AN ELEMENTARY PROOF OF COHEN-GABBER THEOREM IN THE EQUAL CHARACTERISTIC $p > 0$ CASE

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Abstract. The aim of this article is to give a new proof of Cohen-Gabber theorem in the equal characteristic $p > 0$ case.

1. Introduction

Cohen proved the structure theorem on complete local rings in [2] and since then, it has been used as a basic tool in commutative algebra. Since our main concern is in rings of positive characteristic, let us recall its statement in the equal characteristic $p > 0$ case, where p is a prime number. Let $(A, \mathfrak{m}, k)$ be a complete local ring of dimension $d \geq 0$ and of equal characteristic $p > 0$. In particular, k is a field of characteristic p. Then there exists a coefficient field $\phi : k \to A$, together with a system of parameters $x_1, \ldots, x_d$ of A such that there is a module-finite extension $\phi(k)[[x_1, \ldots, x_d]] \subset A$, where $\phi(k)[[x_1, \ldots, x_d]]$ is a complete regular local ring of dimension d. The aim of this article is to give a new and elementary proof of Cohen-Gabber theorem which is stated as follows:

Theorem 1.1 (Cohen-Gabber)**.** *Assume that $(A, \mathfrak{m}, k)$ is a complete local ring of dimension $d \geq 0$ and of equal characteristic $p > 0$ and let $\pi : A \to k = A/\mathfrak{m}$ be the quotient map. Then there exists a system of parameters $y_1, \ldots, y_d$ of A and a ring map $\phi : k \to A$ such that the following hold: $\pi \circ \phi = \mathrm{id}_k$, the natural map*

$$\phi(k)[[y_1, \ldots, y_d]] \subset A$$

is module-finite, and $\mathrm{Frac}(\phi(k)[[y_1, \ldots, y_d]]) \to \mathrm{Frac}(A/P)$ is a separable field extension for any minimal prime P of A such that $\dim A/P = d$.

The above theorem is seen as a strengthened version of Cohen structure theorem in that the module-finite extension $\phi(k)[[y_1, \ldots, y_d]] \subset A$ can be made to be generically étale when the local ring R is reduced and equi-dimensional. Theorem 1.1 is formulated and proved by Gabber in [3, Théorème 7.1] and [5, Exposé IV, Théorème 2.1.1]. It plays a

Key words and phrases. Coefficient field, complete local ring, differentials, p-basis.

2000 *Mathematics Subject Classification*: Primary 12F10; Secondary 13B40, 13J10.

The first author was partially supported by JSPS KAKENHI Grant 15K04828. The second author was partially supported by Grant-in-Aid for Young Scientists (B) 25800028.

role in the proof of Gabber's alteration theorem with applications to étale cohomology. It is also essential in the proof of the Bertini-type theorem for reduced hyperplane quotients of complete local rings of characteristic $p > 0$. This result is proved in [7]. Finally, we mention that there is a version of Theorem 1.1 for an affine domain over a perfect field (see [8, Theorem 4.2.2]).

2. Preliminaries

We collect some facts that we shall use in this paper. All rings are assumed to be commutative and Noetherian with unity. A local ring is a Noetherian ring with a unique maximal ideal and it is denoted by the symbol $(A, \mathfrak{m}, k)$. We denote by $\mathrm{Frac}(A)$ the total ring of fractions of a commutative ring A.

Definition 2.1. (1) A *coefficient field* of a complete local ring $(A, \mathfrak{m}, k)$ is a ring map $\phi : k \to A$ such that $\pi \circ \phi = \mathrm{id}_k$, where $\pi : A \to k = A/\mathfrak{m}$ is the quotient map. In particular, if a complete local ring has a coefficient field, this local ring contains the field of rationals $\mathbb{Q}$ or the finite field $\mathbb{F}_p$ for some prime number p.

(2) Let K/k be a field extension such that the characteristic of k is p and let $\{\alpha_i\}_{i \in \Lambda}$ be a set of elements of K. Then say that $\{\alpha_i\}_{i \in \Lambda}$ is a *p-basis* of K over k, if $\{d\alpha_i\}_{i \in \Lambda}$ form a basis of the K-vector space $\Omega_{K/k}$, where $\Omega_{K/k}$ is the module of differentials of K over k. Note that K is a perfect field if $\Omega_{K/\mathbb{F}_p} = 0$.

We recall a fundamental theorem on the existence of a coefficient field. For the proof, see [1, Chapitre IX, § 3, n° 3, Théorème 1 b].

Theorem 2.2 (Cohen). *Let $(A, \mathfrak{m}, k)$ be a complete local ring of equal characteristic $p > 0$. Let $\{\alpha_i\}_{i \in \Lambda}$ be a set of elements of A and let $\{\overline{\alpha_i}\}_{i \in \Lambda}$ be its image in $A/\mathfrak{m} = k$. If $\{\overline{\alpha_i}\}_{i \in \Lambda}$ is a p-basis of k over $\mathbb{F}_p$, then there exists the unique coefficient field $\phi : k \to A$ such that $\phi(\overline{\alpha_i}) = \alpha_i$ for each $i \in \Lambda$. Moreover, if k is a perfect field, then A has the unique coefficient field.*

We recall Weierstrass Preparation Theorem. An element $f \in A[[X]]$ over a local ring $(A, \mathfrak{m}, k)$ is called a *distinguished polynomial of degree n*, if we can write $f = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ for some integer $n \geq 0$ and $a_0, \ldots, a_{n-1} \in \mathfrak{m}$.

