 says that there exists a curve D such that $D \cap C = \emptyset$ and $D \sim abA - uE$ if and only if $\mathcal{R}_s(\mathfrak{p}_{\mathbb{Q}}(a,b,c))$ is a Noetherian ring, where A is an Weil divisor on X satisfying $\mathcal{O}_X(A) = \pi^*\mathcal{O}_{\mathbb{P}}(1)$.

We emphasize that it is possible to verify whether there exists η in $[\mathfrak{p}^{(u)}]_{ab}$ satisfying (1.3) as above using computers. We shall prove this theorem using the mod p reduction as in Goto-Nishida-Watanabe [7], and Cutkosky's methods [2] in characteristic $p > 0$. The most important point is that a negative curve is isomorphic to $\mathbb{P}^1_K$ in this case.

In Section 4, we introduce the condition EU. In Ebina [4] and Uchisawa [12], the condition EU was defined and they proved that the condition EU is a sufficient condition for finite generation under the assumptions (i), (ii), (iii) in Theorem 1.1. For the convenience of the reader, we shall give a proof of it in this paper. Furthermore, in the case where $u \le 6$, we show that the condition EU is a necessary and sufficient condition for the finite generation of the symbolic Rees ring of $\mathfrak{p}$ in Proposition 4.10.

In Section 5 we give an example of infinitely generated symbolic Rees ring of $\mathfrak{p}$ where the homogeneous element of the minimal degree in $\mathfrak{p}^{(2)}$ is a negative curve in Example 5.7. We emphasize that one of the minimal generators of $\mathfrak{p}$ is a negative curve in all known examples of infinitely generated $\mathcal{R}_s(\mathfrak{p}_K(a,b,c))$, except for this example.

2. Huneke's condition and mod p reduction

Let $S = K[x,y,z]$, where K is a field and x, y, z are indeterminates. We regard S as a $\mathbb{Z}$-graded ring putting suitable weights on x, y and z. We set $\mathfrak{m} = (x,y,z)S$ and $R = S_{\mathfrak{m}}$. Let I be a homogeneous proper ideal of S satisfying the following conditions;

- $(x) + I$ is $\mathfrak{m}$-primary,

- $\mathrm{Ass}_S\, S/I = \mathrm{Assh}_S\, S/I := \{\mathfrak{p} \in \mathrm{Ass}_S\, S/I \mid \dim S/\mathfrak{p} = \dim S/I\}$, and
- $I_\mathfrak{p}$ is generated by 2 elements for any $\mathfrak{p} \in \mathrm{Assh}_S\, S/I$.

Then S/I is a $\mathbb{Z}$-graded Cohen-Macaulay ring of $\dim S/I = 1$. If we replace x in the first assumption stated above with y or z, it can play the same role as x in the arguments of this section. So homogeneous prime ideals of height 2 are typical examples of I. For any $n \in \mathbb{Z}$, we set

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \mathrm{Assh}_S\, S/I} (I_\mathfrak{p}^n \cap S),$$

where $I_\mathfrak{p}^n$ denotes the ideal $(I^n)_\mathfrak{p} = (I_\mathfrak{p})^n$ of $S_\mathfrak{p}$. Then we have $\mathrm{Ass}_S\, S/I^{(n)} = \mathrm{Assh}_S\, S/I$ if $n > 0$, and the equality $I^{(n)} = I^n :_S x^i$ holds for $i \gg 0$, which means that $I^{(n)}$ is a homogeneous ideal of S and $(I^{(n)})_x = I_x^n$, where I_x^n denotes the ideal $(I^n)_x = (I_x)^n$ of S_x. Moreover, we set

$$\begin{aligned} \mathcal{R}_s(I) &= \sum_{n \geq 0} I^{(n)} T^n \subset S[T], \\ \mathcal{R}'_s(I) &= \sum_{n \in \mathbb{Z}} I^{(n)} T^n \subset S[T, T^{-1}], \text{ and} \\ \mathcal{G}_s(I) &= \mathcal{R}'_s(I)/T^{-1}\mathcal{R}'_s(I) = \oplus_{n \geq 0}\, I^{(n)}/I^{(n+1)}, \end{aligned}$$

where T is an indeterminate and $I^{(n)} = S$ for $n \leq 0$. Let us call $\mathcal{R}_s(I)$ *the symbolic Rees ring of* I.

We set $\mathfrak{a} = I_\mathfrak{m} = IR$. It is easy to see that $R/\mathfrak{a}$ is a Cohen-Macaulay local ring of $\dim R/\mathfrak{a} = 1$ and $\mathrm{Ass}_R\, R/\mathfrak{a} = \mathrm{Assh}_R\, R/\mathfrak{a} = \{\mathfrak{p}R \mid \mathfrak{p} \in \mathrm{Assh}_S\, S/I\}$. Moreover, for $\mathfrak{p} \in \mathrm{Assh}_S\, S/I$, we have $\mathfrak{a}_{\mathfrak{p}R} = I_\mathfrak{p}$, which becomes a parameter ideal of $R_{\mathfrak{p}R} = S_\mathfrak{p}$. For any $n \in \mathbb{Z}$, we set

$$\mathfrak{a}^{(n)} = \bigcap_{P \in \mathrm{Assh}_R\, R/\mathfrak{a}} (\mathfrak{a}_P^n \cap R).$$

Then we have $\mathfrak{a}^{(n)} = I^{(n)}R$ and $\mathrm{Ass}_R\, R/\mathfrak{a}^{(n)} = \mathrm{Assh}_R\, R/\mathfrak{a}$ if $n > 0$. As $\mathfrak{a}^{(n)} = \mathfrak{a}^n :_R x^i$ holds for $i \gg 0$, we have $(\mathfrak{a}^{(n)})_x = \mathfrak{a}_x^n$. The R-algebras $\mathcal{R}_s(\mathfrak{a})$ and $\mathcal{G}_s(\mathfrak{a})$ are derived from $\mathcal{R}_s(I)$ and $\mathcal{G}_s(I)$ respectively applying $R \otimes_S *$. If $\mathcal{R}_s(\mathfrak{a})$ is finitely generated, then there exists $0 < m \in \mathbb{Z}$ such that $\mathfrak{a}^{(mn)} = (\mathfrak{a}^{(m)})^n$ for any $n \in \mathbb{Z}$. This equality implies $I^{(mn)} = (I^{(m)})^n$ since $I^{(mn)} \supseteq (I^{(m)})^n$ and $(I^{(m)})^n$ is a homogeneous ideal. Thus we see that $\mathcal{R}_s(I)$ is finitely generated if so is $\mathcal{R}_s(\mathfrak{a})$. The converse of this assertion holds obviously.

For a proper ideal J of S such that S/J is Artinian, we have $\ell_S(\, S/J\,) \geq \ell_R(\, R/JR\,)$, and the equality holds if and only if J is $\mathfrak{m}$-primary, which holds if J is homogeneous.

The purpose of this section is to review the condition on I for its symbolic Rees ring to be finitely generated, which was originally given by Huneke [10] in the case where I is a prime ideal of a 3-dimensional regular local ring. Furthermore, using mod p reduction technique for prime numbers $p \gg 0$, we give a condition on I for

$\mathcal{R}_s(I)$ to be infinitely generated, which is a modification to the method introduced in [7].

Let us begin with the following

Proposition 2.1. *Let $0 < k, \ell \in \mathbb{Z}$, $\xi \in I^{(k)}$ and $\eta \in I^{(\ell)}$. Then we have*

$$\ell_R(\, R/(x, \xi, \eta)R\,) \geq kl \cdot \ell_S(\, S/(x) + I\,),$$

and the equality holds if and only if $\mathfrak{a} \subseteq \sqrt{(\xi, \eta)R}$ and

$$\ell_{S_\mathfrak{p}}(\, S_\mathfrak{p}/(\xi, \eta)S_\mathfrak{p}\,) = k\ell \cdot \ell_{S_\mathfrak{p}}(\, S_\mathfrak{p}/I_\mathfrak{p}\,)$$

for all $\mathfrak{p} \in \operatorname{Assh}_S S/I$.

In order to prove Proposition 2.1, let us recall the following fact.

Lemma 2.2. *Let A be a 2-dimensional Cohen-Macaulay local ring and Q a parameter ideal of A. Let $0 < k, \ell \in \mathbb{Z}$, $\xi \in Q^k$ and $\eta \in Q^\ell$. We assume that ξ, η is an sop for A. Then we have*

$$\ell_A(\, A/(\xi, \eta)\,) \geq k\ell \cdot \ell_A(\, A/Q\,),$$

and the equality holds if and only if one of the following conditions, which are equivalent to each other, is satisfied;

1. $QT \subseteq \sqrt{(\xi T^k, \eta T^\ell)\mathcal{R}(Q)}$, *where* $\mathcal{R}(Q) = \sum_{n \geq 0} Q^n T^n \subset A[T]$,
2. $\xi T^k, \eta T^\ell$ *is an sop for* $\mathcal{G}(Q) = \mathcal{R}(Q)/Q\mathcal{R}(Q)$,
3. $Q^{k+\ell-1} = \xi Q^{\ell-1} + \eta Q^{k-1}$,
4. $Q^n \cap (\xi, \eta)A = \xi Q^{n-k} + \eta Q^{n-\ell}$ *for any* $n \in \mathbb{Z}$.

Proof. We set $J = (\xi^\ell, \eta^k)A$. Then we have

$$k\ell \cdot \ell_A(\, A/(\xi, \eta)\,) = \ell_A(\, A/J\,) = \mathrm{e}_J(A),$$

where $\mathrm{e}_J(A)$ denotes the multiplicity of A with respect to J. Because $J \subseteq Q^{k\ell}$, it follows that

$$\mathrm{e}_J(A) \geq \mathrm{e}_{Q^{k\ell}}(A) = (k\ell)^2 \cdot \mathrm{e}_Q(A) = (k\ell)^2 \cdot \ell_A(\, A/Q\,).$$

Hence we get the required inequality. Moreover, we see that the equality holds if and only if J is a reduction of $Q^{k\ell}$, which is a condition equivalent to (1). The equivalence of the conditions (1) and (2) is obvious. Let us notice that $\mathcal{G}(Q)$ is isomorphic to a polynomial ring with 2 variables over A/Q, so its homogeneous sop is always a regular sequence, which implies the equivalence of the conditions (2) and (4). Moreover, if the condition (2) is satisfied, it follows that $\mathcal{G}(Q)/(\xi T^k, \eta T^\ell)\mathcal{G}(Q)$ is an Artinian $\mathbb{Z}$-graded ring whose a-invariant is $k + \ell - 2$ (cf. [8]), so the equality of the condition (3) holds. Finally, if the condition (3) is satisfied, we have

$$(QT)^{k+\ell-1} \subseteq \xi T^k \cdot Q^{\ell-1}T^{\ell-1} + \eta T^\ell \cdot Q^{k-1}T^{k-1} \subseteq (\xi T^k, \eta T^\ell)\mathcal{R}(Q),$$

and hence the condition (1) is satisfied. □

Proof of Proposition 2.1. We may assume that $(x, \xi, \eta)R$ is $\mathfrak{m}R$-primary. As $R/(\xi, \eta)R$ is a Cohen-Macaulay R-module, for which x is an sop, we have

$$\ell_R(\, R/(x, \xi, \eta)R\,) = \mathrm{e}_{xR}(R/(\xi, \eta)R).$$

Here we notice that $\mathfrak{p}R \in \operatorname{Assh}_R R/(\xi,\eta)R$ for any $\mathfrak{p} \in \operatorname{Assh}_S S/I$. Hence, using additive formula of multiplicity and Lemma 2.2, we get

$$\begin{aligned}
\mathrm{e}_{xR}(R/(\xi,\eta)R)) &= \sum_{P \in \operatorname{Assh}_R R/(\xi,\eta)R} \ell_{R_P}(\, R_P/(\xi,\eta)R_P\,) \cdot \mathrm{e}_{xR}(R/P) \\
&\geq \sum_{\mathfrak{p} \in \operatorname{Assh}_S S/I} \ell_{S_\mathfrak{p}}(\, S_\mathfrak{p}/(\xi,\eta)S_\mathfrak{p}\,) \cdot \mathrm{e}_{xR}(R/\mathfrak{p}R) \\
&\geq \sum_{\mathfrak{p} \in \operatorname{Assh}_S S/I} k\ell \cdot \ell_{S_\mathfrak{p}}(\, S_\mathfrak{p}/I_\mathfrak{p}\,) \cdot \mathrm{e}_{xR}(R/\mathfrak{p}R) \\
&= k\ell \cdot \mathrm{e}_{xR}(R/\mathfrak{a}) \\
&= k\ell \cdot \ell_R(\, R/(x) + \mathfrak{a}\,) \\
&= k\ell \cdot \ell_S(\, S/(x) + I\,).
\end{aligned}$$

Thus we get the required inequality. Moreover, we see that the equality holds if and only if $\operatorname{Assh}_R R/(\xi,\eta)R = \operatorname{Assh}_R R/\mathfrak{a}$ and $\ell_{S_\mathfrak{p}}(\, S_\mathfrak{p}/(\xi,\eta)S_\mathfrak{p}\,) = k\ell \cdot \ell_{S_\mathfrak{p}}(\, S_\mathfrak{p}/I_\mathfrak{p}\,)$ for all $\mathfrak{p} \in \operatorname{Assh}_S S/I$. Since $\mathfrak{a} \subseteq \sqrt{(\xi,\eta)R}$ holds if and only if $\operatorname{Assh}_R R/(\xi,\eta)R = \operatorname{Assh}_R R/\mathfrak{a}$, the proof is complete. □

Definition 2.3. Let $0 < k, \ell \in \mathbb{Z}, \xi \in I^{(k)}$ and $\eta \in I^{(\ell)}$. We say that ξ and η satisfy Huneke's condition on I (with respect to x) if

$$\ell_R(\, R/(x,\xi,\eta)R\,) = k\ell \cdot \ell_S(\, S/(x) + I\,).$$

When this is the case, for any $0 < i, j \in \mathbb{Z}$, $\xi^i \in I^{(ki)}$ and $\eta^j \in I^{(\ell j)}$ also satisfy Huneke's condition on I.

Even if there exist elements satisfying Huneke's condition, those elements may not be homogeneous. Although the existence of homogeneous elements satisfying Huneke's condition is not clear, it can be verified easily in special cases. For example, the following remark implies that if ξ and η satisfy Huneke's condition and $\xi \equiv y^i \mod xS$ for some $0 < i \in \mathbb{Z}$, then we can choose homogeneous parts of ξ and η so that they also satisfy Huneke's condition.

Lemma 2.4. *Suppose $\xi \in S$ and $\xi \equiv y^i \mod xS$, where $0 < i \in \mathbb{Z}$. Let ξ' be the homogeneous part of ξ containing y^i as a term. Then the following assertions hold.*

(1) $(x,\xi)S = (x,\xi')S = (x,y^i)S$.

(2) *For any $\eta \in S$, we can choose its homogeneous part η' so that*

$$\ell_S(\, S/(x,\xi',\eta')\,) = \ell_R(\, R/(x,\xi,\eta)R\,).$$

Proof. The assertion (1) holds obviously. Let us verify the assertion (2). We may assume $\eta \notin (x,y)S$. Then, as x, y, η is an R-regular sequence, we have

$$\ell_R(\, R/(x,\xi,\eta)R\,) = \ell_R(\, R/(x,y^i,\eta)R\,) = i \cdot \ell_R(\, R/(x,y,\eta)R\,).$$

We write

$$\eta \equiv \alpha_j z^j + \alpha_{j+1} z^{j+1} + \cdots \mod (x,y)S,$$

where $0 < j \in \mathbb{Z}$ and $\alpha_j, \alpha_{j+1}, \ldots$ are elements of K with $\alpha_j \neq 0$. Since $\alpha_j + \alpha_{j+1} z + \cdots$ is a unit of R, we have

$$\ell_R(\, R/(x,y,\eta)R\,) = \ell_R(\, R/(x,y,z^j)R\,) = j.$$

Thus we get

$$\ell_R(\, R/(x,\xi,\eta)R\,) = ij.$$

Let η' be the homogeneous part of η containing $\alpha_j z^j$ as a term. Then, as $\eta' \equiv \alpha_j z^j$ mod $(x,y)S$, it follows that

$$\ell_S(\, S/(x,\xi',\eta')\,) = \ell_S(\, S/(x,y^i,\eta')\,) = i \cdot \ell_S(\, S/(x,y,z^j)\,) = ij.$$

Thus we get the required equality. □

Theorem 2.5. *The symbolic Rees algebra $\mathcal{R}_s(I)$ is finitely generated over R if and only if there exist elements in $I^{(k)}$ and $I^{(\ell)}$ satisfying Huneke's condition on I for some $0 < k, \ell \in \mathbb{Z}$.*

Proof. First, let us assume that $\mathcal{R}_s(I)$ is finitely generated. Then there exists a positive integer m such that $I^{(mn)} = (I^{(m)})^n$ for any $n \in \mathbb{Z}$. We set $\mathfrak{b} = \mathfrak{a}^{(m)}$. Then, for any $0 < n \in \mathbb{Z}$, we have $\mathfrak{a}^{(mn)} = \mathfrak{b}^n$, which means $\operatorname{depth}_R R/\mathfrak{b}^n = 1$. Hence, by Burch's theorem (cf. [1]), we see

$$2 = \operatorname{ht}_R \mathfrak{b} \leq \lambda(\mathfrak{b}) \leq 3 - \inf_{n>0}\{\operatorname{depth} R/\mathfrak{b}^n\} = 2,$$

where $\lambda(\mathfrak{b})$ denotes the Krull dimension of $R/\mathfrak{m} \otimes \mathcal{G}(\mathfrak{b})$, which is called the analytic spread of $\mathfrak{b}$. Thus we get $\lambda(\mathfrak{b}) = 2$. Hence we can choose $0 < i, j \in \mathbb{Z}$, $\xi \in I^{(mi)}$ and $\eta \in I^{(mj)}$ such that $\xi T^i, \eta T^j$ is an sop for $R/\mathfrak{m} \otimes \mathcal{G}(\mathfrak{b})$. (Here, we notice that we don't have to assume that K is infinite since we don't require $i = j = 1$.) Let us take $r \gg 0$. Then we have $\mathfrak{b}^r = \xi \mathfrak{b}^{r-i} + \eta \mathfrak{b}^{r-j}$, which means $\mathfrak{a}^{mr} \subseteq (\xi, \eta)R$, and so $\mathfrak{a} \subseteq \sqrt{(\xi,\eta)R}$. Moreover, if $\mathfrak{p} \in \operatorname{Assh}_S S/I$ and $mr < n$, we have

$$I_\mathfrak{p}^n = \mathfrak{b}_{\mathfrak{p}R}^r I_\mathfrak{p}^{n-mr} = (\xi \mathfrak{b}_{\mathfrak{p}R}^{r-i} + \eta \mathfrak{b}_{\mathfrak{p}R}^{r-j}) I_\mathfrak{p}^{n-rm} = \xi I_\mathfrak{p}^{n-mi} + \eta I_\mathfrak{p}^{n-mj},$$

which means that $\xi T^{mi}, \eta T^{mj}$ is an sop for $\mathcal{G}(I_\mathfrak{p})$. Therefore by Proposition 2.1 and Lemma 2.2, it follows that ξ and η satisfy Huneke's condition on I.

