On the limit of Frobenius in the Grothendieck group

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Dedicated to Professor Ngô Viêt Trung for his 60th birthday.

Abstract

Considering the Grothendieck group modulo numerical equivalence, we obtain the finitely generated lattice $G_0(R)$ for a Noetherian local ring $R$. Let $C_{CM}(R)$ be the cone in $G_0(R)_R$ spanned by cycles of maximal Cohen-Macaulay $R$-modules. We shall define the fundamental class $\overline{\mu_R}$ of $R$ in $G_0(R)_R$, which is the limit of the Frobenius direct images (divided by their rank) $[e^p R]/pde$ in the case $\text{ch}(R) = p > 0$. The homological conjectures are deeply related to the problems whether $\overline{\mu_R}$ is in the Cohen-Macaulay cone $C_{CM}(R)$ or the strictly nef cone $SN(R)$ defined below. In this paper, we shall prove that $\overline{\mu_R}$ is in $C_{CM}(R)$ in the case where $R$ is FFRT or F-rational.

1 Introduction

We shall define the Cohen-Macaulay cone $C_{CM}(R)$, the strictly nef cone $SN(R)$, and the fundamental class $\overline{\mu_R}$ for a Noetherian local domain $R$. They satisfy

$$
\bigcup_{\mathbb{Q}} G_0(R) \supset SN(R) \supset C_{CM}(R) - \{0\}
$$

where $G_0(R)$ is the Grothendieck group of finitely generated $R$-modules, $G_0(R)_R$ is the Grothendieck group modulo numerical equivalence, and $G_0(R)_K = G_0(R) \otimes_{\mathbb{Z}} K$. By [8], $G_0(R)_R$ is a finitely generated free $\mathbb{Z}$-module. We define $C_{CM}(R)$ to be the cone in $G_0(R)_R$ spanned by cycles corresponding to maximal Cohen-Macaulay $R$-modules. If $R$ is F-finite with residue class field algebraically closed, the fundamental class $\overline{\mu_R}$ is the limit of the Frobenius direct images (divided by their rank) $[e^p R]/pde$ as in Remark 8 (3). In the case where $R$ contains a regular local ring $S$ such that $R$ is contained in a Galois extension $B$ of $S$, then $\overline{\mu_R}$ is described in terms of $B$ as in Remark 8 (2).

The fundamental class is deeply related to the homological conjectures as in Fact 10. The fundamental class $\overline{\mu_R}$ is in $C_{CM}(R)$ for any complete local domain $R$ if and only if

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the small Mac conjecture is true. Roberts proved $\overline{\mu_R} \in SN(R)$ for any Noetherian local ring $R$ of characteristic $p > 0$ in order to show the new intersection theorem in the mixed characteristic case [12]. In order to extend these results, we are mainly interested in the problem whether $\overline{\mu_R}$ is in such cones or not.

**Problem 1** If $R$ is an excellent Noetherian local domain, is $\overline{\mu_R}$ in $C_{CM}(R)$?

Problem 1 is affirmative if $R$ is a complete intersection. However, even if $R$ is a Gorenstein ring which contains a field, Problem 1 is an open question.

The following theorem is the main result in this paper. We define the terminologies later.

**Theorem 2** Assume that $R$ is an $F$-finite Cohen-Macaulay local domain of characteristic $p > 0$ with residue class field algebraically closed.

(1) If $R$ is FFRT, then there exist a natural number $n$ and a maximal Cohen-Macaulay $R$-module $N$ such that $n\mu_R = [N]$ in $G_0(R)_{\mathbb{Q}}$. In particular, $\overline{\mu_R}$ is contained in $C_{CM}(R)$.

(2) If $R$ is F-rational, then $\overline{\mu_R}$ is contained in $Int(C_{CM}(R))$.

In the case FFRT, we shall show that the cone generated by $[M_1], \ldots, [M_s]$ (in Definition 17) contains $\mu_R$. In the case of F-rational, the key point in our proof is to use the dual F-signature defined by Sannai [14].

Finally we shall give a corollary (Corollary 22), which was first proved in [1].

## 2 Cohen-Macaulay cone

In this paper, let $R$ be a $d$-dimensional Noetherian local domain such that one of the following conditions are satisfied:

(a) $R$ is a homomorphic image of an excellent regular local ring containing $\mathbb{Q}$.

(b) $R$ is essentially of finite type over a field, $\mathbb{Z}$ or a complete DVR.

If either (a) or (b) is satisfied, there exists a regular alteration of $\text{Spec} R$ by de Jong’s theorem [5].

We always assume that modules are finitely generated.