Theorem 2.3 (Weierstrass Preparation Theorem). *Let $(A, \mathfrak{m}, k)$ be a complete local ring and let $B = A[[X]]$. Let $f = \sum_{i=0}^{\infty} a_i X^i \in B$ be a non-zero element with $a_i \in A$. If there exists a natural number $n \in \mathbb{N}$ such that $a_i \in \mathfrak{m}$ for all $i < n$ and $a_n \notin \mathfrak{m}$, then we have $f = u \cdot f_0$, where u is a unit in B and $f_0 \in B$ is a distinguished polynomial of degree n. Furthermore, u and f_0 are uniquely determined by f.*

See [4] for a short proof of this theorem.

Remark 2.4. (1) Let $(A, \mathfrak{m}, k)$ be a d-dimensional complete local ring, let $\phi : k \to A$ be a coefficient field and let $x_1, \ldots, x_d \in \mathfrak{m}$. Then there is a natural injective ring map

$$f : \phi(k)[[x_1, \ldots, x_d]] \subset A.$$

Here, $\phi(k)[[x_1, \ldots, x_d]]$ is the image of the map $\phi(k)[[X_1, \ldots, X_d]] \to A$ defined by $X_i \mapsto x_i$ for $i = 1, \ldots, d$, where $\phi(k)[[X_1, \ldots, X_d]]$ is the formal power series ring over $\phi(k)$ with variables $X_1, \ldots, X_d$. The map f is module-finite and $\phi(k)[[x_1, \ldots, x_d]]$ is isomorphic to a formal power series ring with d variables if and only if $x_1, \ldots, x_d$ is a system of parameters of A.

Suppopse that $x_1, \ldots, x_d, x_{d+1}, \ldots, x_{d+h}$ generate $\mathfrak{m}$ such that $x_1, \ldots, x_d$ is a system of parameters of A. Then there is a module-finite ring map

$$\phi(k)[[x_1, \ldots, x_d]] \subset A = \phi(k)[[x_1, \ldots, x_d, x_{d+1}, \ldots, x_{d+h}]].$$

Since $A/(x_1, \ldots, x_d)$ is a finite dimensional $\phi(k)$-vector space spanned by monomials on $x_{d+1}, \ldots, x_{d+h}$, A is a finitely generated $\phi(k)[[x_1, \ldots, x_d]]$-module also spanned by monomials on $x_{d+1}, \ldots, x_{d+h}$ (cf. [6, Theorem 8.4]), i.e.,

$$A = \phi(k)[[x_1, \ldots, x_d, x_{d+1}, \ldots, x_{d+h}]] = \phi(k)[[x_1, \ldots, x_d]][x_{d+1}, \ldots, x_{d+h}].$$

(2) Let $R := k[[X_1, \ldots, X_r]]$ be a formal power series ring over a field k and choose $f \neq 0 \in R$. Write

$$f = \sum_{i=0}^{\infty} b_i X_r^i$$

with $b_i \in k[[X_1, \ldots, X_{r-1}]]$. Let $\mathfrak{n}$ be the maximal ideal of $k[[X_1, \ldots, X_{r-1}]]$ and assume that $b_0, \ldots, b_{\ell-1} \in \mathfrak{n}$ and $b_\ell \notin \mathfrak{n}$ for some $\ell > 0$. By Theorem 2.3, there is a unit $u \in R^\times$ together with a distinguished polynomial $g = X_r^\ell + a_{\ell-1}X_r^{\ell-1} + \cdots + a_0$ with $a_0, \ldots, a_{\ell-1} \in \mathfrak{n}$ such that

$$f = u \cdot g.$$

The ring injection $k[[X_1, \ldots, X_{r-1}]] \subset k[[X_1, \ldots, X_r]]$ induces an injection

$$S := k[[X_1, \ldots, X_{r-1}]] \to R/(f) =: A.$$

Here, A is the S-free module with free basis $1, X_r, \ldots, X_r^{\ell-1}$. In particular, the ideal $(f) = (g)$ of R does not contain any non-zero polynomial in $S[X_r]$ of degree strictly less than ℓ.

We give an example to illustrate the situation. Let $A := \mathbb{F}_p[[X, Y]]/(f)$ with $f = (X^p + tY^p)(X + 1)$, where t is transcendental over $\mathbb{F}_p$ and X, Y are variables. Then we have

$$\frac{\partial f}{\partial X} = X^p + tY^p = 0 \text{ in } A.$$

We note that f is a not a distinguished polynomial with respect to X.

Remark 2.5. Let $(A, \mathfrak{m}, k)$ be a d-dimensional local ring such that $\mathfrak{m}$ is minimally generated by $d+h$ elements. By the prime avoidance theorem, we can find $x_1, \ldots, x_{d+h} \in \mathfrak{m}$ such that

- $\mathfrak{m} = (x_1, \ldots, x_{d+h})$, and
- any d elements in $\{x_1, \ldots, x_{d+h}\}$ form a system of parameters of A.