Next, we assume that there exist $0 < k, \ell \in \mathbb{Z}$, $\xi \in I^{(k)}$ and $\eta \in I^{(\ell)}$ such that ξ and η satisfy Huneke's condition on I. We set $m = k\ell$, $\mathfrak{b} = \mathfrak{a}^{(m)}$ and $\mathfrak{c} = (\xi^\ell, \eta^k)R \subseteq \mathfrak{b}$. Let us look at the exact sequence

$$0 \longrightarrow \mathfrak{c}^r/\mathfrak{b}\mathfrak{c}^r \longrightarrow R/\mathfrak{b}\mathfrak{c}^r \longrightarrow R/\mathfrak{c}^r \longrightarrow 0$$

of R-modules, where r is any non-negative integer. Since ξ^ℓ, η^k is an R-regular sequence, $\mathfrak{c}^r/\mathfrak{c}^{r+1}$ is $R/\mathfrak{c}$-free, so $\mathfrak{c}^r/\mathfrak{b}\mathfrak{c}^r \cong R/\mathfrak{b} \otimes_R \mathfrak{c}^r/\mathfrak{c}^{r+1}$ is $R/\mathfrak{b}$-free, which means

$$\operatorname{Ass}_R \mathfrak{c}^r/\mathfrak{b}\mathfrak{c}^r = \operatorname{Ass}_R R/\mathfrak{b} = \operatorname{Assh}_R R/\mathfrak{a}.$$

On the other hand, by Proposition 2.1 we have

$$\operatorname{Ass}_R R/\mathfrak{c}^r = \operatorname{Assh}_R R/\mathfrak{c} = \operatorname{Assh}_R R/\mathfrak{a}$$

since ξ^ℓ and η^k also satisfy Huneke's condition on I. Thus we see

$$\operatorname{Ass}_R R/\mathfrak{b}\mathfrak{c}^r = \operatorname{Assh}_R R/\mathfrak{a}.$$

Now we take any $P \in \operatorname{Assh}_R R/\mathfrak{a}$, and write $P = \mathfrak{p}R$, where $\mathfrak{p} \in \operatorname{Assh}_S S/I$. Then by Proposition 2.1 and Lemma 2.2, we have $I_{\mathfrak{p}}^{2m-1} = \xi^{\ell} I_{\mathfrak{p}}^{m-1} + \eta^k I_{\mathfrak{p}}^{m-1}$, which means $\mathfrak{b}_P^2 = (\mathfrak{b}\mathfrak{c})_P$, and so $\mathfrak{b}_P^{r+1} = (\mathfrak{b}\mathfrak{c}^r)_P$. Hence we get

$$\mathfrak{a}^{(mr+m)} = \bigcap_{P \in \operatorname{Assh}_R R/\mathfrak{a}} (\mathfrak{b}_P^{r+1} \cap R) = \bigcap_{P \in \operatorname{Assh}_R R/\mathfrak{a}} ((\mathfrak{b}\mathfrak{c}^r)_P \cap R) = \mathfrak{b}\mathfrak{c}^r \subseteq \mathfrak{b}^{r+1} \subseteq \mathfrak{a}^{(mr+m)},$$

and hence $\mathfrak{a}^{(mr+m)} = \mathfrak{b}\mathfrak{c}^r = \mathfrak{b}^{r+1}$. Thus we see that the m-th Veronese subring of $\mathcal{R}_s(\mathfrak{a})$ is generated in degree one. Therefore $\mathcal{R}_s(\mathfrak{a})$ is Noetherian by [6, Lemma (2.4)]. Then $\mathcal{R}_s(I)$ itself must be Noetherian. ☐

Lemma 2.6. *Let $0 < k, \ell \in \mathbb{Z}$, $\xi \in I^{(k)}$ and $\eta \in I^{(\ell)}$. Suppose that ξ and η satisfy Huneke's condition on I. Then the following assertions hold.*

(1) $\mathcal{R}_s(\mathfrak{a})_+ = \sqrt{(\xi T^k, \eta T^{\ell})\mathcal{R}_s(\mathfrak{a})}$, *and hence* $\mathcal{G}_s(\mathfrak{a})_+ \subseteq \sqrt{(\xi T^k, \eta T^{\ell})\mathcal{G}_s(\mathfrak{a})}$.
(2) $\mathfrak{a}^{(k+\ell-1)} \subseteq (\xi, \eta)R$.
(3) $\mathfrak{a}_x^n \cap (\xi, \eta)R_x = \xi\mathfrak{a}_x^{n-k} + \eta\mathfrak{a}_x^{n-\ell}$ *for any* $n \in \mathbb{Z}$.
(4) $\mathfrak{a}^{(n)} \cap (\xi, \eta)R = \xi\mathfrak{a}^{(n-k)} + \eta\mathfrak{a}^{(n-\ell)}$ *if* $n \leq k + \ell$.
(5) *If $k = 1$ or 2, then we have*

$$\mathfrak{a}^{(n)} \cap (\xi, \eta)R = \xi\mathfrak{a}^{(n-k)} + \eta\mathfrak{a}^{(n-\ell)}$$

for any $n \in \mathbb{Z}$, which means that $\xi T^k, \eta T^{\ell}$ is a regular sequence on $\mathcal{G}_s(\mathfrak{a})$, and hence $\operatorname{grade} \mathcal{G}_s(\mathfrak{a})_+ = 2$.

Proof. (1) We set $m = k\ell, \mathfrak{b} = \mathfrak{a}^{(m)}$ and $\mathfrak{c} = (\xi^{\ell}, \eta^k)R$. Then, as is stated in the proof of Theorem 2.5, we have $\mathfrak{a}^{(mr+m)} = \mathfrak{b}^{r+1} = \mathfrak{b}\mathfrak{c}^r$ for any $0 \leq r \in \mathbb{Z}$. Let us take any $0 < n \in \mathbb{Z}$ and $\rho \in \mathfrak{a}^{(n)}$. Then we have $\rho^{2m} \in \mathfrak{a}^{(m(2n-1)+m)} = \mathfrak{b}^{2n} = \mathfrak{b}\mathfrak{c}^{2n-1} \subseteq \mathfrak{c}\mathfrak{b}^{2n-1} = \xi^{\ell}\mathfrak{b}^{2n-1} + \eta^k\mathfrak{b}^{2n-1} \subseteq \xi\mathfrak{a}^{(2mn-k)} + \eta\mathfrak{a}^{(2mn-\ell)}$, so

$$(\rho T^n)^{2m} \in \xi T^k \cdot \mathfrak{a}^{(2mn-k)} T^{2mn-k} + \eta T^{\ell} \cdot \mathfrak{a}^{(2mn-\ell)} T^{2mn-\ell}.$$

Hence we get the assertion (1).

(2) Let us take any $P \in \operatorname{Assh}_R R/\mathfrak{a}$ and write $P = \mathfrak{p}R$, where $\mathfrak{p} \in \operatorname{Assh}_S S/I$. Then, as $R_P = S_{\mathfrak{p}}$ and $\mathfrak{a}_P = I_{\mathfrak{p}}$, by Proposition 2.1 and Lemma 2.2, we have $\mathfrak{a}_P^{k+\ell-1} = \xi\mathfrak{a}_P^{\ell-1} + \eta\mathfrak{a}_P^{k-1} \subseteq (\xi, \eta)R_P$. Therefore we get

$$\mathfrak{a}^{(k+\ell-1)} = \bigcap_{P \in \operatorname{Assh}_R R/\mathfrak{a}} (\mathfrak{a}_P^{k+\ell-1} \cap R) \subseteq \bigcap_{P \in \operatorname{Assh}_R R/\mathfrak{a}} ((\xi, \eta)R_P \cap R) = (\xi, \eta)R.$$

(3) Since $\xi \in I_x^k$ and $\eta \in I_x^{\ell}$, the inclusion $\mathfrak{a}_x^n \supseteq \xi\mathfrak{a}_x^{n-k} + \eta\mathfrak{a}_x^{n-\ell}$ holds obviously. So it is enough to show

$$\mathfrak{a}_P^n \cap (\xi, \eta)R_P = \xi\mathfrak{a}_P^{n-k} + \eta\mathfrak{a}_P^{n-\ell}$$

for any $P \in \operatorname{Spec} R$ satisfying $\xi\mathfrak{a}^{n-k} + \eta\mathfrak{a}^{n-\ell} \subseteq P$ and $x \notin P$. Such a P must contain $\mathfrak{a}$ since $\mathfrak{a} \subseteq \sqrt{(\xi, \eta)R}$, so there exists $\mathfrak{p} \in \operatorname{Assh}_S S/I$ such that $P = \mathfrak{p}R$. Then by Proposition 2.1 and Lemma 2.2, we get the required equality as $R_P = S_{\mathfrak{p}}$ and $\mathfrak{a}_P = I_{\mathfrak{p}}$.

(4) Let $n \leq k + \ell$ and $\varphi \in \mathfrak{a}^{(n)} \cap (\xi, \eta)R$. We write $\varphi = \xi u + \eta v$, where $u, v \in R$. Since $\varphi \in \mathfrak{a}_x^n \cap (\xi, \eta)R_x = \xi\mathfrak{a}_x^{n-k} + \eta\mathfrak{a}_x^{n-\ell}$ by (3), there exist $\alpha \in \mathfrak{a}_x^{n-k}$ and $\beta \in \mathfrak{a}_x^{n-\ell}$ such that $\varphi = \xi\alpha + \eta\beta$. Here, we take $i \gg 0$ so that $x^i\alpha \in \mathfrak{a}^{n-k}$ and $x^i\beta \in \mathfrak{a}^{n-\ell}$.

Then we have $x^i(\xi u + \eta v) = x^i\varphi = x^i(\xi\alpha + \eta\beta)$, so $\xi(x^i u - x^i\alpha) = \eta(x^i\beta - x^i v)$. Since ξ, η is an R-regular sequence, it follows that $x^i u - x^i\alpha \in \eta R \subseteq \mathfrak{a}^{(\ell)} \subseteq \mathfrak{a}^{(n-k)}$ and $x^i\beta - x^i v \in \xi R \subseteq \mathfrak{a}^{(k)} \subseteq \mathfrak{a}^{(n-\ell)}$. Hence $x^i u \in \mathfrak{a}^{(n-k)}$ and $x^i v \in \mathfrak{a}^{(n-\ell)}$, which means $u \in \mathfrak{a}^{(n-k)}$ and $v \in \mathfrak{a}^{(n-\ell)}$. Thus we get $\varphi \in \xi\mathfrak{a}^{(n-k)} + \eta\mathfrak{a}^{(n-\ell)}$.

(5) Let $k = 1$ or 2. By (2) and (4), it is enough to show

$$\mathfrak{a}^{(n)} = \xi\mathfrak{a}^{(n-k)} + \eta\mathfrak{a}^{(n-\ell)}$$

assuming $n > k + \ell$. We take positive integers m and r such that $n - \ell = km - r$ and $0 \le r < k$. Then $m \ge 2$ and r is 0 or 1. Since $\xi^m \in I^{(km)}$ and $\eta \in I^{(\ell)}$ also satisfy Huneke's condition on I and $km + \ell - 1 \le km + \ell - r = n \le km + \ell$, we have $\mathfrak{a}^{(n)} \subseteq (\xi^m, \eta)R$ by (2) and $\mathfrak{a}^{(n)} \cap (\xi^m, \eta)R = \xi^m\mathfrak{a}^{(\ell-r)} + \eta\mathfrak{a}^{(n-\ell)}$ by (4). Let us notice that $\xi^{m-1}\mathfrak{a}^{(\ell-r)} \subseteq \mathfrak{a}^{(n-k)}$ as $k(m-1) + (\ell - r) = n - k$. Thus we see $\mathfrak{a}^{(n)} \subseteq \xi\mathfrak{a}^{(n-k)} + \eta\mathfrak{a}^{(n-\ell)}$. Since the converse inclusion is obvious, we get the required equality. ☐

Definition 2.7. Let $0 < k \in \mathbb{Z}$ and $\xi \in I^{(k)}$. We denote by $\mathrm{HC}(I; k, \xi)$ the set of positive integers ℓ for which there exists $\eta \in I^{(\ell)}$ such that ξ and η satisfy Huneke's condition on I.

Remark 2.8. Let k and ξ be as in Definition 2.7. If $\xi \equiv y^i \mod xS$, where $0 < i \in \mathbb{Z}$, and ξ' is the homogeneous part of ξ containing y^i as a term, we have $\mathrm{HC}(I; k, \xi) = \mathrm{HC}(I; k, \xi')$ by Lemma 2.4 (1).

Proposition 2.9. *Let $k = 1$ or 2, and let $\xi \in I^{(k)}$. Suppose that $\xi \equiv y^i \mod xS$ for some $0 < i \in \mathbb{Z}$ and $\mathrm{HC}(I; k, \xi) \neq \phi$. We set $m = \min \mathrm{HC}(I; k, \xi)$. Then the following assertions hold.*

1. $\mathrm{HC}(I; k, \xi) = \{m, 2m, 3m, \cdots\}$.
2. $S[\{I^{(n)}T^n \mid 1 \le n \le m - 1\}] \subsetneq \mathcal{R}_s(I)$.
3. *If there exist elements in $I^{(k')}$ and $I^{(\ell')}$ satisfying Huneke's condition on I for $0 < k', \ell' \in \mathbb{Z}$, we have*

$$S[\{I^{(n)}T^n \mid 1 \le n \le \max\{k', \ell', k' + \ell' - 2\}\}] = \mathcal{R}_s(I).$$

In particular,

$$S[\{I^{(n)}T^n \mid 1 \le n \le \max\{k, m\}\}] = \mathcal{R}_s(I).$$

Proof. By Remark 2.8, we may assume that ξ is homogeneous. Then by Lemma 2.4 (2), we can choose a homogeneous element $\eta \in I^{(m)}$ such that ξ and η satisfy Huneke's condition on I.

(1) We obviously have $\mathrm{HC}(I; k, \xi) \supseteq \{m, 2m, 3m, \cdots\}$: see the remark after Definition 2.3. In order to show the converse inclusion, we suppose that there exists $\ell \in \mathrm{HC}(I; k, \xi)$ which is not a multiple of m. Let us choose such ℓ as small as possible. Then there exists a homogeneous element $\rho \in I^{(\ell)}$ such that ξ and ρ satisfy Huneke's condition on I. Since $m < \ell$, by Lemma 2.6 (2) and (5), we have $\mathfrak{a}^{(\ell)} = \xi\mathfrak{a}^{(\ell-k)} + \eta\mathfrak{a}^{(\ell-m)}$, which implies

$$I^{(\ell)} = \xi I^{(\ell-k)} + \eta I^{(\ell-m)}$$

as ξ and η are homogeneous. Hence there exists a homogeneous element $\rho' \in I^{(\ell-m)}$ such that

$$\rho \equiv \eta\rho' \mod \xi I^{(\ell-k)}.$$

Then $\rho \in (\xi, \rho')S$, and hence we get

$$\mathfrak{a} \subseteq \sqrt{(\xi, \rho')R}$$

as $\mathfrak{a} \subseteq \sqrt{(\xi, \rho)R}$ by Proposition 2.1. Now we take any $\mathfrak{p} \in \mathrm{Assh}_S\, S/I$ and $n \gg 0$. Then by Proposition 2.1 and Lemma 2.2, we have

$$I_{\mathfrak{p}}^n = \xi I_{\mathfrak{p}}^{n-k} + \rho I_{\mathfrak{p}}^{n-\ell} = \xi I_{\mathfrak{p}}^{n-k} + \eta\rho' I_{\mathfrak{p}}^{n-\ell} \subseteq \xi I_{\mathfrak{p}}^{n-k} + \rho' I_{\mathfrak{p}}^{n-(\ell-m)} \subseteq I_{\mathfrak{p}}^n,$$

so we get

$$I_{\mathfrak{p}}^n = \xi I_{\mathfrak{p}}^{n-k} + \rho' I_{\mathfrak{p}}^{n-(\ell-m)}.$$

Therefore ξ and ρ' satisfy Huneke's condition on I, so $\ell - m \in \mathrm{HC}(I; k, \xi)$, which contradicts to the minimality of ℓ as $\ell - m$ is not a multiple of m. Consequently, we see that any $\ell \in \mathrm{HC}(I; k, \xi)$ is a multiple of m.