Let $G_0(R)$ be the Grothendieck group of finitely generated $R$-modules, that is,

$$G_0(R) = \bigoplus_{\text{M: f. g. R-module}} \mathbb{Z}[M] \text{ subject to } < [M] - [L] - [N] | 0 \to L \to M \to N \to 0 \text{ is exact }>,$$

where $[M]$ denotes the generator corresponding to an $R$-module $M$. Let $C(R)$ be the category of bounded complexes of finitely generated $R$-free modules such that every homology is of finite length. Let $C_d(R)$ be the subcategory of $C(R)$ consisting of complexes of length $d$ with $H_0(\mathcal{F}) \neq 0$. A complex $\mathcal{F}$ in $C_d(R)$ is of the form

$$0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0.$$
For example, the Koszul complex of a parameter ideal belongs to $C_d(R)$.

For $F \in C(R)$, we have a well-defined map

$$
\chi_F : G_0(R) \to \mathbb{Z}
$$

by $\chi_F([M]) = \sum_i (-1)^i \ell_R(H_i(F \otimes_R M))$. We have the induced maps $\chi_F : G_0(R)_Q \to \mathbb{Q}$ and $\chi_F : G_0(R)_R \to \mathbb{R}$. We say that $\alpha \in G_0(R)$ ($\alpha \in G_0(R)_Q$ or $\alpha \in G_0(R)_R$) is numerically equivalent to 0 if $\chi_F(\alpha) = 0$ for any $F \in C(R)$. We define the Grothendieck group modulo numerical equivalence as follows:

$$
\overline{G_0(R)} = G_0(R)/\{\alpha \in G_0(R) \mid \chi_F(\alpha) = 0 \text{ for any } F \in C(R)\}.
$$

Then, by Theorem 3.1 and Remark 3.5 in [8], we know that $\overline{G_0(R)}$ is a non-zero finitely generated $\mathbb{Z}$-free module.

**Example 3**

1. If $d \leq 2$, then $\overline{G_0(R)} = \mathbb{Z}$ (Proposition 3.7 in [8]). If $d \geq 3$, there exists an example of $d$-dimensional Noetherian local domain $R$ such that $\dim \overline{G_0(R)} = m$ for any positive integer $m$ as in (2) (b) (i) below.

2. Let $X$ be a smooth projective variety with embedding $X \hookrightarrow \mathbb{P}^n$. Let $R$ (resp. $D$) be the affine cone (resp. the very ample divisor) of this embedding. Let $A_*(R)$ be the Chow group of $R$. By [8], we can define numerical equivalence also on $A_*(R)$, that is compatible with the Riemann-Roch theory as below. Let $CH(X)$ (resp. $CH_{num}(X)$) be the Chow ring (resp. Chow ring modulo numerical equivalence) of $X$. It is well-known that $CH_{num}(X)_Q$ is a finite dimensional $\mathbb{Q}$-vector space. Then, we have the following commutative diagram:

$$
\begin{array}{ccc}
G_0(R)_Q & \xrightarrow{\tau_R} & A_*(R)_Q \\
\downarrow & & \downarrow \\
\overline{G_0(R)}_Q & \xrightarrow{\tau_R} & A_*(R)_Q \\
\end{array}
\quad \begin{array}{ccc}
& \phi & \quad \begin{array}{c}
CH(X)_Q/D \cdot CH(X)_Q \\
\downarrow & & \downarrow \\
CH_{num}(X)_Q/D \cdot CH_{num}(X)_Q \\
\end{array}
\end{array}
$$

(a) By the commutativity of this diagram, $\phi$ is a surjection. Therefore, we have

$$
\text{rank } \overline{G_0(R)} \leq \dim_Q CH_{num}(X)_Q/D \cdot CH_{num}(X)_Q. \tag{1}
$$

(b) If $CH(X)_Q \simeq CH_{num}(X)_Q$, then we can prove that $\phi$ is an isomorphism ([8], [13]). In this case, the equality holds in (1). Using it, we can show the following:

(i) If $X$ is a blow-up at $n$ points of $\mathbb{P}^k$ ($k \geq 2$), then $\text{rank } \overline{G_0(R)} = n + 1$.

(ii) If $X = \mathbb{P}^m \times \mathbb{P}^n$, then $\text{rank } \overline{G_0(R)} = \min\{m, n\}$.

(c) There exists an example such that $\phi$ is not an isomorphism [13].

Further, Roberts and Srinivas [13] proved the following: Assume that the standard conjecture and Bloch-Beilinson conjecture are true. Then $\phi$ is an isomorphism if the defining ideal of $R$ is generated by polynomials with coefficients in the algebraic closure of the prime field.

\footnote{We need the existence of a regular alteration in the proof of this result.}
Consider the groups $G_0(\mathbb{R}) \subset G_0(\mathbb{R})_\mathbb{Q} \subset G_0(\mathbb{R})_\mathbb{R}$. We shall define some cones in $G_0(\mathbb{R})_\mathbb{R}$.