3. Proof of Cohen-Gabber theorem

We shall prove Theorem 1.1 in this section.

Let $(A, \mathfrak{m}, k)$ be a d-dimensional complete local ring containing a field of characteristic $p > 0$. Let $P_1, \ldots, P_r$ be the set of minimal prime ideals of A of coheight d. If Theorem 1.1 is proved for $\tilde{A} := A/P_1 \cap \cdots \cap P_r$, then the same is true for A. Indeed, $\tilde{A}$ is an equi-dimensional reduced local ring of dimension d. Then, if we can find a required coefficient field together with a system of parameters for $\tilde{A}$, we can lift them to A by Theorem 2.2. Therefore, we may assume that

(3.1) $(A, \mathfrak{m}, k)$ is a d-dimensional reduced equi-dimensional complete local ring containing a field of characteristic $p > 0$.

First we give a proof of the hardest case of Cohen-Gabber theorem.

Proposition 3.1. *Let $(A, \mathfrak{m}, k)$ be a ring as in (3.1). Assume that the ideal $\mathfrak{m}$ is generated by $d+1$ elements. Then there exists a system of parameters $y_1, \ldots, y_d$ of A and a ring map $\phi : k \to A$ such that the following hold: $\pi \circ \phi = \mathrm{id}_k$, the natural map*

$$\phi(k)[[y_1, \ldots, y_d]] \subset A$$

is module-finite, and $\mathrm{Frac}(\phi(k)[[y_1, \ldots, y_d]]) \to \mathrm{Frac}(A/P)$ is a separable field extension for any minimal prime $P \subset A$.

Proof. We fix a coefficient field $\phi : k \to A$ together with a set of elements $x_1, \ldots, x_{d+1} \in \mathfrak{m}$ which satisfy the conclusion of Remark 2.5. Then we have a module-finite injection

$$k[[X_1, \ldots, X_d]] \to A$$

by mapping k to $\phi(k)$ and each X_i to x_i. Since A is reduced and equi-dimensional, after embedding $k[[X_1, \ldots, X_d]]$ to $R := k[[X_1, \ldots, X_d, X_{d+1}]]$ in the natural way, we get a presentation:

$$k[[X_1, \ldots, X_d]] \subset R \twoheadrightarrow R/(f) = A,$$

where $f = f_1 \cdots f_r$ such that f_i is irreducible for $i = 1, \ldots, r$. Furthermore, we may assume that each f_i is a distinguished polynomial with respect to X_{d+1}. Here, remark that, since $X_1, \ldots, X_d, f_i$ is a system of parameters of R for $i = 1, \ldots, r$, each f_i satisfies the assumption of Theorem 2.3. In summary,

(1) any d elements in $\{x_1, \ldots, x_{d+1}\}$ form a system of parameters of A,
(2) $f = f_1 \cdots f_r$ is a factorization, where each f_i is a prime element of R, and
(3) each f_i is a distinguished polynomial with respect to X_{d+1}.

We claim the following fact.

Claim 3.2. *After replacing a coefficient field of A and $x_1, \ldots, x_{d+1}$ if necessary, the following formula together with (1), (2) and (3) above holds:*

$$\frac{\partial f_i}{\partial X_1} \neq 0 \text{ in } R$$

for all $i = 1, \ldots, r$.

Before proving this claim, let us see how Proposition 3.1 follows from it. Since f_i is a distinguished polynomial with respect to X_{d+1}, it follows from Claim 3.2 that

$$\frac{\partial f_i}{\partial X_1} \notin (f_i) \text{ in } R \tag{3.2}$$

since $\deg_{X_{d+1}} \frac{\partial f_i}{\partial X_1}$ is strictly less than $\deg_{X_{d+1}} f_i$ (see Remark 2.4 (2)). Since $x_2, \ldots, x_{d+1}$ form a system of parameters of A by (1), the composite ring map

$$k[[X_2, \ldots, X_{d+1}]] \subset R \twoheadrightarrow R/(f) = A \twoheadrightarrow R/(f_i)$$

is module-finite. Then by Remark 2.4, we can find a unit $u_i \in R^\times$ such that $g_i := f_i \cdot u_i$ is a distinguished polynomial with respect to X_1 for $i = 1, \ldots, r$. Moreover, g_i is a minimal polynomial of $x_1 \in A$ over $\mathrm{Frac}(k[[X_2, \ldots, X_{d+1}]])$. By Leibniz rule, we get

$$\frac{\partial g_i}{\partial X_1} = \frac{\partial f_i}{\partial X_1} u_i + f_i \frac{\partial u_i}{\partial X_1}.$$

Then by (3.2),

$$\frac{\partial g_i}{\partial X_1} \notin (f_i)$$

and in particular,

$$\frac{\partial g_i}{\partial X_1} \neq 0 \text{ in } R.$$

Since g_i is a distinguished polynomial with respect to X_1 satisfying the above, it follows that g_i is a separable polynomial over $\mathrm{Frac}(k[[X_2, \ldots, X_{d+1}]])$. Therefore, the field extension

$$\mathrm{Frac}(k[[X_2, \ldots, X_{d+1}]]) \subset \mathrm{Frac}(R/(f_i)) = \mathrm{Frac}(R/(g_i))$$

is finite separable and this proves Proposition 3.1.