(2) The assertion holds obviously if $m = 1$, so let us consider the case where $m \geq 2$. Suppose

$$\eta T^m \in S[\{I^{(n)}T^n \mid 1 \leq n \leq m-1\}].$$

Then we have

$$\eta \in \sum_{\alpha=1}^{m-1} I^{(\alpha)} I^{(m-\alpha)}.$$

We set $\overline{S} = S/(x, y) \cong K[z]$. Since any homogeneous ideal of $\overline{S}$ is a power of $z\overline{S}$,

$$\sum_{\alpha=1}^{m-1} I^{(\alpha)} I^{(m-\alpha)} \overline{S} = I^{(\beta)} I^{(m-\beta)} \overline{S}$$

holds for some $\beta = 1, 2, \ldots, m-1$. Moreover, we can choose homogeneous elements $\rho \in I^{(\beta)}$ and $\rho' \in I^{(m-\beta)}$ such that η and $\rho\rho'$ have the same class in $\overline{S}$, which is equivalent to

$$\eta \equiv \rho\rho' \mod (x, y).$$

Then by Proposition 2.1, we have

$$\begin{aligned} \ell_S(\, S/(x, \xi, \rho)\,) &\geq k\beta \cdot \ell_S(\, S/(x) + I\,) \quad \text{and} \\ \ell_S(\, S/(x, \xi, \rho')\,) &\geq k(m-\beta) \cdot \ell_S(\, S/(x) + I\,). \end{aligned}$$

Since (x,y,η), (x,y,ρ) and (x,y,ρ') are all homogeneous $\mathfrak{m}$-primary ideals, we have

$$\begin{aligned}
\ell_S(\,S/(x,\xi,\eta)\,) &= \ell_S(\,S/(x,y^i,\eta)\,)\\
&= i\cdot \ell_S(\,S/(x,y,\eta)\,)\\
&= i\cdot \ell_S(\,S/(x,y,\rho\rho')\,)\\
&= i\cdot \{\ell_S(\,S/(x,y,\rho)\,)+\ell_S(\,S/(x,y,\rho')\,)\}\\
&= \ell_S(\,S/(x,y^i,\rho)\,)+\ell_S(\,S/(x,y^i,\rho')\,)\\
&= \ell_S(\,S/(x,\xi,\rho)\,)+\ell_S(\,S/(x,\xi,\rho')\,)\\
&\geq k\beta\cdot \ell_S(\,S/(x)+I\,)+k(m-\beta)\cdot \ell_S(\,S/(x)+I\,)\\
&= \{k\beta+k(m-\beta)\}\cdot \ell_S(\,S/(x)+I\,)\\
&= km\cdot \ell_S(\,S/(x)+I\,)\\
&= \ell_S(\,S/(x,\xi,\eta)\,).
\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}
\ell_S(\,S/(x,\xi,\rho)\,) &= k\beta\cdot \ell_S(\,S/(x)+I\,) \quad \text{and}\\
\ell_S(\,S/(x,\xi,\rho')\,) &= k(m-\beta)\cdot \ell_S(\,S/(x)+I\,).
\end{aligned}$$

Hence we get $\beta, m-\beta \in \mathrm{HC}(I;k,\xi)$, which contradicts to the minimality of m. Thus we see

$$\eta T^m \notin S[\{I^{(n)}T^n \mid 1 \leq n \leq m-1\}].$$

(3) Let $0 < k', \ell' \in \mathbb{Z}$, $\xi' \in I^{(k')}$ and $\eta' \in I^{(\ell')}$. Suppose that ξ' and η' satisfy Huneke's condition on I. Then by Lemma 2.6 (1), we have

$$\mathcal{G}_s(\mathfrak{a})_+ \subseteq \sqrt{(\xi' T^{k'}, \eta' T^{\ell'})\mathcal{G}_s(\mathfrak{a})}.$$

On the other hand, from the existence of ξ and η, we see $\operatorname{grade}\mathcal{G}_s(\mathfrak{a})_+ = 2$ by Lemma 2.6 (5). Hence it follows that $\xi' T^{k'}, \eta' T^{\ell'}$ is a regular sequence on $\mathcal{G}_s(\mathfrak{a})$. If $k'+\ell'-1 \leq n$, we have $\mathfrak{a}^{(n)} \subseteq (\xi',\eta')R$ by Lemma 2.6 (2), so

$$\mathfrak{a}^{(n)} = \mathfrak{a}^{(n)} \cap (\xi',\eta')R = \xi'\mathfrak{a}^{(n-k')} + \eta'\mathfrak{a}^{(n-\ell')}.$$

Thus we see

$$\begin{aligned}
\mathcal{R}_s(\mathfrak{a}) &= S[\xi' T^{k'}, \eta' T^{\ell'}, \{\mathfrak{a}^{(n)}T^n \mid 1 \leq n \leq k'+\ell'-2\}]\\
&= S[\{\mathfrak{a}^{(n)}T^n \mid 1 \leq n \leq \max\{k',\ell',k'+\ell'-2\}\}],
\end{aligned}$$

which means that the first assertion of (3) holds. We get the last assertion taking k and m as k' and ℓ', respectively. □

In the rest of this section, let $S_{\mathbb{Z}} = \mathbb{Z}[x,y,z]$. Moreover, for a field K, we denote $K[x,y,z]$ by S_K instead of S in order to emphasize that the coefficient field is K. Putting suitable weights on x, y and z, we regard $S_{\mathbb{Z}}$ and S_K as $\mathbb{Z}$-graded rings. We set $\mathfrak{m}_{\mathbb{Z}} = (x,y,z)S_{\mathbb{Z}}$, $\mathfrak{m}_K = (x,y,z)S_K$ and $R_K = (S_K)_{m_K}$. When we denote an ideal of $S_{\mathbb{Z}}$ by $J_{\mathbb{Z}}$, the ideal $J_{\mathbb{Z}}S_K$ is denoted by J_K. Similarly, when we denote an element of $S_{\mathbb{Z}}$ by $\xi_{\mathbb{Z}}$, its image in S_K is denoted by ξ_K. For a prime number p, we set $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Of course, $S_{\mathbb{Q}} = (\mathbb{Z}\setminus\{0\})^{-1}S_{\mathbb{Z}}$ and $S_{\mathbb{F}_p} = S_{\mathbb{Z}}/pS_{\mathbb{Z}}$.

Lemma 2.10. *Let $J_{\mathbb{Z}}$ be an ideal of $S_{\mathbb{Z}}$. Then, we have*

$$\ell_{R_{\mathbb{Q}}}(\, R_{\mathbb{Q}}/(J_{\mathbb{Q}})_{\mathfrak{m}_{\mathbb{Q}}}\,) = \ell_{R_{\mathbb{F}_p}}(\, R_{\mathbb{F}_p}/(J_{\mathbb{F}_p})_{\mathfrak{m}_{\mathbb{F}_p}}\,)$$

for any prime number $p \gg 0$. If $J_{\mathbb{Z}}$ is homogeneous, we may replace $R_{\mathbb{Q}}, (J_{\mathbb{Q}})_{\mathfrak{m}_{\mathbb{Q}}}, R_{\mathbb{F}_p}$ and $(J_{\mathbb{F}_p})_{\mathfrak{m}_{\mathbb{F}_p}}$ with $S_{\mathbb{Q}}, J_{\mathbb{Q}}, S_{\mathbb{F}_p}$ and $J_{\mathbb{F}_p}$, respectively.

Proof. First, let us consider the case where $R_{\mathbb{Q}}/(J_{\mathbb{Q}})_{\mathfrak{m}_{\mathbb{Q}}}$ is Artinian. We prove the required equality by induction on $\ell_{R_{\mathbb{Q}}}(\, R_{\mathbb{Q}}/(J_{\mathbb{Q}})_{\mathfrak{m}_{\mathbb{Q}}}\,)$.

If $\ell_{R_{\mathbb{Q}}}(\, R_{\mathbb{Q}}/(J_{\mathbb{Q}})_{\mathfrak{m}_{\mathbb{Q}}}\,) = 0$, then $J_{\mathbb{Q}}$ contains an element which does not belong to $\mathfrak{m}_{\mathbb{Q}}$, so there exists $\xi_{\mathbb{Z}} \in J_{\mathbb{Z}} \setminus \mathfrak{m}_{\mathbb{Z}}$. Let us take a prime number $p \gg 0$ so that the constant term of $\xi_{\mathbb{Z}}$, which is non-zero, is not a multiple of p. Then $\xi_{\mathbb{F}_p} \in J_{\mathbb{F}_p} \setminus \mathfrak{m}_{\mathbb{F}_p}$. Hence $(J_{\mathbb{F}_p})_{\mathfrak{m}_{\mathbb{F}_p}} = R_{\mathbb{F}_p}$, so $\ell_{R_{\mathbb{F}_p}}(\, R_{\mathbb{F}_p}/(J_{\mathbb{F}_p})_{\mathfrak{m}_{\mathbb{F}_p}}\,) = 0$.

Now we suppose $\ell_{R_{\mathbb{Q}}}(\, R_{\mathbb{Q}}/(J_{\mathbb{Q}})_{\mathfrak{m}_{\mathbb{Q}}}\,) > 0$. Then, as $\mathfrak{m}_{\mathbb{Z}} \in \operatorname{Ass}_{S_{\mathbb{Z}}} S_{\mathbb{Z}}/J_{\mathbb{Z}}$, there exists $\eta_{\mathbb{Z}} \in S_{\mathbb{Z}}$ such that $J_{\mathbb{Z}} : \eta_{\mathbb{Z}} = \mathfrak{m}_{\mathbb{Z}}$. We set $L_{\mathbb{Z}} = J_{\mathbb{Z}} + (\eta_{\mathbb{Z}})$. Let us notice $L_{\mathbb{Q}}/J_{\mathbb{Q}} \cong S_{\mathbb{Q}}/\mathfrak{m}_{\mathbb{Q}}$. Hence we have $\ell_{R_{\mathbb{Q}}}(\, (L_{\mathbb{Q}}/J_{\mathbb{Q}})_{\mathfrak{m}_{\mathbb{Q}}}\,) = 1$, so

$$\ell_{R_{\mathbb{Q}}}(\, R_{\mathbb{Q}}/(L_{\mathbb{Q}})_{\mathfrak{m}_{\mathbb{Q}}}\,) = \ell_{R_{\mathbb{Q}}}(\, R_{\mathbb{Q}}/(J_{\mathbb{Q}})_{\mathfrak{m}_{\mathbb{Q}}}\,) - 1.$$

Here, we take a prime number $p \gg 0$. Then the hypothesis of induction implies

$$\ell_{R_{\mathbb{Q}}}(\, R_{\mathbb{Q}}/(L_{\mathbb{Q}})_{\mathfrak{m}_{\mathbb{Q}}}\,) = \ell_{R_{\mathbb{F}_p}}(\, R_{\mathbb{F}_p}/(L_{\mathbb{F}_p})_{\mathfrak{m}_{\mathbb{F}_p}}\,).$$

Moreover, by taking larger p if necessary, we may assume that p is regular on $S_{\mathbb{Z}}/L_{\mathbb{Z}}$. If $\eta_{\mathbb{F}_p} \in J_{\mathbb{F}_p}$, we have $\eta_{\mathbb{Z}} \in J_{\mathbb{Z}} + pS_{\mathbb{Z}}$, so there exists $\rho_{\mathbb{Z}} \in S_{\mathbb{Z}}$ such that $\eta_{\mathbb{Z}} \equiv p \cdot \rho_{\mathbb{Z}}$ mod $J_{\mathbb{Z}}$. Since p is regular on $S_{\mathbb{Z}}/L_{\mathbb{Z}}$, we have $\rho_{\mathbb{Z}} \in L_{\mathbb{Z}}$, so there exists $\sigma_{\mathbb{Z}} \in S_{\mathbb{Z}}$ such that $\rho_{\mathbb{Z}} \equiv \eta_{\mathbb{Z}}\sigma_{\mathbb{Z}}$ mod $J_{\mathbb{Z}}$. Then we have $\eta_{\mathbb{Z}} \equiv p \cdot \eta_{\mathbb{Z}}\sigma_{\mathbb{Z}}$ mod $J_{\mathbb{Z}}$, and hence $1 - p \cdot \sigma_{\mathbb{Z}} \in J_{\mathbb{Z}} : \eta_{\mathbb{Z}} = \mathfrak{m}_{\mathbb{Z}}$, which is impossible. Thus we see $\eta_{\mathbb{F}_p} \notin J_{\mathbb{F}_p}$. Hence we have $\mathfrak{m}_{\mathbb{F}_p} = J_{\mathbb{F}_p} : \eta_{\mathbb{F}_p}$ since $\eta_{\mathbb{F}_p} \cdot \mathfrak{m}_{\mathbb{F}_p} \subseteq J_{\mathbb{F}_p}$ holds obviously. Then we get $L_{\mathbb{F}_p}/J_{\mathbb{F}_p} \cong R_{\mathbb{F}_p}/\mathfrak{m}_{\mathbb{F}_p}$, so $\ell_{R_{\mathbb{F}_p}}(\, L_{\mathbb{F}_p}/J_{\mathbb{F}_p}\,) = 1$. Consequently,

$$\ell_{R_{\mathbb{F}_p}}(\, R_{\mathbb{F}_p}/(L_{\mathbb{F}_p})_{\mathfrak{m}_{\mathbb{F}_p}}\,) = \ell_{R_{\mathbb{F}_p}}(\, R_{\mathbb{F}_p}/(J_{\mathbb{F}_p})_{\mathfrak{m}_{\mathbb{F}_p}}\,) - 1.$$

Therefore the required equality follows.

Next, we assume $\dim R_{\mathbb{Q}}/(J_{\mathbb{Q}})_{\mathfrak{m}_{\mathbb{Q}}} > 0$, and aim to prove $\dim R_{\mathbb{F}_p}/(J_{\mathbb{F}_p})_{\mathfrak{m}_{\mathbb{F}_p}} > 0$ for $p \gg 0$. In this case, there exists $P_{\mathbb{Z}} \in \operatorname{Spec} S_{\mathbb{Z}}$ such that $J_{\mathbb{Z}} \subseteq P_{\mathbb{Z}} \subsetneq \mathfrak{m}_{\mathbb{Z}}$. Let us take any $\tau_{\mathbb{Z}} \in \mathfrak{m}_{\mathbb{Z}} \setminus P_{\mathbb{Z}}$ and choose a prime number $p \gg 0$ so that p is regular on $S_{\mathbb{Z}}/(\tau_{\mathbb{Z}}) + P_{\mathbb{Z}}$. Then, as $p, \tau_{\mathbb{Z}}$ is a regular sequence on $(S_{\mathbb{Z}}/P_{\mathbb{Z}})_{(pS_{\mathbb{Z}}+\mathfrak{m}_{\mathbb{Z}})}$, it follows that $\tau_{\mathbb{Z}}$ is regular on $(S_{\mathbb{Z}}/pS_{\mathbb{Z}} + P_{\mathbb{Z}})_{(pS_{\mathbb{Z}}+\mathfrak{m}_{\mathbb{Z}})} \cong R_{\mathbb{F}_p}/(P_{\mathbb{F}_p})_{\mathfrak{m}_{\mathbb{F}_p}}$. Hence we have $\dim R_{\mathbb{F}_p}/(P_{\mathbb{F}_p})_{\mathfrak{m}_{\mathbb{F}_p}} > 0$, and so $\dim R_{\mathbb{F}_p}/(J_{\mathbb{F}_p})_{\mathfrak{m}_{\mathbb{F}_p}} > 0$ as $J_{\mathbb{F}_p} \subseteq P_{\mathbb{F}_p}$. □

In the rest of this section, let $I_{\mathbb{Z}}$ be a homogeneous ideal of $S_{\mathbb{Z}}$ contained in $\mathfrak{m}_{\mathbb{Z}}$. We assume that the following conditions are satisfied for any field K;

- $(x) + I_K$ is $\mathfrak{m}_K$-primary,
- $\operatorname{Ass}_{S_K} S_K/I_K = \operatorname{Assh}_{S_K} S_K/I_K$, and
- $(I_K)_{\mathfrak{p}}$ is generated by 2 elements for any $\mathfrak{p} \in \operatorname{Assh}_{S_K} S_K/I_K$.

Furthermore, for any $n \in \mathbb{Z}$, we set $(I^{(n)})_{\mathbb{Z}} = \bigcup_{i>0}((I_{\mathbb{Z}})^n :_{S_{\mathbb{Z}}} x^i)$, which is a homogeneous ideal of $S_{\mathbb{Z}}$. Let us denote $(I^{(n)})_{\mathbb{Z}} S_K$ by $(I^{(n)})_K$ for any field K.

Lemma 2.11. *The following assertions hold for any $n \in \mathbb{Z}$.*

(1) $(I_{\mathbb{Q}})^{(n)} = (I^{(n)})_{\mathbb{Q}}$.

(2) $(I_{\mathbb{F}_p})^{(n)} = (I^{(n)})_{\mathbb{F}_p}$ *for any prime number* $p \gg 0$.

Proof. First, let us notice that, for any field K, we have $(I_K)^{(n)} = \bigcup_{i>0} ((I_K)^n :_{S_K} x^i)$, and hence $(I_K)^{(n)} \supseteq (I^{(n)})_K$ holds. The converse inclusion holds obviously if $K = \mathbb{Q}$. So we have to prove $(I_{\mathbb{F}_p})^{(n)} \subseteq (I^{(n)})_{\mathbb{F}_p}$ for $p \gg 0$.