**Definition 4** Let $C_{CM}(R)$ be the cone (in $G_0(\mathbb{R})_\mathbb{R}$) spanned by all maximal Cohen-Macaulay $R$-modules.

$$C_{CM}(R) = \sum_{M: MCM} \mathbb{R}_{\geq 0}[M] \subset G_0(\mathbb{R})_\mathbb{R}.$$  

We call it the **Cohen-Macaulay cone** of $R$. Thinking a free basis of $G_0(\mathbb{R})_\mathbb{R}$ as an orthonormal basis of $G_0(\mathbb{R})_\mathbb{R}$, we think $G_0(\mathbb{R})_\mathbb{R}$ as a metric space. Let $C_{CM}(R)^- \supset C_{CM}(R)$ be the closure of $C_{CM}(R)$ with respect to this topology on $G_0(\mathbb{R})_\mathbb{R}$.

We define the **strictly nef cone** by

$$SN(R) = \{ \alpha \mid \chi_F(\alpha) > 0 \text{ for any } F \in C_d(R) \}.$$

By the depth sensitivity, $\chi_F([M]) = \ell_R(H_0(F \otimes M)) > 0$ for any maximal Cohen-Macaulay module $M \neq 0$ and $F \in C_d(R)$. Therefore,

$$SN(R) \supset C_{CM}(R) - \{0\}.$$

**Remark 5** Assume that $R$ is a Cohen-Macaulay local domain. Let $M$ be a torsion $R$-module. Taking sufficiently high syzygies of $M$, we know

$$\pm [M] + n[R] \in C_{CM}(R) \text{ for } n \gg 0.$$  

Therefore, we have $\dim C_{CM}(R) = \text{rank } G_0(\mathbb{R})$ and

$$C_{CM}(R)^- \supset C_{CM}(R) \supset \text{Int}(C_{CM}(R)^-) = \text{Int}(C_{CM}(R)) \ni [R],$$

where $\text{Int}(\ )$ denotes the interior.

**Example 6** The following examples are given in [2]. Assume that $k$ is an algebraically closed field of characteristic zero.

1. Put $R = k[x, y, z, w]/(xy - f_1 f_2 \cdots f_t)$. Here, we assume that $f_1, f_2, \ldots, f_t$ are pairwise coprime linear forms in $k[z, w]$ with $t \geq 2$. In this case, we have rank $G_0(\mathbb{R}) = t$. We know (see [2]) that the Cohen-Macaulay cone is minimally spanned by the following $2^t - 2$ maximal Cohen-Macaulay modules of rank one:

$$\{(x, f_{i_1} f_{i_2} \cdots f_{i_s}) \mid 1 \leq s < t, \ 1 \leq i_1 < i_2 < \cdots < i_s \leq t\}$$

Here, remark that this ring is of finite representation type if and only if $t \leq 3$.

2. The Cohen-Macaulay cone of $k[x_1, x_2, \ldots, x_6]/(x_1 x_2 + x_3 x_4 + x_5 x_6)$ is not spanned by maximal Cohen-Macaulay modules of rank one. It is of finite representation type since it has a simple singularity.
3 Fundamental class

**Definition 7** Let $R$ be a $d$-dimensional Noetherian local domain. We put

$$
\mu_R = \tau_R^{-1}([\text{Spec } R]) \in G_0(R)_\mathbb{Q},
$$

where $\tau_R : G_0(R)_\mathbb{Q} \sim A_*(R)_\mathbb{Q}$ is the singular Riemann-Roch map, and $[\text{Spec } R]$ denotes the cycle in $A_*(R)$ corresponding to the scheme Spec $R$ itself.

$$
\begin{align*}
G_0(R)_\mathbb{Q} & \rightarrow G_0(R)_\mathbb{Q} \\
\mu_R & \mapsto \overline{\mu_R}
\end{align*}
$$

We call the image of $\mu_R$ in $G_0(R)_\mathbb{Q}$ the fundamental class of $R$, and denote it by $\overline{\mu_R}$.

Remark that $\overline{\mu_R} \neq 0$ since $\text{rank}_R \mu_R = 1$.

Put $R = T/I$, where $T$ is a regular local ring. The map $\tau_R$ is defined using not only $R$ but also $T$. Therefore, $\mu_R$ may depend on the choice of $T$. However, we can prove that $\overline{\mu_R}$ is independent of $T$ (Theorem 5.1 in [8]).

We shall explain the reason why we call $\overline{\mu_R}$ the fundamental class of $R$.

**Remark 8** (1) If $X (= \text{Spec } R)$ is a $d$-dimensional affine variety over $\mathbb{C}$, we have the cycle map $cl$ such that $cl([\text{Spec } R])$ coincides with the fundamental class $\mu_X$ in $H_{2d}(X, \mathbb{Q})$ in the usual sense, where $H_*(X, \mathbb{Q})$ is the Borel-Moore homology. Here $\mu_X$ is the generator of $H_{2d}(X, \mathbb{Q}) \cong \mathbb{Z}$.