Proof of Claim 3.2. The point is to make a good choice of a coefficient field of A. Consider the following condition for some $s \geq 1$:

$$\frac{\partial f_i}{\partial X_1} \neq 0 \text{ for } i = 1, \ldots, s-1 \text{ and } \frac{\partial f_s}{\partial X_1} = 0 \text{ in } R. \tag{3.3}$$

Assume (3.3). Then we shall prove that

$$\begin{array}{l}\text{after replacing a coefficient field } \phi : k \to A \text{ and } X_1, \ldots, X_{d+1}, \\ \dfrac{\partial f_i}{\partial X_1} \neq 0 \text{ holds for every } i = 1, \ldots, s.\end{array} \tag{3.4}$$

We prove (3.4) in 2 steps below. Keep in mind that we assume (3.3).

Step1 Let us assume that the following condition holds.

$$\text{There exists some } j \geq 2 \text{ such that } \frac{\partial f_s}{\partial X_j} \neq 0 \text{ in } R. \tag{3.5}$$

Write

$$f_s = \sum_{a,b} F_{a,b} X_1^a X_j^b$$

where

$$F_{a,b} := F_{a,b}(X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{d+1}) \in k[[X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{d+1}]].$$

By hypothesis (3.3), if $F_{a,b} \neq 0$, then we have $p|a$. Define

$$b_0 := \min\{b \mid p \nmid b \text{ and } F_{a,b} \neq 0 \text{ for some } a \geq 0\}$$

and

$$a_0 := \min\{a \mid F_{a,b_0} \neq 0\}.$$

Note that $p|a_0$. We prove the following claim.

- Any choice of d elements from the set $\{x_1, \ldots, x_{j-1}, x_j - x_1^n, x_{j+1}, \ldots, x_{d+1}\}$ forms a system of parameters of A for $n \gg 0$.

It suffices to take care of the d elements $x_2, \ldots, x_{j-1}, x_j - x_1^n, x_{j+1}, \ldots, x_{d+1}$. Let $P \in \operatorname{Min}(A/(x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{d+1}))$. Then we have $x_j - x_1^n \notin P$ for $n \gg 0$. Indeed, if this is not the case, there exist n_1 and n_2 such that $n_1 < n_2$ and $x_j - x_1^{n_1}, x_j - x_1^{n_2} \in P$. Then we would have that

$$x_1^{n_1}(1 - x_1^{n_2 - n_1}) \in P \text{ and thus } x_1 \in P.$$

This is a contradiction. Hence the claim follows. By (3.3), we may assume that the following condition is satisfied.

$$\text{For } i = 1, \ldots, s-1, \text{ the coefficient of } X_1^{c_{i,1}} \cdots X_{d+1}^{c_{i,d+1}} \text{ in } f_i \text{ is not zero and } p \nmid c_{i,1}. \tag{3.6}$$

For the sequence $c_{1,1}, c_{2,1}, \ldots, c_{s-1,1}$ as above, let us make a choice of an integer $q > 0$ such that the following condition holds.

- Let $n := qp + 1$. Furthermore, n is strictly greater than any element in the set $\{a_0, c_{1,1}, c_{2,1}, \ldots, c_{s-1,1}\}$, and any choice of d elements from the set $\{x_1, \ldots, x_{j-1}, x_j - x_1^n, x_{j+1}, \ldots, x_{d+1}\}$ forms a system of parameters of A.

We put

$$Y_t := X_t \text{ for } t = 1, \ldots, j-1, j+1, \ldots, d+1 \text{ and } Y_j := X_j - X_1^n.$$

Then

$$g_i(Y_1, \ldots, Y_{d+1}) := f_i(Y_1, \ldots, Y_{j-1}, Y_j + Y_1^n, Y_{j+1}, \ldots, Y_{d+1}) \in R = k[[Y_1, \ldots, Y_{d+1}]]$$

is still a distinguished polynomial with respect to Y_{d+1}. Let us look at the partial derivative of g_i with respect to Y_1 for $i = 1, \ldots, s-1$. Note that

$$g_i(Y_1, \ldots, Y_{d+1}) = Y_1^n \cdot h(Y_1, \ldots, Y_{d+1}) + f_i(Y_1, \ldots, Y_{d+1})$$

for some $h(Y_1, \ldots, Y_{d+1}) \in R = k[[Y_1, \ldots, Y_{d+1}]]$. Since the coefficient of $Y_1^{c_{i,1}} Y_2^{c_{i,2}} \cdots Y_{d+1}^{c_{i,d+1}}$ in $f_i(Y_1, \ldots, Y_{d+1})$ is not zero and $c_{i,1} < n$, the coefficient of $Y_1^{c_{i,1}} Y_2^{c_{i,2}} \cdots Y_{d+1}^{c_{i,d+1}}$ in $g_i(Y_1, \ldots, Y_{d+1})$ turns out to be not zero. Since $p \nmid c_{i,1}$ by (3.6), it follows that

$$\frac{\partial g_i}{\partial Y_1} \neq 0 \text{ in } R \text{ for } i = 1, \ldots, s-1. \tag{3.7}$$