Let us take a prime number $p \gg 0$ so that p is regular on $S_{\mathbb{Z}}/(x) + (I^{(n)})_{\mathbb{Z}}$. Moreover, we take any $\xi_{\mathbb{Z}} \in S_{\mathbb{Z}}$ satisfying $\xi_{\mathbb{F}_p} \in (I_{\mathbb{F}_p})^{(n)}$. Then there exists $0 < i \in \mathbb{Z}$ such that $x^i \xi_{\mathbb{F}_p} \in (I_{\mathbb{F}_p})^n$, which means $x^i \xi_{\mathbb{Z}} \in pS_{\mathbb{Z}} + (I_{\mathbb{Z}})^n \subseteq pS_{\mathbb{Z}} + (I^{(n)})_{\mathbb{Z}}$. Hence there exists $\eta_{\mathbb{Z}} \in S_{\mathbb{Z}}$ such that $x^i \xi_{\mathbb{Z}} \equiv p\eta_{\mathbb{Z}} \mod (I^{(n)})_{\mathbb{Z}}$. Since x, p is a regular sequence on $S_{\mathbb{Z}}/(I^{(n)})_{\mathbb{Z}}$, so is x^i, p. Hence $\eta_{\mathbb{Z}} \in (x^i) + (I^{(n)})_{\mathbb{Z}}$, so there exists $\rho_{\mathbb{Z}} \in S_{\mathbb{Z}}$ such that $\eta_{\mathbb{Z}} \equiv x^i \rho_{\mathbb{Z}} \mod (I^{(n)})_{\mathbb{Z}}$. Then we have $x^i \xi_{\mathbb{Z}} \equiv p x^i \rho_{\mathbb{Z}} \mod (I^{(n)})_{\mathbb{Z}}$, which means $\xi_{\mathbb{Z}} - p\rho_{\mathbb{Z}} \in (I^{(n)})_{\mathbb{Z}}$. Thus we get $\xi_{\mathbb{F}_p} \in (I^{(n)})_{\mathbb{F}_p}$. □

Proposition 2.12. *Let $0 < k, \ell \in \mathbb{Z}, \xi_{\mathbb{Z}} \in (I^{(k)})_{\mathbb{Z}}$ and $\eta_{\mathbb{Z}} \in (I^{(\ell)})_{\mathbb{Z}}$. Suppose that $\xi_{\mathbb{Q}} \in (I_{\mathbb{Q}})^{(k)}$ and $\eta_{\mathbb{Q}} \in (I_{\mathbb{Q}})^{(\ell)}$ satisfy Huneke's condition on $I_{\mathbb{Q}}$. Then for any prime number $p \gg 0$, $\xi_{\mathbb{F}_p} \in (I_{\mathbb{F}_p})^{(k)}$, $\eta_{\mathbb{F}_p} \in (I_{\mathbb{F}_p})^{(\ell)}$, and these elements satisfy Huneke's condition on $I_{\mathbb{F}_p}$.*

Proof. Let $p \gg 0$. Then by Lemma 2.11, we have $\xi_{\mathbb{F}_p} \in (I_{\mathbb{F}_p})^{(k)}$ and $\eta_{\mathbb{F}_p} \in (I_{\mathbb{F}_p})^{(\ell)}$. Moreover, by Lemma 2.10, we have

$$\begin{aligned} \ell_{R_{\mathbb{F}_p}}(\, R_{\mathbb{F}_p}/(x, \xi_{\mathbb{F}_p}, \eta_{\mathbb{F}_p})\,) &= \ell_{R_{\mathbb{Q}}}(\, R_{\mathbb{Q}}/(x, \xi_{\mathbb{Q}}, \eta_{\mathbb{Q}})\,) \\ &= k\ell \cdot \ell_{S_{\mathbb{Q}}}(\, S_{\mathbb{Q}}/(x) + I_{\mathbb{Q}}\,) \\ &= k\ell \cdot \ell_{S_{\mathbb{F}_p}}(\, S_{\mathbb{F}_p}/(x) + I_{\mathbb{F}_p}\,). \end{aligned}$$

Thus we get the required assertion. □

Theorem 2.13. *Let $k = 1$ or 2. Let $\xi_{\mathbb{Z}} \in (I^{(k)})_{\mathbb{Z}}$ and $\xi_{\mathbb{Z}} \equiv y^i \mod xS_{\mathbb{Z}}$ for some $0 < i \in \mathbb{Z}$. Suppose that there exists a positive integer r such that, for any prime number $p \gg 0$, we have $rp^{e_p} \in \mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p})$ for some $0 < e_p \in \mathbb{Z}$. Then the following conditions are equivalent:*

(1) $\mathcal{R}_s(I_{\mathbb{Q}})$ *is finitely generated.*

(2) $\mathrm{HC}(I_{\mathbb{Q}}; k, \xi_{\mathbb{Q}}) \neq \emptyset$.

(3) $r \in \mathrm{HC}(I_{\mathbb{Q}}; k, \xi_{\mathbb{Q}})$.

(4) $r \in \mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p})$ *for any prime number* $p \gg 0$.

Proof. Let $\xi'_{\mathbb{Z}}$ be the homogeneous part of $\xi_{\mathbb{Z}}$ containing y^i as a term. Then, as $\xi'_{\mathbb{Z}} \in (I^{(k)})_{\mathbb{Z}}$, we have $\xi'_{\mathbb{Q}} \in (I_{\mathbb{Q}})^{(k)}$ and $\xi'_{\mathbb{F}_p} \in (I_{\mathbb{F}_p})^{(k)}$ for $p \gg 0$ by Lemma 2.11. Moreover, by Remark 2.8, we have $\mathrm{HC}(I_{\mathbb{Q}}; k, \xi_{\mathbb{Q}}) = \mathrm{HC}(I_{\mathbb{Q}}; k, \xi'_{\mathbb{Q}})$ and $\mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p}) = \mathrm{HC}(I_{\mathbb{F}_p}; k, \xi'_{\mathbb{F}_p})$ for $p \gg 0$. Hence by replacing $\xi_{\mathbb{Z}}$ with $\xi'_{\mathbb{Z}}$, we may assume that $\xi_{\mathbb{Z}}$ is homogeneous from the beginning. It is easy to see (3) ⇒ (2) ⇒ (1).

Now, we start to prove (1) ⇒ (4). By Theorem 2.5 and Lemma 2.11 (1), there exist $0 < k', \ell' \in \mathbb{Z}, \zeta_{\mathbb{Z}} \in (I^{(k')})_{\mathbb{Z}}$ and $\rho_{\mathbb{Z}} \in (I^{(\ell')})_{\mathbb{Z}}$ such that $\zeta_{\mathbb{Q}} \in (I_{\mathbb{Q}})^{(k')}$ and $\rho_{\mathbb{Q}} \in (I_{\mathbb{Q}})^{(\ell')}$ satisfy Huneke's condition on $I_{\mathbb{Q}}$. Here we take a prime number $p \gg 0$

such that $\zeta_{\mathbb{F}_p} \in (I_{\mathbb{F}_p})^{(k')}$, $\rho_{\mathbb{F}_p} \in (I_{\mathbb{F}_p})^{(\ell')}$, and these elements satisfy Huneke's condition on $I_{\mathbb{F}_p}$, which is possible by Proposition 2.12. By taking larger p if necessary, we may assume $p > \max\{k', \ell', k' + \ell' - 2\}$ and our assumption on $\mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p})$ is satisfied. Then, as $\mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p}) \neq \phi$, we have

$$S_{\mathbb{F}_p}[\{(I_{\mathbb{F}_p})^{(n)}T^n \mid 1 \leq n \leq p-1\}] = \mathcal{R}_s(I_{\mathbb{F}_p})$$

by Proposition 2.9 (3). We set $m = \min \mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p})$ and take $0 < e_p \in \mathbb{Z}$ such that $rp^{e_p} \in \mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p})$. Then by Proposition 2.9 (1), there exists $m' \in \mathbb{Z}$ such that $rp^{e_p} = mm'$. Since Proposition 2.9 (2) implies $m < p$, m is not a multiple of p, so m' is a multiple of p^{e_p}. Hence r is a multiple of m, which means $r \in \mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p})$.

Next, we shall prove (4) $\Rightarrow$ (3). Let us take a prime number $p \gg 0$ such that $r \in \mathrm{HC}(I_{\mathbb{F}_p}; k, \xi_{\mathbb{F}_p})$, $\ell_{S_{\mathbb{Q}}}(\, S_{\mathbb{Q}}/(x) + I_{\mathbb{Q}}\,) = \ell_{S_{\mathbb{F}_p}}(\, S_{\mathbb{F}_p}/(x) + I_{\mathbb{F}_p}\,)$ and $(I_{\mathbb{F}_p})^{(r)} = (I^{(r)})_{\mathbb{F}_p}$, which is possible by Lemma 2.10 and Lemma 2.11. Then by Lemma 2.4 (2) and Lemma 2.11, there exists a homogeneous element $\eta_{\mathbb{Z}} \in (I^{(r)})_{\mathbb{Z}}$ such that $\xi_{\mathbb{F}_p} \in (I_{\mathbb{F}_p})^{(k)}$ and $\eta_{\mathbb{F}_p} \in (I_{\mathbb{F}_p})^{(r)}$ satisfy Huneke's condition on $I_{\mathbb{F}_p}$. We write

$$\eta_{\mathbb{Z}} \equiv \alpha z^j \mod (x, y)S_{\mathbb{Z}},$$

where j is a positive integer and α is an integer which is not a multiple of p. Let $K = \mathbb{Q}$ or $\mathbb{F}_p$. Then, as the image of α in K is not zero, we have $(x, y, \eta_K)S_K = (x, y, z^j)S_K$. Hence we get

$$\begin{aligned} \ell_{S_K}(\, S_K/(x, \xi_K, \eta_K)\,) &= \ell_{S_K}(\, S_K/(x, y^i, \eta_K)\,) \\ &= i \cdot \ell_{S_K}(\, S_K/(x, y, \eta_K)\,) \\ &= i \cdot \ell_{S_K}(\, S_K/(x, y, z^j)\,) \\ &= ij. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \ell_{S_{\mathbb{Q}}}(\, S_{\mathbb{Q}}/(x, \xi_{\mathbb{Q}}, \eta_{\mathbb{Q}})\,) &= \ell_{S_{\mathbb{F}_p}}(\, S_{\mathbb{F}_p}/(x, \xi_{\mathbb{F}_p}, \eta_{\mathbb{F}_p})\,) \\ &= kr \cdot \ell_{S_{\mathbb{F}_p}}(\, S_{\mathbb{F}_p}/(x) + I_{\mathbb{F}_p}\,) \\ &= kr \cdot \ell_{S_{\mathbb{Q}}}(\, S_{\mathbb{Q}}/(x) + I_{\mathbb{Q}}\,), \end{aligned}$$

which means $r \in \mathrm{HC}(I_{\mathbb{Q}}; k, \xi_{\mathbb{Q}})$. $\square$

3. Proof of Theorem 1.1

We shall prove Theorem 1.1 in this section.

Let K be a field and a, b, c be pairwise coprime positive integers. We regard the polynomial ring $S = K[x, y, z]$ as a $\mathbb{Z}$-graded ring by $\deg(x) = a$, $\deg(y) = b$ and $\deg(z) = c$.

We denote by $\mathbb{P}_K(a, b, c)$ the weighted projective space $\mathrm{Proj}\, S$. Let

$$\pi : X_K(a, b, c) \longrightarrow \mathbb{P}_K(a, b, c)$$

be the blow-up at the point corresponding to $\mathfrak{p}_K(a, b, c)$. We remark that $\mathbb{P}_K(a, b, c)$ is non-singular at the point $V_+(\mathfrak{p}_K(a, b, c))$ (e.g., Lemma 9 in [2]). If no confusion is possible, we denote $\mathfrak{p}_K(a, b, c)$ (resp. $X_K(a, b, c)$, $\mathbb{P}_K(a, b, c)$) simply by $\mathfrak{p}$ (resp. X, $\mathbb{P}$).

Let E be the exceptional divisor of π. Let A be a Weil divisor on X which satisfies $\mathcal{O}_X(A) = \pi^*\mathcal{O}_{\mathbb{P}}(1)$. Since a, b, c are pairwise coprime, we have $\mathcal{O}_X(nA) = \pi^*\mathcal{O}_{\mathbb{P}}(n)$ for any $n \in \mathbb{Z}$ (e.g. Lemma 1.6 in [11] [1]). Then

$$\mathrm{Cl}(X) = \mathbb{Z}A + \mathbb{Z}E \simeq \mathbb{Z}^2$$

and the intersection pairing is given by

$$A^2 = \frac{1}{abc}, \quad E^2 = -1, \quad A.E = E.A = 0.$$

Definition 3.1. A curve C on $X_K(a,b,c)$ is called a *negative curve* on $X_K(a,b,c)$ if $C^2 < 0$ and $C \neq E$.

An irreducible homogeneous polynomial ξ in $[\mathfrak{p}_K(a,b,c)^{(r)}]_d$ is called a *negative curve* in $\mathfrak{p}_K(a,b,c)^{(r)}$ if $d/r < \sqrt{abc}$. Note that in this case the proper transform of $V_+(\xi)$ is a negative curve C which is linearly equivalent to $dA - rE$.

Since X is $\mathbb{Q}$-factorial and the Picard rank of X is two, the Kleiman-Mori cone $\overline{NE}(X)$ is a cone in $\mathbb{R}^2$. If a negative curve C exists, then $\mathbb{R}_{\geq 0}[C]$ and $\mathbb{R}_{\geq 0}[E]$ are boundary rays of $\overline{NE}(X)$. Assume that there exists another negative curve C'. Then C' is linearly equivalent to $\alpha C + \beta E$ for some $\alpha, \beta \geq 0$. Then we have

$$0 > C'^2 = C'.(\alpha C + \beta E) = \alpha(C'.C) + \beta(C'.E) \geq 0.$$

It is a contradiction. Therefore, if a negative curve C on $X_K(a,b,c)$ exists, then it is unique. If a negative curve ξ in $[\mathfrak{p}_K(a,b,c)^{(r)}]_d$ exists, then r and d are uniquely determined, and ξ is also unique up to multiplication by an element in $K^\times$. A negative curve C on $X_K(a,b,c)$ is the proper transform of $V_+(\xi)$.

Lemma 3.2. *Let K be a field and a, b, c be pairwise coprime positive integers. We assume that $\mathfrak{p}_K(a,b,c)$ is minimally generated by the three elements in (1.2).*

Then the curve $V_+(z^u - x^{s_3}y^{t_3})$ in $\mathbb{P}_K(a,b,c)$ is isomorphic to $\mathbb{P}^1_K$. The proper transform C (in X) of this curve is also isomorphic to $\mathbb{P}^1_K$.

Proof. First of all, we remark that $z^u - x^{s_3}y^{t_3}$ is an irreducible polynomial by definition of u (see (1.1)). We put $v = x^{s_2}z^{u_2}/y^t$ and $w = x^{s_3}y^{t_3}/z^u$.

The ring $S[x^{-1}, y^{-1}, z^{-1}]$ still has a structure of a $\mathbb{Z}$-graded ring. By definition, $S[x^{-1}, y^{-1}, z^{-1}]_0$ contains $K[v^{\pm 1}, w^{\pm 1}]$. Let us prove the opposite inclusion. Let M and N be monomials in x, y, z with the same degree. We shall prove $M/N \in K[v^{\pm 1}, w^{\pm 1}]$. Since $M - N \in \mathfrak{p}_K(a,b,c)$, there exists a sequence of monomials (in x, y, z)

$$M = M_1, \ M_2, \ \ldots, \ M_n = N$$

such that, for each $i = 1, 2, \ldots, n-1$, M_i/M_{i+1} is equal to one of

$$\frac{x^s}{y^{t_1}z^{u_1}}, \ \frac{y^{t_1}z^{u_1}}{x^s}, \ \frac{y^t}{x^{s_2}z^{u_2}}, \ \frac{x^{s_2}z^{u_2}}{y^t}, \ \frac{z^u}{x^{s_3}y^{t_3}}, \ \frac{x^{s_3}y^{t_3}}{z^u}.$$

[1] Since a, b, c are pairwise coprime, the closed set $\cup_{1<k}S_k$ in [11] is a subset of $\{V_+(x,y), V_+(y,z), V_+(z,x)\}$. Since the codimension of $\cup_{1<k}S_k$ is 2, there exists a Weil divisor (with integer coefficients) F of $\mathbb{P}$ such that $\mathcal{O}_{\mathbb{P}}(n) = \mathcal{O}_{\mathbb{P}}(nF)$ for any $n \in \mathbb{Z}$ by Lemma 1.6 in [11]. Since X is a blow-up at a smooth point of $\mathbb{P}$, $\mathcal{O}_X(n\pi^{-1}(F)) = \pi^*\mathcal{O}_{\mathbb{P}}(n)$ for any $n \in \mathbb{Z}$.