$\begin{align*}
G_0(R)_\mathbb{Q} & \xrightarrow{\mu_R} A_*(R)_\mathbb{Q} \xrightarrow{cl} H_*(X, \mathbb{Q}) \\
[\text{Spec } R] & \mapsto \mu_X
\end{align*}$

The map $cl$ induces the map $\overline{A_d(R)}_\mathbb{Q} \rightarrow H_{2d}(X, \mathbb{Q})$ such that the fundamental class $\mu_X$ is the image of $\overline{\tau_R(\overline{\mu_R})}$. Hence, we call $\overline{\mu_R}$ the fundamental class of $R$.

(2) Let $R$ have a subring $S$ such that $S$ is a regular local ring and $R$ is a localization of a finite extension of $S$. Let $L$ be a finite-dimensional normal extension of $Q(S)$ containing $Q(R)$. Let $B$ be the integral closure of $R$ in $L$. Then, we have

$$
\mu_R = \frac{1}{\text{rank}_R B} [B] \text{ in } G_0(R)_\mathbb{Q}.
$$

In particular, $\overline{\mu_R} = \frac{|B|}{\text{rank}_R B} \text{ in } G_0(R)_\mathbb{Q}$ (see the proof of Theorem 1.1 in [6]).

(3) Assume that $R$ is of characteristic $p > 0$ and $F$-finite. Assume that the residue class field is algebraically closed. By the singular Riemann-Roch theorem, we have

$$
\overline{\mu_R} = \lim_{e \to \infty} \frac{[e R]}{p^de} \text{ in } G_0(R)_\mathbb{R},
$$

where $eR$ is the $e$-th Frobenius direct image (see Definition 13, 14 below). It immediately follows from the equations (7) and (9) below.

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There is no example that the map $\tau_R$ actually depend on the choice of $T$. For some excellent rings, it had been proved that $\tau_R$ is independent of the choice of $T$ (Proposition 1.2 in [7]).
Example 9  
(1) If \( R \) is a complete intersection, then \( \mu_R \) is equal to \( [R] \) in \( G_0(R)_Q \), therefore \( \overline{\mu_R} = [R] \) in \( \overline{G_0(R)}_Q \). There exists a Gorenstein ring such that \( \overline{\mu_R} \not= [R] \). However there exist many examples of rings satisfying \( \overline{\mu_R} = [R] \) ([7]). Roberts ([10], [11]) proved the vanishing property of intersection multiplicities for rings satisfying \( \overline{\mu_R} = [R] \).

(2) Let \( R \) be a normal domain. Then, we have

\[
\begin{align*}
G_0(R)_Q & \xrightarrow{\tau_R} A_*(R)_Q = A_d(R)_Q \oplus A_{d-1}(R)_Q \oplus \cdots \\
[R] & \mapsto \text{Spec } R - \frac{K_R}{2} + \cdots \\
[\omega_R] & \mapsto \text{Spec } R + \frac{K_R}{2} + \cdots 
\end{align*}
\]

where \( K_R \) is the Weil divisor corresponding to the canonical module \( \omega_R \). If \( \tau_R^{-1}(K_R) \not\equiv 0 \) in \( \overline{G_0(R)}_Q \), then \( [R] \not= \overline{\mu_R} \). Although the equality

\[
\overline{\mu_R} = \frac{1}{2}([R] + [\omega_R])
\]

is sometimes satisfied, it is not true in general.

(3) Let \( R = k[x_{ij}]/I_2(x_{ij}) \), where \( (x_{ij}) \) is the generic \((m + 1) \times (n + 1)\)-matrix, and \( k \) is a field. Suppose \( 0 < m \leq n \). Then, we have

\[
\begin{align*}
G_0(R)_Q & \simeq \overline{G_0(R)}_Q \\
[R] & \mapsto (\frac{a}{1-e^{-a}})^m (\frac{-a}{1-e^{-a}})^n \\
& = 1 + \frac{1}{2}(m - n)a + \frac{1}{24}(\cdots)a^2 + \cdots \\
[\omega_R] & \mapsto (\frac{-a}{1-e^{-a}})^m \\
\overline{\mu_R} & \mapsto 1 \\
\tau_R^{-1}(K_R) & \mapsto (n - m) \overline{\mu_R}
\end{align*}
\]

(4) By Remark 2.9 in [1], if \( \overline{\mu_R} \in CM(R) \), then there exists a maximal Cohen-Macaulay \( R \)-module \( N \) such that \( [N] = \text{rank}_R N \cdot \overline{\mu_R} \) in \( G_0(R)_Q \).

Here, we shall explain the connection between the fundamental class \( \overline{\mu_R} \) and the homological conjectures.

Fact 10  
(1) The small Mac conjecture is true if and only if \( \overline{\mu_R} \in CM(R) \) for any complete local domain \( R \) (Theorem 1.3 in [6]). We give an outline of the proof here.