Next, define $G_{a,b}(Y_2, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_{d+1}) \in k[[Y_2, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_{d+1}]]$ by the following equation:

$$\begin{aligned} g_s(Y_1, \ldots, Y_{d+1}) &= \sum_{a,b} F_{a,b}(Y_2, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_{d+1}) Y_1^a (Y_j + Y_1^n)^b \\ &= \sum_{a,b} G_{a,b}(Y_2, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_{d+1}) Y_1^a Y_j^b. \end{aligned}$$

Then we claim that $G_{a_0+n, b_0-1} \neq 0$ by the choice of n. Indeed, if $p \mid b$, $F_{a,b} Y_1^a (Y_j + Y_1^n)^b$ does not contribute to $G_{a_0+n,b_0-1} Y_1^{a_0+n} Y_j^{b_0-1}$. Since $n > a_0$, $F_{a,b} Y_1^a (Y_j + Y_1^n)^b$ does not contribute to $G_{a_0+n,b_0-1} Y_1^{a_0+n} Y_j^{b_0-1}$ if $b > b_0$. Therefore, only $F_{a_0,b_0} Y_1^{a_0} (Y_j + Y_1^n)^{b_0}$ contributes to $G_{a_0+n,b_0-1} Y_1^{a_0+n} Y_j^{b_0-1}$. We have $G_{a_0+n,b_0-1} = \binom{b_0}{1} F_{a_0,b_0} = b_0 F_{a_0,b_0} \neq 0$.

Since $p \nmid (a_0 + n)$, it follows that

$$\frac{\partial g_s}{\partial Y_1} \neq 0 \text{ in } R. \tag{3.8}$$

Combining (3.7) and (3.8) together, we complete the proof of (3.4) under the condition (3.5).

Now we move on to **Step2**.

Step2 Next, let us assume that the following condition holds.

$$\frac{\partial f_s}{\partial X_j} = 0 \text{ for all } j = 1, \ldots, d+1. \tag{3.9}$$

In this case, the coefficient of some monomial on $X_1, \ldots, X_{d+1}$ in f_s does not belong to k^p. If not, f_s must be a p-th power of some element of $R = k[[X_1, \ldots, X_{d+1}]]$. In this case $A = R/(f)$ is not reduced, which contradicts to our hypothesis. Thus, we have $k^p \subsetneq k$ and in particular,

$$k \text{ is an infinite field.} \tag{3.10}$$

Consider the set

$T = \{(\ell_1, \ldots, \ell_{d+1}) \mid \text{the coefficient of } X_1^{\ell_1} X_2^{\ell_2} \cdots X_{d+1}^{\ell_{d+1}} \text{ in } f_s \text{ is not in } k^p\}.$

Let $(\ell'_1, \ldots, \ell'_{d+1})$ be an element of T such that $\ell'_1 + \cdots + \ell'_{d+1}$ is the minimum element in

$$\{\ell_1 + \cdots + \ell_{d+1} \mid (\ell_1, \ldots, \ell_{d+1}) \in T\}.$$

Note that

$$\text{each of } \ell'_1, \ldots, \ell'_{d+1} \text{ is divisible by } p. \tag{3.11}$$

Let α be the coefficient of $X_1^{\ell'_1} X_2^{\ell'_2} \cdots X_{d+1}^{\ell'_{d+1}}$ in f_s, and take a p-basis of $k/\mathbb{F}_p$:

$$\{\alpha\} \cup \{\beta_\lambda\}_{\lambda \in \Lambda}. \tag{3.12}$$

Let $\pi : k[[X_1]] \to k$ be the natural surjection. Let δ be an element in k. Since $\{\pi(\alpha + \delta X_1)\} \cup \{\pi(\beta_\lambda)\}_{\lambda \in \Lambda}$ is a p-basis of $k/\mathbb{F}_p$, we have a map $\psi_\delta : k \to k[[X_1]]$ such that $\psi_\delta(\beta_\lambda) = \beta_\lambda$ for $\lambda \in \Lambda$ and $\psi_\delta(\alpha) = \alpha + \delta X_1$ by Theorem 2.2. Here, ψ_δ naturally induces an isomorphism $\tilde{\psi_\delta} : k[[X_1]] \to k[[X_1]]$ such that $\tilde{\psi_\delta}(X_1) = X_1$ and $\tilde{\psi_\delta}|_k = \psi_\delta$, where $\tilde{\psi_\delta}|_k$ is the restriction of $\tilde{\psi_\delta}$ to k. Let ϕ_δ be the composite map of

$$k \subset k[[X_1]] \xrightarrow{\tilde{\psi_\delta}^{-1}} k[[X_1]].$$

Let

$$\Phi_\delta : R \to R$$

be a ring isomorphism such that $\Phi_\delta(X_i) = X_i$ for $i = 1, \ldots, d+1$ and $\Phi_\delta|_{k[[X_1]]} = \tilde{\psi_\delta}$. We have the following commutative diagram

$$\begin{array}{ccc} k & & \\ \downarrow & \searrow \phi_\delta & \\ k[[X_1]] & \xleftarrow{\tilde{\psi_\delta}} & k[[X_1]] \\ \downarrow & & \downarrow \\ R & \xleftarrow{\Phi_\delta} & R \end{array}$$

where the vertical maps are the natural inclusions.