They coincide with

$$vw,\ v^{-1}w^{-1},\ v^{-1},\ v,\ w^{-1},\ w$$

respectively. Therefore, we have

$$\frac{M}{N} = \frac{M_1}{M_2}\frac{M_2}{M_3}\cdots\frac{M_{n-1}}{M_n} \in K[v^{\pm 1}, w^{\pm 1}].$$

Thus we know

$$S[x^{-1}, y^{-1}, z^{-1}]_0 = K[v^{\pm 1}, w^{\pm 1}].$$

Then we have

$$S[y^{-1}]_0 = K\left[v^{\alpha}w^{\beta} \;\middle|\; \alpha, \beta \in \mathbb{Z},\quad \alpha \geq 0,\quad -\frac{s_2}{s_3}\alpha \leq \beta \leq \frac{u_2}{u}\alpha \right]. \tag{3.1}$$

Taking the degree 0 component of

$$\frac{S[y^{-1}]}{(z^u - x^{s_3}y^{t_3})S[y^{-1}]} \subset \frac{S[x^{-1}, y^{-1}, z^{-1}]}{(w-1)S[x^{-1}, y^{-1}, z^{-1}]},$$

we obtain

$$\frac{S[y^{-1}]_0}{(w-1)K[v^{\pm 1}, w^{\pm 1}] \cap S[y^{-1}]_0} \subset \frac{K[v^{\pm 1}, w^{\pm 1}]}{(w-1)K[v^{\pm 1}, w^{\pm 1}]}.$$

Let $\phi : K[v^{\pm 1}, w^{\pm 1}] \to K[v^{\pm 1}]$ be the map given by $\phi(w) = 1$. The kernel of the map ϕ is $(w-1)K[v^{\pm 1}, w^{\pm 1}]$. By (3.1), we have $\phi(S[y^{-1}]_0) = K[v]$. Hence, we have

$$\left[\frac{S[y^{-1}]}{(z^u - x^{s_3}y^{t_3})S[y^{-1}]}\right]_0 \simeq K[v].$$

In the same way, we know that

$$\left[\frac{S[x^{-1}]}{(z^u - x^{s_3}y^{t_3})S[x^{-1}]}\right]_0$$

is also isomorphic to a polynomial ring over K with one variable. Hence, the curve $V_+(z^u - x^{s_3}y^{t_3})$ in $\mathbb{P}_K(a,b,c)$ is isomorphic to $\mathbb{P}^1_K$.

Since the map $C \to V_+(z^u - x^{s_3}y^{t_3})$ is a finite birational map, C is also isomorphic to $\mathbb{P}^1_K$. □

We use Cutkosky's method [2] to prove the following two lemmas.

Lemma 3.3. *Let K be a field of prime characteristic p. Let a, b, c be pairwise coprime positive integers. We assume the conditions (ii) and (iii) in Theorem 1.1. Let C be the proper transform of $V_+(z^u - x^{s_3}y^{t_3})$, that is, C is a negative curve.*

Then there exists a positive integer r satisfying the following property: For any positive integer n and any $\mathcal{L}$ in the kernel of the natural map $\mathrm{Pic}(nC) \to \mathrm{Pic}(C)$, $\mathcal{L}^{\otimes p^r}$ is isomorphic to $\mathcal{O}_{nC}$.

Proof. We have the natural exact sequence

$$0 \longrightarrow \mathcal{O}_X(-nC)/\mathcal{O}_X(-(n+1)C) \longrightarrow \mathcal{O}_{(n+1)C} \longrightarrow \mathcal{O}_{nC} \longrightarrow 0.$$

It induces the exact sequences [2]

$$0 \longrightarrow \mathcal{O}_X(-nC)/\mathcal{O}_X(-(n+1)C) \longrightarrow \mathcal{O}_{(n+1)C}^{\times} \longrightarrow \mathcal{O}_{nC}^{\times} \longrightarrow 1$$

and

$$\begin{array}{ccccccc} H^1(X, \mathcal{O}_X(-nC)/\mathcal{O}_X(-(n+1)C)) & \longrightarrow & H^1(X, \mathcal{O}_{(n+1)C}^{\times}) & \longrightarrow & H^1(X, \mathcal{O}_{nC}^{\times}) & \to 0 \\ & & \| & & \| & \\ & & \mathrm{Pic}((n+1)C) & \longrightarrow & \mathrm{Pic}(nC) & \end{array}.$$

Therefore we know that the order of an element in the kernel of the map $\mathrm{Pic}((n+1)C) \to \mathrm{Pic}(nC)$ is 1 or p. Therefore, for any element $\mathcal{L}$ in the kernel of the map $\mathrm{Pic}((n+1)C) \to \mathrm{Pic}(C)$, $\mathcal{L}^{\otimes p^n}$ is isomorphic to $\mathcal{O}_{(n+1)C}$.

Since C is linearly equivalent to $cuA - E$, abC is linearly equivalent to $abcuA - abE$. Therefore abC is a Cartier divisor by Lemma 1.3 in [11]. Since $\mathcal{O}_X(-abC)$ is invertible, we have

$$\frac{\mathcal{O}_X(-nC)}{\mathcal{O}_X(-(n+1)C)} \otimes \mathcal{O}_C(-abC) = \frac{\mathcal{O}_X(-nC)}{\mathcal{O}_X(-(n+1)C)} \otimes \mathcal{O}_X(-abC) = \frac{\mathcal{O}_X(-(n+ab)C)}{\mathcal{O}_X(-(n+ab+1)C)}.$$

Since $C^2 < 0$, $\mathcal{O}_C(-abC)$ is an ample Cartier divisor on C since $C \simeq \mathbb{P}^1_K$ by Lemma 3.2. Thus there exists a positive integer r such that

$$H^1(X, \mathcal{O}_X(-nC)/\mathcal{O}_X(-(n+1)C)) = 0$$

for any $n > r$.

Thus, for $n > r$, we know that the natural map $\mathrm{Pic}((n+1)C) \to \mathrm{Pic}(nC)$ is an isomorphism.

It is easy to see that r satisfies our requirement. □

Lemma 3.4. *Let K be a field of prime characteristic p. Let a, b, c be pairwise coprime positive integers. We assume the conditions (ii) and (iii) in Theorem 1.1.*

Then there exist $e > 0$ and $\eta \in [\mathfrak{p}^{(p^e u)}]_{p^e ab}$ such that $z^u - x^{s_3} y^{t_3}$ and η satisfy Huneke's condition on $\mathfrak{p}$, that is,

$$\ell_S(\, S/(x, z^u - x^{s_3} y^{t_3}, \eta)\,) = p^e u \cdot \ell_S(\, S/(x) + \mathfrak{p}\,)$$

holds. (The above integer e depends on a, b, c and p.)

Proof. Let C be a negative curve on X. A negative curve exists by the condition (iii) of Theorem 1.1. We have $C \sim cuA - E$.

Consider the reflexive sheaf $\mathcal{O}_{\mathbb{P}}(ab)$. Since S_{ab} contains both x^b and y^a, $\mathcal{O}_{\mathbb{P}}(ab)$ is invertible away from the point $V_+(x, y)$. Therefore $\mathcal{O}_X(abA)$ is invertible away from the point $\pi^{-1}(V_+(x, y))$. Since C does not contain the point $\pi^{-1}(V_+(x, y))$, $\mathcal{O}_X(abA - uE) \otimes \mathcal{O}_{nC}$ is an invertible sheaf on nC for any $n > 0$.

Consider the invertible sheaf $\mathcal{O}_X(abA - uE) \otimes \mathcal{O}_C$. Since $(abcA - cuE).C = 0$, the degree of $\mathcal{O}_X(abcA - cuE) \otimes \mathcal{O}_C$ is 0. Here remark that $\mathcal{O}_X(abcA - cuE)$ is

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[2]Suppose that I is an ideal of a ring A with $I^2 = (0)$. Consider the map $I \to A^{\times}$ defined by $a \mapsto 1 + a$. It induces the exact sequence $0 \to I \to A^{\times} \to (A/I)^{\times} \to 1$.

an invertible sheaf. Since C is isomorphic to $\mathbb{P}^1_K$, $\mathcal{O}_X(abcA - cuE) \otimes \mathcal{O}_C \simeq \mathcal{O}_C$. Therefore,

$$\mathcal{O}_X(abA - uE) \otimes \mathcal{O}_C \simeq \mathcal{O}_C \tag{3.2}$$

since $\mathrm{Pic}(\mathbb{P}^1_K) \simeq \mathbb{Z}$.

Since $(abA - uE).C = 0$, we have

$$(abA - uE - C)^2 = (abA - uE)^2 + C^2 = \frac{(ab - c)(ab - cu^2)}{abc} > 0$$

by the condition (iii) in Theorem 1.1. By the condition (ii) in Theorem 1.1, we have $u \geq 2$. Therefore, we have

$$\begin{aligned} (abA - uE - C).E &= u - 1 > 0 \\ (abA - uE - C).C &= -C^2 > 0. \end{aligned}$$

Since $\overline{NE}(X)$ is spanned by $[C]$ and $[E]$, $(abA - uE - C).F > 0$ for any curve F on X. Here remark that $abcA$ and E are Cartier divisors. Then we know that $abc(abA-uE-C)$ is an ample Cartier divisor on X by the Nakai-Moishezon criterion. Therefore we have $H^1(X, \mathcal{O}_X(\ell(abA - uE - C))) = 0$ for $\ell \gg 0$.

We choose $e > 0$ such that

$$H^1(X, \mathcal{O}_X(p^e(abA - uE - C))) = 0 \tag{3.3}$$

and $e \geq r$, where r is a positive integer that satisfies the requirement in Lemma 3.3.

Since

$$0 \longrightarrow \mathcal{O}_X(-p^eC) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{p^eC} \longrightarrow 0$$

is exact and $e \geq r$, we have an exact sequence

$$\begin{array}{c} 0 \to \mathcal{O}_X(p^e(abA - uE - C)) \to \mathcal{O}_X(p^e(abA - uE)) \to \mathcal{O}_X(p^e(abA - uE)) \otimes \mathcal{O}_{p^eC} \to 0 \\ \hfill \| \quad\;\; \\ \hfill \mathcal{O}_{p^eC} \quad\;\; \end{array}$$

by Lemma 3.3. Then, by (3.3), we have a surjection

$$\begin{array}{ccccc} H^0(X, \mathcal{O}_X(p^e(abA - uE))) & \longrightarrow & H^0(X, \mathcal{O}_X(p^e(abA - uE)) \otimes \mathcal{O}_{p^eC}) & \longrightarrow & 0. \\ & & \| & & \\ & & H^0(X, \mathcal{O}_{p^eC}) & & \end{array}$$

Therefore, the natural map

$$\mathcal{O}_X(p^e(abA - uE)) \longrightarrow \mathcal{O}_X(p^e(abA - uE)) \otimes \mathcal{O}_C \simeq \mathcal{O}_C$$

induces the surjection

$$H^0(X, \mathcal{O}_X(p^e(abA - uE))) \longrightarrow H^0(C, \mathcal{O}_C) = K.$$

Thus there exists an effective Weil divisor D such that $D \sim p^e(abA - uE)$ and the support of D does not intersect C. Let η be the equation of $\pi(D)$. The degree of η is p^eab. Since $D \cap C = \emptyset$, $V_+(z^u - x^{s_3}y^{t_3}) \cap V_+(\eta) \subset V_+(\mathfrak{p})$ as a set. Therefore, $\mathfrak{p}$ is the only one minimal prime ideal of $(z^u - x^{s_3}y^{t_3}, \eta)$. Hence x, $z^u - x^{s_3}y^{t_3}$, η form a regular sequence of S. We obtain

$$\ell_S(\, S/(x, z^u - x^{s_3}y^{t_3}, \eta)\,) = \ell_S(\, S/(x, z^u, y^{p^e a})\,) = p^e au = p^e u \cdot \ell_S(\, S/(x) + \mathfrak{p}\,),$$

which is the required equality. The last equality above follows from

$$\ell_S(S/(x) + \mathfrak{p}) = \mathrm{e}_{(x)}(S/\mathfrak{p}) = \mathrm{e}_{(t^a)}(K[t^a, t^b, t^c]) = \mathrm{e}_{(t^a)}(K[t]) = \ell_{K[t]}(K[t]/t^a K[t]) = a.$$

□

Proof of Theorem 1.1. By Lemma 3.4, for any prime number p, there exists a positive integer e such that

$$up^e \in \mathrm{HC}(\mathfrak{p}_{\mathbb{F}_p}(a,b,c); 1, z^u - x^{s_3}y^{t_3}).$$

Then, by Theorem 2.13, we know that $\mathcal{R}_s(\mathfrak{p}_{\mathbb{Q}}(a,b,c))$ is Noetherian if and only if

$$u \in \mathrm{HC}(\mathfrak{p}_{\mathbb{Q}}(a,b,c); 1, z^u - x^{s_3}y^{t_3}).$$

We have completed the proof of Theorem 1.1. □

Remark 3.5. Let a, b, c be pairwise coprime positive integers.

Let $\xi \in [\mathfrak{p}_K(a,b,c)^{(k)}]_d$ be a negative curve with $d/k < \sqrt{abc}$. Then $\mathcal{R}_s(\mathfrak{p}_K(a,b,c))$ is Noetherian if and only if $\mathrm{HC}(\mathfrak{p}_K(a,b,c); k, \xi) \neq \emptyset$.

Let ξ be a homogeneous element in $S_{\mathbb{Z}}$. Then $\xi_{\mathbb{Q}} \in [\mathfrak{p}_{\mathbb{Q}}(a,b,c)^{(k)}]_d$ is a negative curve with $d/k < \sqrt{abc}$ if and only if, for $p \gg 0$, $\xi_{\mathbb{F}_p} \in [\mathfrak{p}_{\mathbb{F}_p}(a,b,c)^{(k)}]_d$ is a negative curve with $d/k < \sqrt{abc}$.

Let ξ be a homogeneous element in $S_{\mathbb{Z}}$. Assume that $\xi_{\mathbb{Q}} \in [\mathfrak{p}_{\mathbb{Q}}(a,b,c)^{(k)}]_d$ is a negative curve with $d/k < \sqrt{abc}$, and $\mathcal{R}_s(\mathfrak{p}_{\mathbb{Q}}(a,b,c))$ is Noetherian. Then $\mathrm{HC}(\mathfrak{p}_{\mathbb{Q}}(a,b,c); k, \xi_{\mathbb{Q}}) = \mathrm{HC}(\mathfrak{p}_{\mathbb{F}_p}(a,b,c); k, \xi_{\mathbb{F}_p})$ for $p \gg 0$.

Let $\xi \in [\mathfrak{p}_K(a,b,c)^{(k)}]_d$ be a negative curve with $d/k < \sqrt{abc}$. Assume that $k = 1$ or 2. Suppose that

$$\xi \equiv y^i \mod xS$$

for some i. Furthermore, assume that $\mathcal{R}_s(\mathfrak{p}_K(a,b,c))$ is Noetherian. Then there exists a positive integer m such that

$$\mathrm{HC}(\mathfrak{p}_K(a,b,c); k, \xi) = \{\ell m \mid \ell \in \mathbb{N}\}$$

by Proposition 2.9 (1).

4. The condition EU

In this section, we introduce a sufficient condition (which is called as "the condition EU" below) for finite generation of $\mathcal{R}_s(\mathfrak{p})$ under the assumption in Theorem 1.1. The condition EU was defined in Ebina [4] and Uchisawa [12]. We shall prove that, if $u \leq 6$, the condition EU is a necessary and sufficient condition for finite generation of $\mathcal{R}_s(\mathfrak{p})$ in Proposition 4.10.

Let us remember the method introduced in González-Karu [5]. Let a, b, c be pairwise coprime positive integers and K be a field. Let $S = K[x, y, z]$ be a $\mathbb{Z}$-graded ring with $\deg(x) = a$, $\deg(y) = b$ and $\deg(z) = c$. Suppose that the prime ideal $\mathfrak{p}_K(a, b, c)$ is minimally generated by the three elements in (1.2).

We put

$$v = x^{s_2} z^{u_2} / y^t, \quad w = x^{s_3} y^{t_3} / z^u.$$

Since $\mathfrak{p}_K(a, b, c)$ is generated by the three elements in (1.2), we have

$$S[x^{-1}, y^{-1}, z^{-1}]_0 = K[v^{\pm 1}, w^{\pm 1}]$$

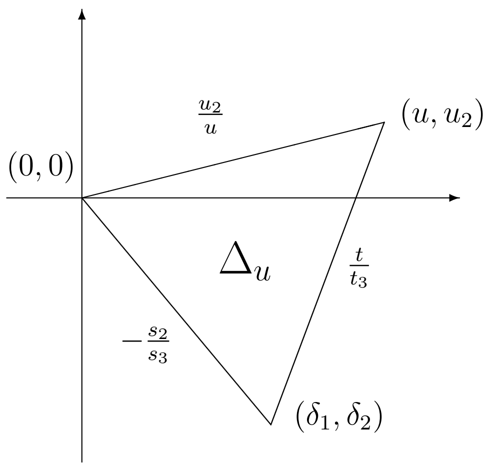
as in Lemma 3.2. Therefore, for each non-negative integer e, we have

$$S[x^{-1}, y^{-1}, z^{-1}]_{eab} = y^{ea} \cdot S[x^{-1}, y^{-1}, z^{-1}]_0 = y^{ea} \cdot K[v^{\pm 1}, w^{\pm 1}] \tag{4.1}$$

Let Δ_u be the domain (with boundary) surrounded by the following three lines

$$\begin{array}{rcl} y & = & -(s_2/s_3)x \\ y & = & (u_2/u)x \\ y & = & (t/t_3)(x - u) + u_2. \end{array}$$

Let $(0, 0)$, (u, u_2), (δ_1, δ_2) be the vertices of Δ_u. Here, δ_1 and δ_2 may not be integers.