“\( \text{If} \)” part is trivial. We shall show “\( \text{only if} \)” part. Suppose that \( S \) is a regular local ring such that \( R \) is a finite extension over \( S \). Let \( L \) be a finite-dimensional normal extension of \( Q(S) \) containing \( Q(R) \). Let \( B \) be the integral closure of \( R \) in \( L \). Then, \( B \) is finite over \( R \), and \( B \) is a complete local domain. Here, assume that there exists an maximal Cohen-Macaulay \( B \)-module \( M \). Put \( \text{Aut}_{Q(S)}(L) = \{ g_1, \ldots, g_t \} \) and \( N = \oplus_i (g_i, M) \), where \( g_i, M \) denotes \( M \) with \( R \)-module structure given by \( a \times m = g_i(a)m \). Then \( N \) is a maximal Cohen-Macaulay \( R \)-module such that \( [N] = \text{rank}_R N \cdot \mu_R \) in \( G_0(R)_Q \). Therefore, \( \overline{\mu_R} = \frac{[N]}{\text{rank}_R N} \in CM(R) \).

Even if \( R \) is an equi-characteristic Gorenstein ring, it is not known whether \( \overline{\mu_R} \) is in \( CM(R) \) or not. If \( R \) is a complete intersection, then \( \overline{\mu_R} = [R] \in CM(R) \) as in (1) in Example 9.
(2) If $\overline{\mu_R} = [R]$ in $G_0(R)_R$, then the vanishing property of intersection multiplicities holds (Roberts [10], [11]).

(3) Roberts [12] proved $\overline{\mu_R} \in SN(R)$ if $ch(R) = p > 0$. Using it, he proved the new intersection theorem in the mixed characteristic case.

(4) If $R$ contains a field, then $\overline{\mu_R} \in SN(R)$ (Kurano-Roberts [9]). Even if $R$ is a Gorenstein ring (of mixed characteristic), we do not know whether $\overline{\mu_R} \in SN(R)$ or not.

(5) If $\overline{\mu_R} \in SN(R)$ for any $R$, then Serre’s positivity conjecture is true in the case where one of two modules is (not necessary maximal) Cohen-Macaulay.

It is well-known that Serre’s positivity conjecture follows from the small Mac conjecture.

Remark 11 (1) If $R$ is Cohen-Macaulay of characteristic $p > 0$, then $^eR$ is a maximal Cohen-Macaulay module. Since $\overline{\mu_R}$ is the limit of $[^eR]/p^{de}$ in $G_0(R)_R$ as in Remark 8 (3), $\overline{\mu_R}$ is contained in $CCM(R)^-$. If we know that $CCM(R)$ is a closed set of $G_0(R)_R$, we have $\overline{\mu_R} \in CCM(R)^- = CCM(R)$. If the cone $CCM(R)$ is finitely generated, then it is a closed subset. We do not know any example that the cone $CCM(R)$ is not finitely generated.

In the case where $R$ is not of characteristic $p > 0$, we do not know whether $\overline{\mu_R}$ is contained in $CCM(R)^-$ even if $R$ is a Gorenstein ring.

(2) As we have already seen in Remark 5, if $R$ is Cohen-Macaulay, then $[R] \in Int(CCM(R)) \subset CCM(R)$.

There is an example of non-Cohen-Macaulay ring $R$ containing a field such that $[R] \not\in SN(R)$. On the other hand, it is expected that $\overline{\mu_R} \in SN(R)$ for any $R$ (Fact 10 (4)). Therefore, for a non-Cohen-Macaulay local ring $R$, $\overline{\mu_R}$ behaves better than $[R]$ in a sense.

4 Main theorem

In Fact 10, we saw that the fundamental class $\overline{\mu_R}$ is deeply related to the homological conjectures. We propose the following question.

Question 12 Assume that $R$ is a “good” Cohen-Macaulay local domain (for example, equicharacteristic, Gorenstein, etc). Is $\overline{\mu_R} \in CCM(R)$?

If $R$ is a Cohen-Macaulay local domain such that the rank of $G_0(R)$ is one, then $[R] = \overline{\mu_R} \in CCM(R)$, therefore Question 12 is true in this case. There are a lot of such examples (for instance, invariant subrings with respect to finite group actions, etc.).

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3It was conjectured above 50 years ago that $[R]$ was in $SN(R)$ for any local ring $R$. Essentially, the famous counter example due to Dutta-Hochster-MacLaughlin [3] gives an example $[R] \not\in SN(R)$. 

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**Definition 13** Let $p$ be a prime number and $R$ be a Noetherian ring of characteristic $p$. Let $e > 0$ be an integer and 

$$F^e : R \rightarrow R$$

be the $e$-th Frobenius map. We denote by $^eR$ the $R$-module $R$ with $R$-module structure given by $r \times x = F^e(r)x$. It is called the $e$-th *Frobenius direct image*.

**Definition 14** Let $p$ be a prime number and $R$ be a Noetherian ring of characteristic $p$. We say that $R$ is *F-finite* if the Frobenius map $F : R \rightarrow R$ is finite.