Considering the composite map of

$$k \xrightarrow{\phi_\delta} k[[X_1]] \subset R = k[[X_1, \dots, X_{d+1}]]$$

as a new coefficient field of R, we have an isomorphism

$$R/(f_i) \simeq R/(\Phi_\delta(f_i)).$$

We claim the following.

- For $i = 1, \dots, s-1$, one can present the coefficient of $X_1^{c_{i,1}} \cdots X_{d+1}^{c_{i,d+1}}$ in $\Phi_\delta(f_i)$ in the form $\xi_i(\delta)$, where $\xi_i(X) \in k[X]$ and $\xi_i(0) \neq 0$, where the sequence $c_{i,1}, \dots, c_{i,d+1}$ is given as in (3.6).

Let us prove this claim. Let $\xi_i(\delta)$ be the coefficient of $X_1^{c_{i,1}} \cdots X_{d+1}^{c_{i,d+1}}$ in $\Phi_\delta(f_i)$. Note by (3.6) that $\xi_i(0) \neq 0$ since Φ_0 is the identity. We shall prove that $\xi_i(\delta)$ is a polynomial function on δ. Pick an element $c \in k \subseteq R$, where k embeds into $R = k[[X_1, \dots, X_{d+1}]]$ in the natural way. Then we have $\Phi_\delta(c) = \tilde{\psi}_\delta(c) \in k[[X_1]]$ and $\tilde{\psi}_\delta(c) - c$ is divisible by X_1. So we can write

$$\Phi_\delta(c) = c + \sum_{i=1}^{\infty} \eta_{c,i}(\delta) X_1^i, \tag{3.13}$$

where $\eta_{c,i}(\delta) \in k$ for $i \geq 1$. It is sufficient to prove that each $\eta_{c,i}(\delta)$ is a polynomial with respect to δ.

For a fixed integer $e > 0$, note that the set

$$\left\{ \alpha^q \cdot \prod_{\lambda \in \Lambda} \beta_\lambda^{q_\lambda} \;\middle|\; \begin{array}{l} q, q_\lambda = 0, 1, \dots, p^e - 1, \\ q_\lambda = 0 \text{ except for finitely many } \lambda \in \Lambda \end{array} \right\}$$

forms a basis of the k^{p^e}-vector space k. We use the symbol $\underline{q}_\lambda$ to denote a vector $\{q_\lambda\}_{\lambda \in \Lambda}$, where $q_\lambda = 0$ except for finitely many $\lambda \in \Lambda$. Suppose

$$c = \sum_{q, \underline{q}_\lambda = 0}^{p^e - 1} (d_{q,\underline{q}_\lambda})^{p^e} \cdot \alpha^q \cdot (\prod_{\lambda \in \Lambda} \beta_\lambda^{q_\lambda})$$

for $d_{q,\underline{q}_\lambda} \in k$. Applying the map Φ_δ, we have

$$\Phi_\delta(c) = \sum_{q, \underline{q}_\lambda = 0}^{p^e - 1} \Phi_\delta(d_{q,\underline{q}_\lambda})^{p^e} \cdot (\alpha + \delta X_1)^q \cdot (\prod_{\lambda \in \Lambda} \beta_\lambda^{q_\lambda}),$$

where

$$\Phi_\delta(d_{q,\underline{q}_\lambda})^{p^e} = \left(d_{q,\underline{q}_\lambda} + X_1 \cdot \gamma\right)^{p^e} = (d_{q,\underline{q}_\lambda})^{p^e} + X_1^{p^e} \cdot \gamma^{p^e}$$

for some $\gamma \in R$. Letting $\eta_{c,i}(\delta)$ be as in (3.13), it follows that, if $e \geq 0$ is an integer satisfying $p^e > i$, then $\eta_{c,i}(\delta)$ is the coefficient of X_1^i in

$$\sum_{q,\underline{q}_\lambda=0}^{p^e-1} (d_{q,\underline{q}_\lambda})^{p^e} \cdot (\alpha + \delta X_1)^q \cdot (\prod_{\lambda \in \Lambda} \beta_\lambda^{q_\lambda}).$$

This description shows that $\eta_{c,i}(\delta)$ is a polynomial with respect to δ. Therefore, $\xi_i(\delta)$ is also a polynomial function on δ for $i = 1, \ldots, s-1$.

Let us write

$$\xi(x) := \xi_1(x) \cdots \xi_{s-1}(x) \in k[x].$$

Then since $\xi(0) \neq 0$, there exists $\delta \in k^\times$ such that $\xi(\delta) \neq 0$ due to the fact that k is an infinite field (3.10). Now we are going to finish the proof of Claim 3.2.