For a non-negative integer e, we put

$$e\Delta_u = \{(e\alpha, e\beta) \mid (\alpha, \beta) \in \Delta_u\}.$$

Then it is easy to see that the equality (4.1) induces

$$S_{eab} = y^{ea} \cdot \left(\bigoplus_{(\alpha,\beta) \in e\Delta_u \cap \mathbb{Z}^2} K v^\alpha w^\beta \right).$$

Since

$$\begin{aligned} x^s - y^{t_1} z^{u_1} &= y^{t_1} z^{u_1}(vw-1) \\ y^t - x^{s_2} z^{u_2} &= y^t(1-v) \\ z^u - x^{s_3} y^{t_3} &= z^u(1-w), \end{aligned}$$

we have

$$\mathfrak{p}S[x^{-1}, y^{-1}, z^{-1}] = (v-1, w-1)S[x^{-1}, y^{-1}, z^{-1}].$$

Since each associated prime ideal of $\mathfrak{p}^n$ is a homogeneous prime ideal containing $\mathfrak{p}$, $(x, y, z)S$ is the unique embedded associated prime ideal of $\mathfrak{p}^n$. Thus we have

$$\mathfrak{p}^{(n)}S[x^{-1}, y^{-1}, z^{-1}] = \mathfrak{p}^n S[x^{-1}, y^{-1}, z^{-1}] = (v-1, w-1)^n S[x^{-1}, y^{-1}, z^{-1}]$$

and

$$\mathfrak{p}^{(n)} = (v-1, w-1)^n S[x^{-1}, y^{-1}, z^{-1}] \cap S$$

for any $n > 0$. Therefore,

$$\begin{aligned} [\mathfrak{p}^{(n)}]_{eab} &= (v-1, w-1)^n S[x^{-1}, y^{-1}, z^{-1}] \cap S_{eab} \\ &= y^{ea} \left[\left(\bigoplus_{(\alpha,\beta)\in e\Delta_u \cap \mathbb{Z}^2} K v^\alpha w^\beta \right) \bigcap (v-1, w-1)^n K[v^{\pm 1}, w^{\pm 1}] \right]. \end{aligned} \tag{4.2}$$

Remark 4.1. Let K be a field of characteristic 0. Let $\varphi(v, w)$ be an element in $K[v^{\pm 1}, w^{\pm 1}]$. Then $\varphi(v, w) \in (v-1, w-1)^n K[v^{\pm 1}, w^{\pm 1}]$ if and only if

$$\frac{\partial^{k+\ell}\varphi}{\partial v^k \partial w^\ell}(1, 1) = 0$$

for $k + \ell < n$.

Remark 4.2. Assume the conditions (i), (ii) and (iii) in Theorem 1.1. Then, by Theorem 1.1, $\mathcal{R}_s(\mathfrak{p})$ is Noetherian if and only if $[\mathfrak{p}^{(u)}]_{ab}$ contains an element whose coefficient of y^a is not 0. By (4.2), it is equivalent to the statement that

$$\left(\bigoplus_{(\alpha,\beta)\in \Delta_u \cap \mathbb{Z}^2} K v^\alpha w^\beta \right) \bigcap (v-1, w-1)^u K[v^{\pm 1}, w^{\pm 1}]$$

contains an element whose constant term is not 0. It is not so difficult to check whether it is satisfied or not using computers.

Now, we introduce the condition EU which is defined by Ebina [4] and Uchisawa [12].

Definition 4.3 (Ebina [4], Uchisawa [12]). Let a, b, c be pairwise coprime positive integers. Suppose that the prime ideal $\mathfrak{p}$ is minimally generated by the three elements in (1.2). For $i = 1, 2, \ldots, u$, we put

$$\ell_i = ^\# \{(\alpha, \beta) \in \Delta_u \cap \mathbb{Z}^2 \mid \alpha = i\}.$$

Note that $\ell_i \geq 1$ for all $i = 1, 2, \dots, u$. We sort the sequence $\ell_1, \ell_2, \dots, \ell_u$ into ascending order

$$\ell'_1 \leq \ell'_2 \leq \cdots \leq \ell'_u.$$

We say that the *condition EU* is satisfied for (a, b, c) if

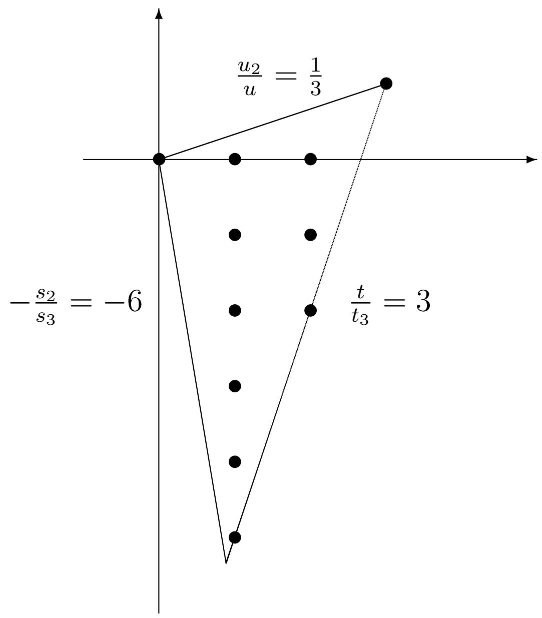
$$\ell'_i \geq i$$

for $i = 1, 2, \dots, u$.

Example 4.4. (I) Assume $(a, b, c) = (8, 19, 9)$. Then

$$\mathfrak{p} = (x^7 - y^2z^2, y^3 - x^6z, z^3 - xy)$$

and the conditions (ii) and (iii) in Theorem 1.1 are satisfied.



Then $u = 3$ and

$$\ell_1 = 6, \quad \ell_2 = 3, \quad \ell_3 = 1.$$

Therefore

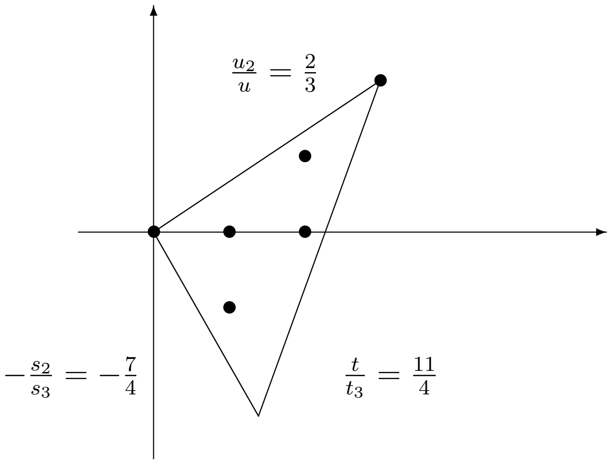
$$\ell'_1 = 1, \quad \ell'_2 = 3, \quad \ell'_3 = 6.$$

The condition EU is satisfied in this case.

(II) Assume $(a, b, c) = (25, 29, 72)$. Then

$$\mathfrak{p} = (x^{11} - y^7z, y^{11} - x^7z^2, z^3 - x^4y^4)$$

and the conditions (ii) and (iii) in Theorem 1.1 are satisfied.



Then $u = 3$ and

$$\ell_1 = 2, \quad \ell_2 = 2, \quad \ell_3 = 1.$$

Therefore

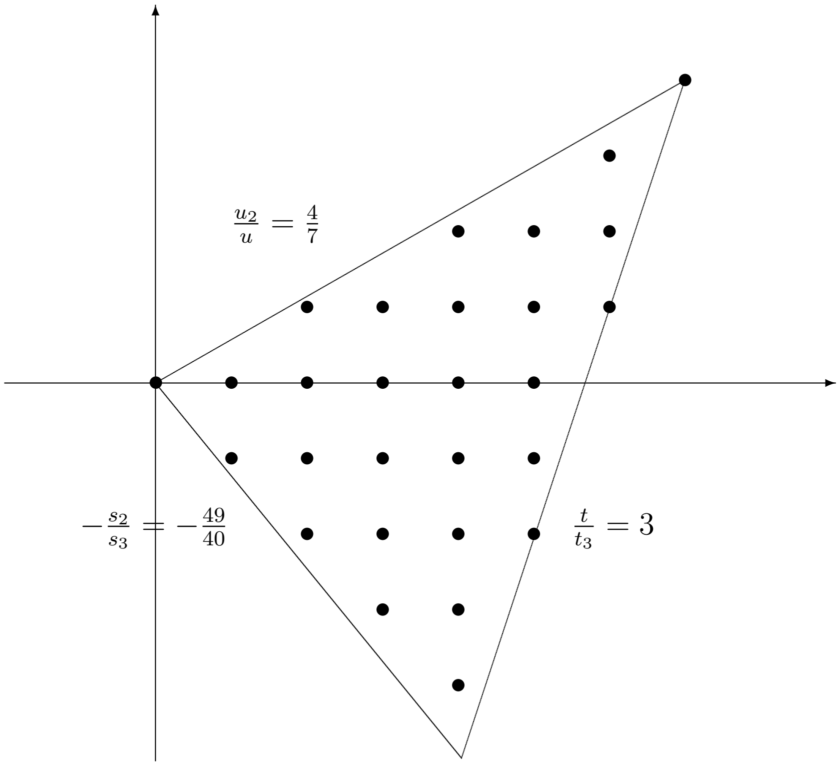
$$\ell'_1 = 1, \quad \ell'_2 = 2, \quad \ell'_3 = 2.$$

The condition EU is not satisfied in this case.

(III) Assume $(a, b, c) = (17, 503, 169)$. Then

$$\mathfrak{p} = (x^{89} - y^2z^3, y^3 - x^{49}z^4, z^7 - x^{40}y)$$

and the conditions (ii) and (iii) in Theorem 1.1 are satisfied.



Then $u = 7$ and

$$\ell_1 = 2, \quad \ell_2 = 4, \quad \ell_3 = 5, \quad \ell_4 = 7, \quad \ell_5 = 5, \quad \ell_6 = 3, \quad \ell_7 = 1.$$

Therefore

$$\ell'_1 = 1, \quad \ell'_2 = 2, \quad \ell'_3 = 3, \quad \ell'_4 = 4, \quad \ell'_5 = 5, \quad \ell'_6 = 5, \quad \ell'_7 = 7.$$

The condition EU is not satisfied in this case.

In order to show that the condition EU is a sufficient condition for finite generation of $\mathcal{R}_s(\mathfrak{p})$ under some assumptions, we need the following lemma ([4], [12]). For the convenience of the reader, we give a proof of it here.

Lemma 4.5 (Ebina [4], Uchisawa [12])**.** *Let K be a field of characteristic* 0 *and v, w be variables.*

Let u be a positive integer and $\alpha_1, \alpha_2, \ldots, \alpha_u$ be mutually distinct integers. For $i = 1, 2, \ldots, u$, consider some integers $\beta_{i1}, \beta_{i2}, \ldots, \beta_{ii}$ satisfying

$$\beta_{i1} < \beta_{i2} < \cdots < \beta_{ii}.$$

Put

$$T = \bigcup_{i=1}^{u} \{(\alpha_i, \beta_{i1}), (\alpha_i, \beta_{i2}), \ldots, (\alpha_i, \beta_{ii})\} \subset \mathbb{Z}^2.$$

Then we have

$$\left(\bigoplus_{(\alpha,\beta)\in T} Kv^{\alpha}w^{\beta} \right) \bigcap (v-1, w-1)^{u} K[v^{\pm 1}, w^{\pm 1}] = 0.$$

Proof. We shall prove it by induction on u.

If $u = 1$, then ${}^{\#}T = 1$. It is easily verified in this case.

Assume $u \geq 2$. Take

$$\varphi(v, w) \in \left(\bigoplus_{(\alpha,\beta)\in T} Kv^{\alpha}w^{\beta} \right) \bigcap (v-1, w-1)^{u} K[v^{\pm 1}, w^{\pm 1}].$$

Considering $v^{-\alpha_u}\varphi(v, w)$, we may assume $\alpha_u = 0$. Then $\frac{\partial\varphi}{\partial v}$ satisfies all the assumptions with $u-1$. Here, recall $\frac{\partial\varphi}{\partial v} \in (v-1, w-1)^{u-1}K[v^{\pm 1}, w^{\pm 1}]$ by Remark 4.1. By induction, we obtain $\frac{\partial\varphi}{\partial v} = 0$. Therefore, we may suppose

$$\varphi(v, w) = \sum_{j=1}^{u} C_j w^{\beta_{uj}}$$

where $C_1, \ldots, C_u \in K$. Since $\varphi(v, w) \in (v-1, w-1)^{u}K[v^{\pm 1}, w^{\pm 1}]$, we have

$$0 = \frac{\partial^k \varphi}{\partial w^k}(1, 1) = \sum_{j=1}^{u} C_j \beta_{uj}(\beta_{uj} - 1) \cdots (\beta_{uj} - k + 1)$$

for $k = 0, 1, \ldots, u-1$. Then we have

$$\sum_{j=1}^{u} C_j \beta_{uj}^{k} = 0$$

for $k = 0, 1, \ldots, u-1$. It is easy to see $C_1 = C_2 = \cdots = C_u = 0$. $\square$

By this lemma, we can prove that the condition EU is a sufficient condition for finite generation of $\mathcal{R}_s(\mathfrak{p})$ under some assumptions.

Proposition 4.6 (Ebina [4], Uchisawa [12])**.** *Let a, b, c be pairwise coprime positive integers. Assume the conditions (i), (ii), (iii) in Theorem 1.1.*

If the condition EU is satisfied, then $\mathcal{R}_s(\mathfrak{p}_K(a, b, c))$ is Noetherian.

Proof. By the condition EU, we can choose a set T as in Lemma 4.5 which satisfies

$$T \subset (\Delta_u - \{(0,0)\}) \cap \mathbb{Z}^2.$$

By (4.2), we obtain

$$[\mathfrak{p}^{(u)}]_{ab} = y^{a} \left[\left(\bigoplus_{(\alpha,\beta)\in \Delta_u \cap \mathbb{Z}^2} Kv^{\alpha}w^{\beta} \right) \bigcap (v-1, w-1)^{u} K[v^{\pm 1}, w^{\pm 1}] \right].$$

By this equality, we know that $[\mathfrak{p}^{(u)}]_{ab}$ is defined by $\frac{u(u+1)}{2}$ linear equations in $y^a\left(\oplus_{(\alpha,\beta)\in\Delta_u\cap\mathbb{Z}^2} K v^\alpha w^\beta\right)$. We put $T' = T \cup \{(0,0)\}$. Recall that $T' \subset \Delta_u \cap \mathbb{Z}^2$ and ${}^\# T' = \frac{u(u+1)}{2} + 1$. Then $[\mathfrak{p}^{(u)}]_{ab}$ contains a non-zero element in the form

$$\eta(x,y,z) = y^a \left(\sum_{(\alpha,\beta)\in T'} C_{(\alpha,\beta)} v^\alpha w^\beta \right),$$

where $C_{(\alpha,\beta)} \in K$.

If $C_{(0,0)} = 0$, we have

$$\eta(x,y,z) \in y^a \left[\left(\bigoplus_{(\alpha,\beta)\in T} K v^\alpha w^\beta \right) \bigcap (v-1, w-1)^u K[v^{\pm 1}, w^{\pm 1}] \right] = 0$$

by Lemma 4.5. It is a contradiction. Therefore, $C_{(0,0)} \neq 0$. Then

$$\ell_S(\, S/(x, z^u - x^{s_3} y^{t_3}, \eta)\,) = ua = u \cdot \ell_S(\, S/(x) + \mathfrak{p}\,)$$

holds. Hence, $\mathcal{R}_s(\mathfrak{p})$ is Noetherian by Huneke's condition. See also Remark 4.2. $\square$

The aim in the rest of this section is to prove the converse of Proposition 4.6 in the case $u \leq 6$.

Definition 4.7. Let a, b, c be pairwise coprime positive integers. Assume the condition (ii) in Theorem 1.1.

We define

$$n = {}^\# \{[-(s_2/s_3), (u_2/u)] \cap \mathbb{Z}\}, \quad m = {}^\# \{[(u_2/u), (t/t_3)] \cap \mathbb{Z}\},$$

where [,] is the closed interval.

We say that the *condition GK* is satisfied if one of the following two conditions is satisfied:

(I) ${}^\# \{(n-1)[(u_2/u), (t/t_3)] \cap \mathbb{Z}\} = n$ and $(u_2/u)n \notin \mathbb{Z}$,

(II) ${}^\# \{(m-1)[-(s_2/s_3), (u_2/u)] \cap \mathbb{Z}\} = m$ and $(u_1/u)m \notin \mathbb{Z}$.

We remark that the above condition (I) is satisfied for a, b, c if and only if the above condition (II) is satisfied for b, a, c.

Let a, b, c be pairwise coprime positive integers. Assume the conditions (i), (ii), (iii) in Theorem 1.1. If the condition GK is satisfied, then $\mathcal{R}_s(\mathfrak{p}_K(a,b,c))$ is not Noetherian by Theorem 1.2 in González-Karu [5].

Proposition 4.8. *Let a, b, c be pairwise coprime positive integers. Assume the conditions (i), (ii), (iii) in Theorem 1.1.*

Then the condition GK is satisfied if and only if one of the following five conditions is satisfied:

(GK1) $n = 1$,

(GK2) $m = 1$,

(GK3) $n = m = 2 < u$,

(GK4) $3 \leq n < u$, $m = 2$ *and* ${}^\# \{(n-1)[(u_2/u), (t/t_3)] \cap \mathbb{Z}\} = n$, (GK5) $n = 2$, $3 \leq m < u$ *and* $^{\#}\{(m-1)[-(s_2/s_3),(u_2/u)] \cap \mathbb{Z}\} = m$.

Proof. Let (δ_1, δ_2) be one of the vertices of Δ_u as in the beginning of this section. First, we remark that, if $0 \leq i < i+1 \leq \delta_1$, then $\ell_{i+1} \geq \ell_i + (n-1)$. In the same way, if $\delta_1 \leq i < i+1 \leq u$, then $\ell_i \geq \ell_{i+1} + (m-1)$. Thus it is easy to see the following:

(4.3) If $n \geq 3$ and $m \geq 3$, then the condition EU is satisfied.