**Remark 15** Let $R$ be a $d$-dimensional F-finite Noetherian local ring. We have the following commutative diagram (2) where the horizontal map $\tau_R$ is the singular Riemann-Roch map and the vertical maps are induced by $F^e$:

$$
\begin{array}{ccc}
G_0(R)_Q & \xrightarrow{\tau_R} & A_*(R)_Q \\
\downarrow F^e & & \downarrow F^e \\
G_0(R)_Q & \xrightarrow{\tau_R} & A_*(R)_Q
\end{array}
$$

By diagram (2), we have

$$\tau_R([eR]) = F^e(\tau_R([R])). \quad (3)$$

We set

$$\tau_R([R]) = \tau_R([R])_d + \tau_R([R])_{d-1} + \cdots + \tau_R([R])_0$$

where $\tau_R([R])_i \in A_i(R)_Q$ for $i = 0, \ldots, d$. Then, by the top term property [4], we know

$$\tau_R([R])_d = [\text{Spec } R] \in A_*(R)_Q. \quad (4)$$

Assume that $(R, \mathfrak{m})$ is a $d$-dimensional F-finite Noetherian local domain with residue class field $R/\mathfrak{m}$ algebraically closed. For $\alpha \in A_i(R)_Q$ we have

$$F_* (\alpha) = p^i \alpha \quad (5)$$

by Lemma 16 below and the definition of $F_*$ [4]. Therefore

$$F^e_* (\tau_R([R])) = p^{de}[\text{Spec } R] + \sum_{0 \leq i \leq d-1} p^i \tau_R([R])_i. \quad (6)$$

Hence, by the equations (3), (6), we have

$$\tau_R([eR])_i = p^e \tau_R([R])_i.$$ 

Therefore,

$$[eR] = p^{de} \tau_R^{-1}([\text{Spec } R]) + \sum_{0 \leq i \leq d-1} p^i \tau_R^{-1}(\tau_R([R])_i) \quad (7)$$

in $G_0(R)_Q$.

The following lemma is well-known. We omit a proof.
Lemma 16 Assume that $R$ is an $F$-finite Noetherian local domain of characteristic $p$ with residue class field algebraically closed. Then, for any $e > 0$, we have

$$\text{rank}_R^e R = p^{(\dim R)e}.$$ 

Definition 17 Let $R$ be a Cohen-Macaulay ring of characteristic $p > 0$. We say that $R$ is FFRT (of finite $F$-representation type) if there exist finitely many indecomposable maximal Cohen-Macaulay $R$-modules $M_1, \ldots, M_s$ such that there exist nonnegative integers $a_{e1}, \ldots, a_{es}$ with

$$^e R \simeq M_1^{ae1} \oplus \cdots \oplus M_s^{aes}$$

for each $e > 0$. 

Definition 18 Let $p$ be a prime number and $R$ be a Noetherian ring of characteristic $p > 0$. Let $R^\circ$ be the set of elements of $R$ that are not contained in any minimal prime ideals of $R$. Let $I$ be an ideal of $R$. Given a natural number $e$, set $q = p^e$. The ideal generated by the $q$-th powers of elements of $I$ is called the $q$-th Frobenius power of $I$, denoted by $I^{[q]}$. We define the tight closure $I^*$ of $I$ as follows:

$$I^* = \{ x \in R \mid \text{there exists } c \in R^\circ \text{ such that } cx^q \in I^{[q]} \text{ for } q \gg 0 \}.$$ 

We say that $I$ is tightly closed if $I = I^*$. 

Definition 19 Let $R$ be a Noetherian local ring of characteristic $p > 0$. We say that $R$ is $F$-rational if every parameter ideal is tightly closed. 

Now, we start to prove Theorem 2 (1). Since $R$ is FFRT, there exist finitely many indecomposable maximal Cohen-Macaulay $R$-modules $M_1, \ldots, M_s$ such that there exist nonnegative integers $a_{e1}, \ldots, a_{es}$ with

$$^e R \simeq M_1^{ae1} \oplus \cdots \oplus M_s^{aes}$$

for each $e > 0$. Let $U$ be the $\mathbb{Q}$-vector subspace of $G_0(R)_{\mathbb{Q}}$ spanned by

$$\{[M_1], \ldots, [M_s]\} \cup \{\tau_R^{-1}(\tau_R([R]))_j \mid 0 \leq j \leq d\}.$$ 

Here, recall that $\mu_R = \tau_R^{-1}(\tau_R([R]))_d \in U$ by the top term property (4). Although we can show that $U$ is spanned by $\{[M_1], \ldots, [M_s]\}$, we do not need it in this proof. Thinking a basis of $U$ as an orthonormal basis of $U_R$, we think $U_R$ as a metric space. Set $C = \sum_{i=1}^s \mathbb{R}_{\geq 0}[M_i] \subset U_R$. Then $C$ is a closed subset of $U_R$. We shall show $\mu_R \in C$. 