- For $i = 1, 2, \ldots, s-1$, we have

$$\frac{\partial \Phi_\delta(f_i)}{\partial X_1} \neq 0,$$

  since the coefficient of $X_1^{c_{i,1}} X_2^{c_{i,2}} \cdots X_{d+1}^{c_{i,d+1}}$ in $\Phi_\delta(f_i)$ is $\xi_i(\delta)$, which is not 0 by the choice of δ.
- For $i = s$, we shall prove that the coefficient of $X_1^{\ell'_1+1} X_2^{\ell'_2} \cdots X_{d+1}^{\ell'_{d+1}}$ in $\Phi_\delta(f_s)$ is not zero. Put

$$f_s = \sum_{\ell_1, \ldots, \ell_{d+1}} c_{\ell_1, \ldots, \ell_{d+1}} X_1^{\ell_1} X_2^{\ell_2} \cdots X_{d+1}^{\ell_{d+1}},$$

  where $c_{\ell_1, \ldots, \ell_{d+1}} \in k$. Then, we have

$$\begin{aligned}
\Phi_\delta(f_s) &= \sum_{\ell_1, \ldots, \ell_{d+1}} \Phi_\delta(c_{\ell_1, \ldots, \ell_{d+1}}) X_1^{\ell_1} X_2^{\ell_2} \cdots X_{d+1}^{\ell_{d+1}} \\
&= \sum_{\ell_1, \ldots, \ell_{d+1}} \left(c_{\ell_1, \ldots, \ell_{d+1}} + \sum_{i=1}^{\infty} \eta_{c_{\ell_1, \ldots, \ell_{d+1}}, i}(\delta) X_1^i \right) X_1^{\ell_1} X_2^{\ell_2} \cdots X_{d+1}^{\ell_{d+1}} \\
&= f_s + \sum_{\ell_1, \ldots, \ell_{d+1}} \sum_{i=1}^{\infty} \eta_{c_{\ell_1, \ldots, \ell_{d+1}}, i}(\delta) X_1^{\ell_1+i} X_2^{\ell_2} \cdots X_{d+1}^{\ell_{d+1}}.
\end{aligned}$$

  Therefore, the coefficient of $X_1^{\ell'_1+1} X_2^{\ell'_2} \cdots X_{d+1}^{\ell'_{d+1}}$ in $\Phi_\delta(f_s)$ is

$$\sum_{i=1}^{\ell'_1+1} \eta_{c_{\ell'_1+1-i, \ell'_2, \ldots, \ell'_{d+1}}, i}(\delta),$$

  since the coefficient of $X_1^{\ell'_1+1} X_2^{\ell'_2} \cdots X_{d+1}^{\ell'_{d+1}}$ in f_s is zero by (3.9) and (3.11). If $\eta_{c_{\ell'_1+1-i, \ell'_2, \ldots, \ell'_{d+1}}, i}(\delta) \neq 0$, then this implies that $c_{\ell'_1+1-i, \ell'_2, \ldots, \ell'_{d+1}} \neq 0$ and thus, $\ell'_1 + 1 - i$ is divisible by p by (3.9). So, we assume that $i \equiv 1 \mod p$ by (3.11). If $i = qp + 1$ with $q > 0$, then $c_{\ell'_1+1-i, \ell'_2, \ldots, \ell'_{d+1}} \in k^p$ by the definition of $(\ell'_1, \ell'_2, \dots, \ell'_{d+1})$. Under the notation as in (3.13), note that $\eta_{\gamma,i}(\delta) = 0$ if $p \nmid i$ and $\gamma \in k^p$, because $\Phi_\delta(\gamma)$ has a p-th root in R. Therefore, we have $\eta_{c_{\ell'_1+1-i,\ell'_2,\dots,\ell'_{d+1}},i}(\delta) = 0$ for any integer $i \geq 2$. Then, the coefficient of $X_1^{\ell'_1+1} X_2^{\ell'_2} \cdots X_{d+1}^{\ell'_{d+1}}$ in $\Phi_\delta(f_s)$ is

$$\eta_{c_{\ell'_1,\ell'_2,\dots,\ell'_{d+1}},1}(\delta) = \eta_{\alpha,1}(\delta) = \delta \neq 0,$$

where $\alpha \in k$ is as in (3.12). Hence, we obtain

$$\frac{\partial \Phi_\delta(f_s)}{\partial X_1} \neq 0.$$

We complete the proof of (3.4) under the condition (3.9).

We have completed the proof of Claim 3.2. □

We have completed the proof of Proposition 3.1, which is the hypersurface case of Cohen-Gabber theorem. □

As noted in (3.1), it suffices to prove Cohen-Gabber theorem (Theorem 1.1) in the reduced equi-dimensional case.

Proof of Theorem 1.1. Let $\mathfrak{m} = (x_1, \dots, x_d, x_{d+1}, \dots, x_{d+h})$ such that $x_1, \dots, x_d$ is a system of parameters of A (see Remark 2.5). We shall prove the reduced equi-dimensional case of Theorem 1.1 by induction on h.

$\mathbf{h = 0}$: Since $\mathfrak{m}$ is generated by d elements, we have $A = k[[x_1, \dots, x_d]]$ and we are done in this case.

$\mathbf{h = 1}$: This is already established as in Proposition 3.1.