(4.4) If $n = 2$ and $m \geq u$, then the condition EU is satisfied.

(4.5) If $n \geq u$ and $m = 2$, then the condition EU is satisfied.

Next, recall $s = s_2 + s_3$, $t = t_1 + t_3$ and $u = u_1 + u_2$ by the condition (ii). Then we have

$$\begin{aligned} a &= \ell_S(\, S/(x) + \mathfrak{p}\,) = \ell_S(\, S/(x, y^t, z^u, y^{t_1} z^{u_1})\,) = tu - t_3 u_2, \\ b &= \ell_S(\, S/(y) + \mathfrak{p}\,) = \ell_S(\, S/(x^s, y, z^u, x^{s_2} z^{u_2})\,) = su - s_3 u_1, \\ c &= \ell_S(\, S/(z) + \mathfrak{p}\,) = \ell_S(\, S/(x^s, y^t, z, x^{s_3} y^{t_3})\,) = st - s_2 t_1. \end{aligned}$$

Since a and b are coprime, u_1, u_2 and u are pairwise coprime. Therefore, $(u_2/u)n \notin \mathbb{Z}$ if and only if $n/u \notin \mathbb{Z}$, and $(u_3/u)m \notin \mathbb{Z}$ if and only if $m/u \notin \mathbb{Z}$.

It is easy to see that, if the condition (GKi) is satisfied for some i, then the condition GK is satisfied.

Conversely, assume that the condition GK is satisfied. Then $\mathcal{R}_s(\mathfrak{p}_K(a,b,c))$ is not Noetherian by Theorem 1.2 in González-Karu [5]. By Theorem 4.6 and (4.3), either $n < 3$ or $m < 3$ is satisfied.

If $n = 1$ (resp. $m = 1$), then (GK1) (resp. (GK2)) holds.

Suppose $n = 2$ and $m \geq 2$. Since the condition EU is not satisfied, we have $m < u$ by (4.4). If (I) of the condition GK is satisfied, then $n = m = 2 < u$, and therefore (GK3) is satisfied. If (II) of the condition GK is satisfied, then (GK3) or (GK5) is satisfied.

Suppose $n \geq 3$ and $m = 2$. We know $n < u$ by (4.5). Then (II) is not satisfied. If (I) is satisfied, then (GK4) is satisfied. □

Lemma 4.9. *Let a, b, c be pairwise coprime positive integers. Assume the conditions (i), (ii), (iii) in Theorem 1.1.*

1) *Assume $n = 2$ and $3 \leq m < u$. If either $u_1 = 1$ or $u_2 = 1$, then either the condition GK or the condition EU is satisfied.*
2) *If $n = 2$ and $u > m \geq (u+1)/2$, then either the condition GK or the condition EU is satisfied.*

Proof. First of all, remark that the condition EU is satisfied for a, b, c if and only if so for b, a, c. Furthermore, the condition GK is satisfied for a, b, c if and only if so for b, a, c.

First, we shall prove 1). Assume $n = 2$, $3 \le m < u$ and $u_2 = 1$. If (GK5) is not satisfied, then

$$\ell_1 = 2,\ \ell_2 \ge 3,\ \ldots,\ \ell_{m-2} \ge m-1,\ \ell_{m-1} \ge m+1,\ \ldots\ , \\ \ell_{2m-3} \ge 2m-1,\ \ell_{2m-2} \ge 2m+1, \ldots$$

and

$$\ell_u = 1,\ \ell_{u-1} \ge m,\ \ell_{u-2} \ge 2m,\ \ell_{u-3} \ge 3m, \ldots$$

since $u_2 = 1$. Thus the condition EU is satisfied.

Next, assume $n = 2$, $3 \le m < u$ and $u_1 = 1$. Considering b, a, c, we may assume $3 \le n < u$, $m = 2$ and $u_2 = 1$. Then we have

$$\ell_u = 1,\ \ell_{u-1} \ge 2,\ \ell_{u-2} \ge 4,\ \ell_{u-3} \ge 6,\ \ell_{u-4} \ge 8, \ldots$$

and

$$\ell_1 = n \ge 3,\ \ell_2 \ge 2n-1 \ge 5,\ \ell_3 \ge 3n-2 \ge 7,\ \ell_4 \ge 4n-3 \ge 9, \ldots.$$

In this case, the condition EU is always satisfied.

We prove 2) next. Assume that (GK5) is not satisfied. Then we have

$$\ell_1 = 2,\ \ell_2 \ge 3,\ \ldots,\ \ell_{m-2} \ge m-1,\ \ell_{m-1} \ge m+1,\ \ell_m \ge m+2,\ \ldots$$

and

$$\ell_u = 1,\ \ell_{u-1} \ge m,\ \ell_{u-2} \ge 2m-1 \ge u,\ \ldots.$$

Thus the condition EU is satisfied. □

Proposition 4.10. *Let a, b, c be pairwise coprime positive integers. Assume the conditions (i), (ii), (iii) in Theorem 1.1.*

If $u \le 6$, then the condition EU is a necessary and sufficient condition for finite generation of $\mathcal{R}_s(\mathfrak{p}_K(a,b,c))$.

If $u \le 6$, then the condition GK is a necessary and sufficient condition for infinite generation of $\mathcal{R}_s(\mathfrak{p}_K(a,b,c))$.

Proof. We shall prove that either the condition GK or the condition EU is satisfied if $u \le 6$.

If $n = 1$ or $m = 1$, then (GK1) or (GK2) is satisfied.

If $n \ge 3$ and $m \ge 3$, then the condition EU is satisfied as in (4.3).

If $u = n = m = 2$, then the condition EU is satisfied.

If $u > n = m = 2$, then (GK3) is satisfied.

If $n = 2$ and $m \ge u$, then the condition EU is satisfied by (4.4).

If $n \ge u$ and $m = 2$, then the condition EU is satisfied by (4.5).

Now assume that $n = 2$ and $3 \le m < u$. If $u > m \ge (u+1)/2$, then either the condition GK or the condition EU is satisfied by Lemma 4.9 2). Assume $3 \le m < (u+1)/2$. If $u \le 5$, then such m does not exist. Suppose $u = 6$ and $m = 3$. Since u, u_1, u_2 are pairwise coprime, either u_1 or u_2 is 1. Then, by Lemma 4.9 1), either the condition GK or the condition EU is satisfied.

Assume $3 \le n < u$ and $m = 2$. If $u \le 6$, we can prove that either the condition GK or the condition EU is satisfied in the same way as above. □

Example 4.11. Let K be a field of characteristic 0.

Remember the three examples in Example 4.4.

Assume $(a, b, c) = (8, 19, 9)$. In this case, $u = 3$, and the conditions (i), (ii) and (iii) in Theorem 1.1 are satisfied. Since the condition EU is satisfied, $\mathcal{R}_s(\mathfrak{p}_K(a, b, c))$ is Noetherian.

Assume $(a, b, c) = (25, 29, 72)$. In this case, $u = 3$, and the conditions (i), (ii) and (iii) in Theorem 1.1 are satisfied. Since the condition EU is not satisfied, $\mathcal{R}_s(\mathfrak{p}_K(a, b, c))$ is not Noetherian. Infinite generation of this ring was proved by Goto-Nishida-Watanabe [7].

Assume $(a, b, c) = (17, 503, 169)$. In this case, $u = 7$, and the conditions (i), (ii) and (iii) in Theorem 1.1 are satisfied. In this case, neither GK nor EU is satisfied. Applying Theorem 1.1, we know that $\mathcal{R}_s(\mathfrak{p}_K(a, b, c))$ is not Noetherian by a calculation using computers (see Remark 4.2).

Let a, b, c be pairwise coprime positive integers. Assume the conditions (i), (ii), (iii) in Theorem 1.1. We do not know any example of finitely generated $\mathcal{R}_s(\mathfrak{p}_K(a, b, c))$ such that the condition EU is not satisfied.

5. An example having negative curve in the second symbolic power

Let $S = K[x, y, z]$, where K is a field and x, y, z are indeterminates. We set $\mathfrak{m} = (x, y, z)S$ and $R = S_{\mathfrak{m}}$. In this section, we first take positive integers $s_2, s_3, t_1, t_3, u_1, u_2$ arbitrarily, and set

$$f = x^s - y^{t_1} z^{u_1},\ g = y^t - x^{s_2} z^{u_2},\ h = z^u - x^{s_3} y^{t_3},$$

where $s = s_2 + s_3, t = t_1 + t_3, u = u_1 + u_2$. Moreover, we set

$$a = t_3 u_1 + t_1 u,\ b = s_3 u_2 + s_2 u,\ c = s_2 t_3 + s_3 t.$$

Let us regard S as a $\mathbb{Z}$-graded ring by setting

$$\deg x = a,\ \deg y = b,\ \deg z = c.$$

Then we can check directly that f, g, h are all homogeneous. We set $I = (f, g, h)S$ and $\mathfrak{a} = IR$.

Lemma 5.1. *We have the following relations;*

(1) $y^{t_3} f + z^{u_1} g + x^{s_2} h = 0$,
(2) $z^{u_2} f + x^{s_3} g + y^{t_1} h = 0$.

Proof. Since f, g, h are the maximal minors of the matrix

$$\begin{pmatrix} y^{t_3} & z^{u_1} & x^{s_2} \\ z^{u_2} & x^{s_3} & y^{t_1} \end{pmatrix},$$

we get the relations stated above. □

Lemma 5.2. *The following assertions hold.*

(1) $(x) + I$ *is* $\mathfrak{m}$*-primary.*

(2) $\mathrm{Ass}_S\, S/I = \mathrm{Assh}_S\, S/I$.
(3) $I_{\mathfrak{p}}$ *is generated by* 2 *elements for any* $\mathfrak{p} \in \mathrm{Assh}_S\, S/I$.
(4) $\ell_S(\, S/(x) + I^{(n)}\,) = (n(n+1)/2) \cdot a$ *for any* $0 < n \in \mathbb{Z}$.
(5) *We have* $I \subseteq \mathfrak{p}_K(a,b,c)$*, and the equality holds if* $\mathrm{GCD}(a,b,c) = 1$.

Proof. (1) This holds as $(x) + I$ contains x, y^t and z^u.

(2) Since $(x) + I$ is $\mathfrak{m}$-primary, we have $\mathrm{grade}\, I = 2$. Hence by Hilbert-Burch's theorem, we see that I is a perfect ideal, which means that S/I is a Cohen-Macaulay S-module. Thus we get the required assertion.

(3) Let us take any $\mathfrak{p} \in \mathrm{Ass}_S\, S/I$. Then, as $x \notin \mathfrak{p}$, we have $h \in (f,g)S_{\mathfrak{p}}$ by Lemma 5.1 (1), so $I_{\mathfrak{p}} = (f,g)S_{\mathfrak{p}}$.

(4) Since $(x)+I = (x, y^t, y^{t_1}z^{u_1}, z^u)$, we have $\mathrm{e}_{xR}(R/\mathfrak{a}) = \ell_S(\, S/(x)+I\,) = a$. Let us take any $0 < n \in \mathbb{Z}$. Then

$$\begin{aligned}\ell_S(\, S/(x) + I^{(n)}\,) &= \ell_R(\, R/xR + \mathfrak{a}^{(n)}\,) = \mathrm{e}_{xR}(R/\mathfrak{a}^{(n)}) \\ &= \sum_{P \in \mathrm{Assh}_R\, R/\mathfrak{a}} \ell_{R_P}(\, R_P/\mathfrak{a}_P^n\,) \cdot \mathrm{e}_{xR}(R/P).\end{aligned}$$

For any $P \in \mathrm{Assh}_R\, R/\mathfrak{a}$, $\mathcal{G}(\mathfrak{a}_P)$ is isomorphic to a polynomial ring with 2 variables over $R_P/\mathfrak{a}_P$, so

$$\begin{aligned}\ell_{R_P}(\, R_P/\mathfrak{a}_P^n\,) &= \sum_{i=1}^{n} \ell_{R_P}(\, \mathfrak{a}_P^{i-1}/\mathfrak{a}_P^i\,) = \sum_{i=1}^{n} i \cdot \ell_{R_P}(\, R_P/\mathfrak{a}_P\,) \\ &= \frac{n(n+1)}{2} \cdot \ell_{R_P}(\, R_P/\mathfrak{a}_P\,).\end{aligned}$$

Thus we get

$$\begin{aligned}\ell_S(\, S/(x) + I^{(n)}\,) &= \frac{n(n+1)}{2} \sum_{P \in \mathrm{Assh}_R\, R/\mathfrak{a}} \ell_{R_P}(\, R_P/\mathfrak{a}_P\,) \cdot \mathrm{e}_{xR}(R/P) \\ &= \frac{n(n+1)}{2} \cdot \mathrm{e}_{xR}(R/\mathfrak{a}) = \frac{n(n+1)}{2} \cdot a.\end{aligned}$$

(5) We set $\mathfrak{p} = \mathfrak{p}_K(a,b,c)$. Since f, g, h are all homogeneous, we have $I \subseteq \mathfrak{p}$. Hence, we have

$$a = \ell_S(\, S/(x) + I\,) \geq \ell_S(\, S/(x) + \mathfrak{p}\,) = \ell_R(\, R/xR + \mathfrak{p}R\,) = \mathrm{e}_{xR}(R/\mathfrak{p}R).$$

Now, we assume $\mathrm{GCD}(a,b,c) = 1$. Then, as is well known, we have $\mathrm{e}_{xR}(R/\mathfrak{p}R) = a$, so we see

$$\ell_S(\, S/(x) + I)\,) = \ell_S(\, S/(x) + \mathfrak{p}\,),$$

which means $(x) + I = (x) + \mathfrak{p}$. Then we have

$$\mathfrak{p} = \mathfrak{p} \cap ((x) + I) = (\mathfrak{p} \cap (x)) + I = x\mathfrak{p} + I,$$

from which the equality $\mathfrak{p} = I$ follows. □

Lemma 5.3. *Let $\mathfrak{b}$ be an ideal of $B = K[y, z]$ generated by the monomials*

$$y^{\alpha_0}, y^{\alpha_1} z^{\beta_1}, \dots, y^{\alpha_{r-1}} z^{\beta_{r-1}}, z^{\beta_r} \qquad (1 \le r \in \mathbb{Z}),$$

where α_i's and β_i's are positive integers such that $\alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_{r-1}$ and $\beta_1 \le \cdots \le \beta_{r-1} \le \beta_r$. Then, setting $\alpha_r = 0$, we have

$$\ell_B(\, B/\mathfrak{b}\,) = \sum_{i=1}^{r} (\alpha_{i-1} - \alpha_i)\beta_i.$$

Proof. We prove by induction on r. If $r = 1$, we have

$$\ell_B(\, B/\mathfrak{b}\,) = \ell_B(\, B/(y^{\alpha_0}, z^{\beta_1})\,) = \alpha_0\beta_1 = (\alpha_0 - \alpha_1)\beta_1.$$

Let us consider the case where $r \ge 2$. We set $\alpha_i' = \alpha_i - \alpha_{r-1}$ for $i = 0, 1, \dots, r-1$ and

$$\mathfrak{b}' = (y^{\alpha_0'}, y^{\alpha_1'} z^{\beta_1}, \dots, y^{\alpha_{r-2}'} z^{\beta_{r-2}}, z^{\beta_{r-1}}).$$

Because $\mathfrak{b} = y^{\alpha_{r-1}}\mathfrak{b}' + (z^{\beta_r})$ and $y^{\alpha_{r-1}}, z^{\beta_r}$ is a B-regular sequence, it follows that

$$(y^{\alpha_{r-1}}, z^{\beta_r})/\mathfrak{b} \cong (y^{\alpha_{r-1}})/(y^{\alpha_{r-1}}\mathfrak{b}' + (y^{\alpha_{r-1}} z^{\beta_r})) \cong B/\mathfrak{b}'.$$

Furthermore, the hypothesis of induction implies

$$\ell_B(\, B/\mathfrak{b}'\,) = \sum_{i=1}^{r-1} (\alpha_{i-1}' - \alpha_i')\beta_i = \sum_{i=1}^{r-1} (\alpha_{i-1} - \alpha_i)\beta_i.$$

Now, looking at the exact sequence

$$0 \longrightarrow (y^{\alpha_{r-1}}, z^{\beta_r})/\mathfrak{b} \longrightarrow B/\mathfrak{b} \longrightarrow B/(y^{\alpha_{r-1}}, z^{\beta_r}) \longrightarrow 0,$$

we get

$$\begin{aligned} \ell_B(\, B/\mathfrak{b}\,) &= \ell_B(\, (y^{\alpha_{r-1}}, z^{\beta_r})/\mathfrak{b}\,) + \ell_B(\, B/(y^{\alpha_{r-1}}, z^{\beta_r})\,) \\ &= \sum_{i=1}^{r-1} (\alpha_{i-1} - \alpha_i)\beta_i + \alpha_{r-1}\beta_r \\ &= \sum_{i=1}^{r} (\alpha_{i-1} - \alpha_i)\beta_i. \end{aligned}$$

□

Lemma 5.4. *Suppose $s_2 > s_3$, $t_1 = t_3 = 1$ and $u_1 < u_2$. Then, the following assertions hold.*

1. *There exists homogeneous $\xi \in I^{(2)}$ such that*
   - (i) $x^{s_3}\xi = z^{u_2 - u_1} f^2 - gh$,
   - (ii) $z^{u_1}\xi = x^{s_2 - s_3} h^2 - fg$, *and*
   - (iii) $\xi \equiv y^3$ *mod* (x).
2. $(x) + I^{(2)} = (x, y^3, y^2 z^{2u_1}, y z^{u+u_1}, z^{2u})$.