Since the residue field is algebraically closed, $\text{rank}_R^e R = p^{de}$ for any $e > 0$ by Lemma 16. Since

$$^e R = a_{e1}[M_1] \oplus \cdots \oplus a_{es}[M_s]$$

by (8), we have

$$\frac{1}{p^{de}} [^e R] \in C$$
for any $e > 0$. By the equation (7),
\[
\frac{1}{p^{de}}e^R = \sum_{0 \leq i \leq d} \frac{1}{p^{de}} \tau_R^{-1}(\tau_R([R])_{d-i}).
\] (9)

By the definition of $U$, every term of the right-hand side is in $U \mathbb{R}$. Hence we have
\[
\lim_{e \to \infty} \frac{1}{p^{de}}e^R = \tau_R^{-1}(\tau_R([R])_{d}) = \tau_R^{-1}([\text{Spec } R]) = \mu_R \text{ in } U \mathbb{R}.
\]

Since $C$ is a closed set of $U \mathbb{R}$, we have $\mu_R \in C$. By the same argument as in Example 9 (4), there exist a natural number $n$ and a maximal Cohen-Macaulay $R$-module $N$ such that $n\mu_R = [N]$ in $G_0(R)Q$.

Next, we shall prove that $[\omega_R] \in \text{Int}(C_{CM}(R))$ if $R$ is Cohen-Macaulay. We have a homomorphism $\xi : G_0(R)_R \to G_0(R)_R$ given by $\xi([M]) = \sum_i (-1)^i[\text{Ext}^i_R(M, \omega_R)]$. For a maximal Cohen-Macaulay module $M$, $\text{Ext}^i_R(M, \omega_R) = 0$ for $i > 0$ and $\text{Hom}_R(\text{Hom}_R(M, \omega_R), \omega_R) \simeq M$. Therefore, $\xi^2$ is equal to the identity, and $\xi$ is an isomorphism. By the definition of $\tau_R$, we have a commutative diagram
\[
\begin{array}{ccc}
G_0(R)_R & \xrightarrow{\tau_R} & A_*(R)_R \\
\xi \downarrow & & \phi \downarrow \\
G_0(R)_R & \xrightarrow{\tau_R} & A_*(R)_R
\end{array}
\]
where $\phi : A_*(R)_R \to A_*(R)_R$ is the map given by
\[
\phi(q_d + q_{d-1} + \cdots + q_i + \cdots + q_0) = q_d - q_{d-1} + \cdots + (-1)^{d-i}q_i + \cdots + (-1)^dq_0
\] (10)
for $q_i \in A_i(R)_R$. Since the numerical equivalence is graded in $A_*(R)_R$ as in Proposition 2.4 in [8], $\phi$ preserves the numerical equivalence. Therefore we have the induced map
\[
\bar{\xi} : \bar{G}_0(R)_R \to \bar{G}_0(R)_R.
\]
Remark that $\bar{\xi}$ is an isomorphism of $\mathbb{R}$-vector spaces since $\bar{\xi}^2$ is the identity. The map $\bar{\xi}$ satisfies $\bar{\xi}([R]) = [\omega_R]$ and $\bar{\xi}(C_{CM}(R)) = C_{CM}(R)$. Since $[R] \in \text{Int}(C_{CM}(R))$ by Remark 5, we obtain $[\omega_R] \in \text{Int}(C_{CM}(R))$.

Assume that $M$ is a maximal Cohen-Macaulay module. For $e > 0$, consider the following exact sequence
\[
0 \to L_e \to F^e_*(M) \to M^{\oplus b_e} \to 0
\]
where $F^e_*(M)$ is the $e$-th Frobenius direct image of $M$. Take $b_e$ as large as possible. Recall that $L_e$ is a maximal Cohen-Macaulay module. Put $r = \text{rank}_R M$.

---

4Put $R = T/I$, where $T$ is a regular local ring. Then, $\xi([M]) = (-1)^{ht(I)}\sum_i (-1)^i[\text{Ext}^i_T(M, T)]$. Let $F_*$ be a $T$-free resolution of $M$. Then, by the definition of $\tau_R$, we have $\tau_R([M]) = \text{ch}(F_*) \cap [\text{Spec } T]$, where $\text{ch}(F_*)$ is the localized Chern character of $F_*$. (§18 in [4]). By the local Riemann-Roch formula (Example 18.3.12 in [4]), $\tau_R([M]) = \text{ch}(F_*^{\bullet}[ht(I)]) \cap [\text{Spec } T]$. By Example 18.1.2, we obtain the equality (10).
Here we define the dual F-signature following Sannai [14] as follows:

\[ s(M) = \limsup_{e \to \infty} \frac{b_e}{r p^{de}} \]

Then, taking a subsequence of \( \{ \frac{b_e}{r p^{de}} \} \), we may assume that \( s(M) = \lim_{e \to \infty} \frac{b_e}{r p^{de}} \).