$\mathbf{h \geq 2}$: With notation as above, fix a coefficient field $\phi : k \to A$. Then $A = \phi(k)[[x_1, \dots, x_{d+h}]] = \phi[[x_1, \dots, x_d]][x_{d+1}, \dots, x_{d+h}]$ by Remark 2.4 (1). We consider the following commutative diagram of complete local rings:

$$\begin{array}{ccccc}
A = \phi(k)[[x_1, \dots, x_{d+h}]] & \longleftarrow & D := \phi'(k)[[y_1, \dots, y_d, x_{d+2}, \dots, x_{d+h}]] & \longleftarrow & E := \phi''(k)[[z_1, \dots, z_d]] \\
\uparrow & & \uparrow & & \\
B := \phi(k)[[x_1, \dots, x_{d+1}]] & \longleftarrow & C := \phi'(k)[[y_1, \dots, y_d]] & & \\
\uparrow & & & & \\
\phi(k)[[x_1, \dots, x_d]] & & & &
\end{array}$$

We explain the structure of the above diagram.

- Let B be the subring $\phi(k)[[x_1, \dots, x_{d+1}]]$ of A. After applying the case $\mathbf{h = 1}$ to B, we can find a coefficient field $\phi' : k \to B$ together with a system of parameters $y_1, \dots, y_d$ to get a formal power series ring $C = \phi'(k)[[y_1, \dots, y_d]]$ such that $\mathrm{Frac}(C) \to \mathrm{Frac}(B/P)$ is a separable field extension for any minimal prime ideal

P of B. Let D be the subring $\phi'(k)[[y_1, \ldots, y_d, x_{d+2}, \ldots, x_{d+h}]]$ of A. Note that $C \subset D \subset A$.

- Since the maximal ideal of D is generated by at most $d+h-1$ elements, we can find, by induction hypothesis on h, a coefficient field $\phi'' : k \to D$ together with a system of parameters $z_1, \ldots, z_d$ to get a formal power series ring $E = \phi''(k)[[z_1, \ldots, z_d]]$ such that $\mathrm{Frac}(E) \to \mathrm{Frac}(D/Q)$ is a separable field extension for any minimal prime ideal Q of D.

All the maps appearing in the diagram are injective and module-finite. We claim that $E \to A$ satisfies the conclusion of Cohen-Gabber theorem. To see this, fix a minimal prime $P \subset A$ and form the following commutative diagram of quotient fields.

$$\begin{array}{ccccc}
\mathrm{Frac}(A/P) & & & & \\
\| & & & & \\
\mathrm{Frac}(D/D \cap P)(x_1, \ldots, x_{d+1}) & \xleftarrow{f_1} & \mathrm{Frac}(D/D \cap P) & \xleftarrow{f_2} & \mathrm{Frac}(E) \\
\uparrow & & \uparrow & & \\
\mathrm{Frac}(C)(x_1, \ldots, x_{d+1}) & \xleftarrow{f_3} & \mathrm{Frac}(C) & & \\
\| & & & & \\
\mathrm{Frac}(B/B \cap P) & & & &
\end{array}$$

Note that E and C are domains over which A is module-finite and torsion free, so we have $E \cap P = (0)$ and $C \cap P = (0)$. We need to prove that $\mathrm{Frac}(E) \to \mathrm{Frac}(A/P)$ is a separable field extension. By construction, f_3 is separable and f_1 is obtained by adjoining $x_1, \ldots, x_{d+1}$ to $\mathrm{Frac}(D/D \cap P)$. Hence f_1 is separable. That is, we proved that $f_1 \circ f_2$ is separable. $\square$

We end this paper with the following example. For a ring map $R \to S$, let $\Omega_{S/R}$ denote the module of differentials of S over R. It is regarded as an S-module.

Example 3.3. Consider the following two integral domains;

$$\begin{aligned} A &= \mathbb{F}_p(t)[[X, Y]]/(tX^p + Y^p), \\ B &= \mathbb{F}_p(t)[X, Y]/(tX^p + Y^p), \end{aligned}$$

where t is transcendental over $\mathbb{F}_p$ and X, Y are variables.

We have $\Omega_{B/\mathbb{F}_p(t)} = BdX + BdY \simeq B^{\oplus 2}$. Here, assume that $\mathbb{F}_p(t)[z] \hookrightarrow B$ is a module-finite map for some $z \in B$. Note that

$$\Omega_{B/\mathbb{F}_p(t)[z]} = \Omega_{B/\mathbb{F}_p(t)}/Bdz.$$

Since it is not a torsion B-module, $\mathrm{Frac}(B)$ is not separable over $\mathrm{Frac}(\mathbb{F}_p(t)[z])$.

Let w be any non-zero element in the maximal ideal of A. Then, $\mathbb{F}_p(t)[[w]] \to A$ is a module-finite extension. Then, we have

$$\Omega_{A/\mathbb{F}_p(t)[[w]]} = AdX + AdY/Adw,$$

and it is not a torsion A-module. Hence, $\mathrm{Frac}(A)$ is not separable over $\mathrm{Frac}(\mathbb{F}_p(t)[[w]])$.

On the other hand, put $s = t + X \in A$. Then, $\mathbb{F}_p(s)$ is another coefficient field of A, and

$$A = \mathbb{F}_p(s)[[X, Y]]/((s - X)X^p + Y^p).$$

Then, $\mathbb{F}_p(s)[[Y]] \to A$ is module-finite and $\mathrm{Frac}(A)$ is a separable field extension over $\mathrm{Frac}(\mathbb{F}_p(s)[[Y]])$.

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