Proof. (1) From the relations (1) and (2) of Lemma 5.1, we get

$$-yfh = z^{u_1}gh + x^{s_2}h^2 \quad \text{and} \quad -yfh = z^{u_2}f^2 + x^{s_3}fg,$$

respectively. Hence we have

$$z^{u_1}gh + x^{s_2}h^2 = z^{u_2}f^2 + x^{s_3}fg,$$

so we get

$$x^{s_3}(x^{s_2-s_3}h^2 - fg) = z^{u_1}(z^{u_2-u_1}f^2 - gh).$$

Since x^{s_3}, z^{u_1} is a regular sequence on S, there exists $\xi \in S$ such that

$$x^{s_3}\xi = z^{u_2-u_1}f^2 - gh \quad \text{and} \quad z^{u_1}\xi = x^{s_2-s_3}h^2 - fg.$$

The first equality implies $x^{s_3}\xi \in I^2$, so $\xi \in I^{(2)}$. The second equality implies $z^{u_1}\xi \equiv -fg \bmod (x)$, so $z^{u_1}\xi \equiv yz^{u_1} \cdot y^2 \bmod (x)$ as $f \equiv -yz^{u_1} \bmod (x)$ and $g \equiv y^2 \bmod (x)$. Hence we get $\xi \equiv y^3 \bmod (x)$ since z^{u_1} is regular on $S/(x)$.

(2) Since $(x) + I = (x, y^2, yz^{u_1}, z^u)$ and $u_1 < u_2$, we have

$$(x) + I^2 = (x, y^4, y^3z^{u_1}, y^2z^{2u_1}, yz^{u+u_1}, z^{2u}).$$

We set $J = (\xi) + I^2 \subseteq I^{(2)}$. Then, as

$$(x) + J = (x, y^3, y^2z^{2u_1}, yz^{u+u_1}, z^{2u}),$$

we have

$$\ell_S(\, S/(x) + J\,) = 3(u + u_1) = 3a = \ell_S(\, S/(x) + I^{(2)}\,)$$

by Lemma 5.3 and Lemma 5.2 (4). Hence we get the required assertion. □

The element ξ exhibited in the above lemma will give the required negative curve in Example 5.7.

Lemma 5.5. *Suppose $s_2 > 2s_3$, $t_1 = t_3 = 1$ and $u_1 < u_2 < 2u_1$. Then, the following assertions hold.*

(1) *There exists homogeneous $\zeta \in I^{(3)}$ such that*
   (i) $x^{s_3}\zeta = f^3 + z^{2u_1-u_2}h\xi$,
   (ii) $z^{u_2-u_1}\zeta = f\xi + x^{s_2-2s_3}h^3$, *and*
   (iii) $\zeta \equiv -y^4z^{2u_1-u_2}$ *mod* (x).

(2) $(x) + I^{(3)} = (x, y^5, y^4z^{2u_1-u_2}, y^3z^u, y^2z^{u+2u_1}, yz^{2u+u_1}, z^{3u})$.

(3) $S[IT, I^{(2)}T^2, I^{(3)}T^3] \subsetneq \mathcal{R}_s(I)$.

Proof. (1) From the relations (i) and (ii) of Lemma 5.4, we get

$$fgh = z^{u_2-u_1}f^3 - x^{s_3}f\xi \quad \text{and} \quad fgh = x^{s_2-s_3}h^3 - z^{u_1}h\xi,$$

respectively. Hence we have

$$z^{u_2-u_1}f^3 - x^{s_3}f\xi = x^{s_2-s_3}h^3 - z^{u_1}h\xi,$$

so we get

$$z^{u_2-u_1}(f^3 + z^{2u_1-u_2}h\xi) = x^{s_3}(f\xi + x^{s_2-2s_3}h^3).$$

Since $x^{s_3}, z^{u_2-u_1}$ is a regular sequence on S, there exists $\zeta \in S$ such that

$$x^{s_3}\zeta = f^3 + z^{2u_1-u_2}h\xi \quad \text{and} \quad z^{u_2-u_1}\zeta = f\xi + x^{s_2-2s_3}h^3.$$

The first equality implies $x^{s_3}\zeta \in II^{(2)} \subseteq I^{(3)}$, so $\zeta \in I^{(3)}$ as x^{s_3} is regular on $S/I^{(3)}$. The second equality implies $z^{u_2-u_1}\zeta \equiv f\xi \bmod (x)$, so $z^{u_2-u_1}\zeta \equiv -yz^{u_1} \cdot y^3 \bmod (x)$ as $f \equiv -yz^{u_1} \bmod (x)$ and $\xi \equiv y^3 \bmod (x)$. Hence we get $\zeta \equiv -y^4 z^{2u_1-u_2} \bmod (x)$ since $z^{u_2-u_1}$ is regular on $S/(x)$.

(2) By Lemma 5.4 (2), we have

$$(x) + II^{(2)} = (x, y^5, y^4 z^{u_1}, y^3 z^u, y^2 z^{u+2u_1}, yz^{2u+u_1}, z^{3u})$$

as $u_1 < u_2 < 2u_1$. We set $J = (\zeta) + II^{(2)} \subseteq I^{(3)}$. Then, as

$$(x) + J = (x, y^5, y^4 z^{2u_1-u_2}, y^3 z^u, y^2 z^{u+2u_1}, yz^{2u+u_1}, z^{3u}),$$

we have

$$\ell_S(\, S/(x) + J\,) = 6(u + u_1) = 6a = \ell_S(\, S/(x) + I^{(3)}\,)$$

by Lemma 5.3 and Lemma 5.2 (4). Hence we get the required assertion.

(3) It is enough to show $I^{(2)}I^{(3)} \subsetneq I^{(5)}$. (We have $(I^{(2)})^2 + II^{(3)} = I^{(4)}$, which can be verified in the same way.) Sorting the products of the monomials of y and z exhibited in Lemma 5.4 (2) and Lemma 5.5 (2), we have

$$(x) + I^{(2)}I^{(3)} = (x) + \left(\begin{array}{c} y^8,\ y^7 z^{2u_1-u_2},\ y^6 z^{\min\{u,\, 4u_1-u_2\}},\ y^5 z^{4u_1}, \\ y^4 z^{4u_1+u_2},\ y^3 z^{3u},\ y^2 z^{3u+2u_1},\ yz^{4u+u_1},\ z^{5u} \end{array} \right),$$

so we get

$$\ell_S(\, S/(x) + I^{(2)}I^{(3)}\,) = \min\{29u_1 + 16u_2, 32u_1 + 14u_2\}$$

by Lemma 5.3. On the other hand, by Lemma 5.2 (4), we have

$$\ell_S(\, S/(x) + I^{(5)}\,) = 15a = 30u_1 + 15u_2.$$

Since $\min\{29u_1 + 16u_2, 32u_1 + 14u_2\} - (30u_1 + 15u_2) = \min\{u_2 - u_1, 2u_1 - u_2\} > 0$, we see

$$\ell_S(\, S/(x) + I^{(2)}I^{(3)}\,) > \ell_S(\, S/(x) + I^{(5)}\,),$$

which means $I^{(2)}I^{(3)} \subsetneq I^{(5)}$. $\square$

In the rest of this section, let us denote S and I by S_K and I_K, respectively, in order to emphasize that the coefficient field is K. Moreover, the elements ξ and ζ constructed in Lemmas 5.4 and 5.5 are denoted by ξ_K and ζ_K, respectively.

Theorem 5.6. *Let us choose any rational numbers α and β such that*

$$1 < \alpha < \frac{5}{4} \quad \textit{and} \quad 2 < \beta < \frac{7}{3} - \frac{\alpha - 1}{2 - \alpha}.$$

Moreover, we choose positive integers s_2, s_3, u_1 and u_2 such that

$$\frac{s_2}{s_3} = \beta \quad \textit{and} \quad \frac{u_2}{u_1} = \alpha.$$

Then, setting $t_1 = t_3 = 1$, we get the following assertions.

(1) *$s_2 > 2s_3$ and $u_1 < u_2 < 2u_1$.*

(2) *Let $0 \ll q \in \mathbb{Z}$. We denote by k the largest integer which is not bigger than $q/3$. Then we have $(x^{s_3}, z^{2u_1-u_2})^k \subseteq \big(x^{q(s_2-2s_3)+1}, z^{q(u_2-u_1)}\big)$.*

(3) *Let p be any prime number. Then $3p^{e_p} \in \mathrm{HC}(I_{\mathbb{F}_p}; 2, \xi_{\mathbb{F}_p})$ for any $e_p \gg 0$.*

(4) $3 \notin \mathrm{HC}(I_K; 2, \xi_K)$ *for any field* K.
(5) $\mathcal{R}_s(I_{\mathbb{Q}})$ *is infinitely generated.*

Proof. (1) These inequalities hold since $s_2/s_3 > 2$ and $1 < u_2/u_1 < 2$.

(2) Since $(x^{s_3}, z^{2u_1-u_2})^k$ is generated my

$$\{x^{(k-i)s_3} z^{i(2u_1-u_2)} \mid i = 0, 1, \ldots, k\},$$

it is enough to show that

$$(k-i)s_3 \le q(s_2 - 2s_3) \quad \Longrightarrow \quad i(2u_1 - u_2) \ge q(u_2 - u_1)$$

holds for any $i = 0, 1, \ldots, k$. So we suppose $(k-i)s_3 \le q(s_2 - 2s_3)$, where $i = 0, 1, \ldots, k$. Then, dividing both sides of this inequality by s_3, we get

$$k - i \le q(\beta - 2).$$

Here we write $q = 3k + \ell$, where $\ell = 0, 1, 2$. Then, as

$$k - i \le 3k(\beta - 2) + \ell(\beta - 2),$$

we have

$$i \ge k - 3k(\beta - 2) - \ell(\beta - 2) = k(7 - 3\beta) - \ell(\beta - 2).$$

Hence we get

$$\begin{aligned} i(2-\alpha) - q(\alpha - 1) &\ge \{k(7-3\beta) - \ell(\beta - 2)\}(2 - \alpha) - (3k + \ell)(\alpha - 1) \\ &= k\{(2-\alpha)(7 - 3\beta) - 3(\alpha - 1)\} + m, \end{aligned}$$

where $m = -\ell\{(\beta - 2)(2 - \alpha) + (\alpha - 1)\}$. Now, we notice that our assumption $\alpha < 5/4$ and $\beta < 7/3 - (\alpha - 1)/(2 - \alpha)$ implies

$$3\beta(2 - \alpha) < 7(2 - \alpha) - 3(\alpha - 1),$$

so we see

$$(2 - \alpha)(7 - 3\beta) - 3(\alpha - 1) > 0.$$

Since $q \gg 0$, we have $k \gg 0$ too, so it follows that

$$i(2 - \alpha) - q(\alpha - 1) > 0$$

as m is a bounded number. Then, multiplying both sides of $i(2 - \alpha) > q(\alpha - 1)$ by u_1, we get

$$i(2u_1 - u_2) > q(u_2 - u_1).$$

(3) By Lemma 5.5 (1) (ii), we have a relation

$$z^{u_2-u_1}\zeta_{\mathbb{F}_p} - x^{s_2-2s_3}h^3 = f\xi_{\mathbb{F}_p}$$

in $S_{\mathbb{F}_p}$. We take $0 \ll e_p \in \mathbb{Z}$ and put $q = p^{e_p}$. Then we have

$$z^{q(u_2-u_1)}\zeta_{\mathbb{F}_p}^q + (-1)^q x^{q(s_2-2s_3)}h^{3q} = f^q\xi_{\mathbb{F}_p}^q.$$

Here we write $q = 3k + \ell$, where $\ell = 0, 1, 2$. Then,

$$f^q\xi_{\mathbb{F}_p}^q = (f^3)^k \cdot f^\ell \xi_{\mathbb{F}_p}^q \in (f^3)^k \cdot I^{(2q+\ell)}.$$

The relation (1) (i) of Lemma 5.5 means $f^3 \in (x^{s_3}, z^{2u_1-u_2})I_{\mathbb{F}_p}^{(3)}$. Hence by (2) of this proposition proved above, we have

$$(f^3)^k \in (x^{q(s_2-2s_3)+1}, z^{q(u_2-u_1)})I_{\mathbb{F}_p}^{(3k)}.$$

Thus we see

$$f^q\xi_{\mathbb{F}_p}^q \in (x^{q(s_2-2s_3)+1}, z^{q(u_2-u_1)})I_{\mathbb{F}_p}^{(3q)},$$

so there exist $\sigma_{\mathbb{F}_p}, \tau_{\mathbb{F}_p} \in I_{\mathbb{F}_p}^{(3q)}$ such that

$$z^{q(u_2-u_1)}\zeta_{\mathbb{F}_p}^q + (-1)^q x^{q(s_2-2s_3)}h^{3q} = x^{q(s_2-2s_3)+1}\sigma_{\mathbb{F}_p} + z^{q(u_2-u_1)}\tau_{\mathbb{F}_p}.$$

Then we have

$$z^{q(u_2-u_1)}\{\zeta_{\mathbb{F}_p}^q - \tau_{\mathbb{F}_p}\} = x^{q(s_2-2s_3)}\{(-1)^{q+1}h^{3q} + x\sigma_{\mathbb{F}_p}\}.$$

Since $x^{q(s_2-2s_3)}, z^{q(u_2-u_1)}$ is a regular sequence on $S_{\mathbb{F}_p}$, there exists $\eta_{\mathbb{F}_p} \in S_{\mathbb{F}_p}$ such that

$$x^{q(s_2-2s_3)}\eta_{\mathbb{F}_p} = \zeta_{\mathbb{F}_p}^q - \tau_{\mathbb{F}_p} \quad \text{and} \quad z^{q(u_2-u_1)}\eta_{\mathbb{F}_p} = (-1)^{q+1}h^{3q} + x\sigma_{\mathbb{F}_p}.$$

The first equality implies $x^{q(s_2-2s_3)}\eta_{\mathbb{F}_p} \in I_{\mathbb{F}_p}^{(3q)}$, so we have $\eta_{\mathbb{F}_p} \in I_{\mathbb{F}_p}^{(3q)}$ as $x^{q(s_2-s_3)}$ is regular on $S/I_{\mathbb{F}_p}^{(3q)}$. The second equality implies $z^{q(u_2-u_1)}\eta_{\mathbb{F}_p} \equiv (-1)^{q+1}h^{3q} \bmod xS_{\mathbb{F}_p}$, so $z^{q(u_2-u_1)}\eta_{\mathbb{F}_p} \equiv (-1)^{q+1}z^{3qu} \bmod xS_{\mathbb{F}_p}$ as $h \equiv z^u \bmod xS_{\mathbb{F}_p}$. Hence we get $\eta_{\mathbb{F}_p} \equiv (-1)z^{2q(u+u_1)} \bmod xS_{\mathbb{F}_p}$ since $z^{q(u_2-u_1)}$ is regular on $S_{\mathbb{F}_p}/xS_{\mathbb{F}_p}$. Then we have

$$\begin{aligned} \ell_{S_{\mathbb{F}_p}}(\, S_{\mathbb{F}_p}/(x, \xi_{\mathbb{F}_p}, \eta_{\mathbb{F}_p})\,) &= \ell_{S_{\mathbb{F}_p}}(\, S_{\mathbb{F}_p}/(x, y^3, z^{2q(u+u_1)})\,) \\ &= 6q(u+u_1) \\ &= 2 \cdot 3q \cdot \ell_{S_{\mathbb{F}_p}}(\, S_{\mathbb{F}_p}/(x) + I_{\mathbb{F}_p}\,), \end{aligned}$$

and hence $3q \in \mathrm{HC}(I_{\mathbb{F}_p}; 2, \xi_{\mathbb{F}_p})$.

(4) If $3 \in \mathrm{HC}(I_K; 2, \xi_K)$, by Proposition 2.9 (3), we have

$$S_K[I_K T, I_K^{(2)}T^2, I_K^{(3)}T^3] = \mathcal{R}_s(I_K),$$

which contradicts to Lemma 5.5 (3).

(5) Let us notice that $\xi_{\mathbb{Q}} \in \mathbb{Z}[x, y, z]$. Then, setting $k = 2$ and $r = 3$ in Theorem 2.13, we see that $\mathcal{R}_s(I_{\mathbb{Q}})$ is not finitely generated. □

Example 5.7. Let $\alpha = 6/5$ and $\beta = 49/24$, which satisfy the assumptions on α and β of Theorem 5.6. We set

$$s_2 = 49m,\ s_3 = 24m,\ t_1 = t_3 = 1,\ u_1 = 5n \text{ and } u_2 = 6n,$$

where m, n are coprime positive integers such that m is odd and n is not a multiple of 97. Then, we have

$$a = 16n,\ b = 683mn \text{ and } c = 97m.$$

Since 683 and 97 are prime numbers, we get $\mathrm{GCD}(a, b, c) = 1$. Hence by Lemma 5.2 (5) and Theorem 5.6 (5), we see that $\mathcal{R}_s(\mathfrak{p}_{\mathbb{Q}}(a, b, c))$ is infinitely generated.

Suppose $m = n = 1$ in the example stated above. Then a, b, c are pairwise coprime, and ξ_K is a negative curve for any field K, because

$$\deg \xi_K = \deg y^3 = 3 \cdot 683 = 2049 \text{ and } 2049/2 = 1024.5 < \sqrt{16 \cdot 683 \cdot 97}.$$

On the other hand, ζ_K is not a negative curve, because

$$\deg \zeta_K = \deg y^4 z^4 = 4 \cdot 683 + 4 \cdot 97 = 3120 \text{ and } 3120/3 = 1040 > \sqrt{16 \cdot 683 \cdot 97}.$$

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