On the other hand, consider

\[ \tau_R([M]) = \tau_R([M])_d + \tau_R([M])_{d-1} + \cdots + \tau_R([M])_0. \]

Here, we have \( \tau_R([M])_d = r[\text{Spec } R] \) since \( [M] - r[R] \) is a sum of cycles of torsion modules. By (2) and (5),

\[ \tau_R([F^*_e(M)]) = F^*_e(\tau_R([M]))_d + \tau_R([M])_{d-1} + \cdots + \tau_R([M])_0. \]

Then, we have

\[ \tau_R(\lim_{e \to \infty} \frac{F^*_e([M])}{r p^{de}}) = \frac{\tau_R([M])_d}{r} = [\text{Spec } R] \text{ in } \overline{A_s(R)}_R. \]

Thus,

\[ \lim_{e \to \infty} \frac{[F^*_e(M)]}{r p^{de}} = \overline{\mu}_R \text{ in } \overline{G_0(R)}_R. \]

Then, \( \frac{[L_e]}{r p^{de}} \) converges to some element in \( \overline{G_0(R)}_R \), say \( \alpha(M) \).

\[ \frac{[F^*_e([M])]}{r p^{de}} \downarrow \frac{b_e[M]}{r p^{de}} \downarrow \frac{[L_e]}{r p^{de}} \downarrow \overline{G_0(R)}_R \]

\[ \overline{\mu}_R = s(M)[M] + \alpha(M) \]

Since \( L_e \) is a maximal Cohen-Macaulay module, we know \( \alpha(M) \in C_{CM}(R)^- \).

Here set \( M = \omega_R \). Then

\[ \overline{\mu}_R = s(\omega_R)[\omega_R] + \alpha(\omega_R) \in \overline{G_0(R)}_R, \quad \text{(11)} \]

where

\[ \alpha(\omega_R) \in C_{CM}(R)^- \quad \text{(12)} \]

and

\[ [\omega_R] \in \text{Int}(C_{CM}(R)) = \text{Int}(C_{CM}(R)^-). \quad \text{(13)} \]

The most important point in this proof is the fact that

\( R \) is F-rational if and only if \( s(\omega_R) > 0 \)

due to Sannai [14].

Therefore, if \( R \) is F-rational, then \( \overline{\mu}_R \in \text{Int}(C_{CM}(R)) \) by (11), (12), (13) and Remark 5.

q.e.d.

Remark 20 If \( R \) is a toric ring (a normal semi-group ring over a field \( k \)), then we can prove \( \overline{\mu}_R \in C_{CM}(R) \) as in the case FFRT without assuming that \( ch(k) \) is positive.
Problem 21  (1) As in the above proof, if there exists a maximal Cohen-Macaulay module in $\text{Int}(C_{CM}(R))$ such that its generalized F-signature or its dual F-signature is positive, then $\overline{\mu_R}$ is in $\text{Int}(C_{CM}(R))$.

Without assuming that $R$ is F-rational, do there exist such a maximal Cohen-Macaulay module?

(2) How do we make mod $p$ reduction? (for example, the case of rational singularity)

(3) If $R$ is Cohen-Macaulay, is $\overline{\mu_R}$ in $C_{CM}(R)^-$? If $R$ is a Cohen-Macaulay ring containing a field of positive characteristic, then $\overline{\mu_R}$ in $C_{CM}(R)^-$ as in (1) in Remark 11.

(4) If $R$ is of finite representation type, is $\overline{\mu_R}$ in $C_{CM}(R)$?

(5) Find more examples of $C_{CM}(R)$ and $SN(R)$.

In order to prove the following corollary, it is enough to construct a $d$-dimensional Cohen-Macaulay local domain $A$ satisfying the following two conditions (Lemma 3.1 in [1]):

(1) $\overline{A_i(A)} \neq 0$ for $d/2 < i \leq d$, and

(2) $\overline{\mu_A}$ is contained in $\text{Int}(C_{CM}(A))$.

The ring $R$ in Corollary 22 is the idealization of $A$ and certain maximal Cohen-Macaulay $A$-module $M$. We can simplify the proof of Corollary 22 using Theorem 2. We know that $k[x_{ij}]_{(x_{ij})}/I_2(x_{ij})$ satisfies the conditions (1) and (2) above, where $(x_{ij})$ is the generic $n \times n$ or $n \times (n+1)$ matrix, and $I_2(x_{ij})$ stands for the ideal generated by 2-minors of $(x_{ij})$. In fact, by Example 3 (2) (b) and Example 9 (3), the condition (1) is satisfied. Since $k[x_{ij}]_{(x_{ij})}/I_2(x_{ij})$ is F-rational, the condition (2) is satisfied by Theorem 2 (2).

**Corollary 22 ([1])** Let $d$ be a positive integer and $p$ a prime number. Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_d$ be integers such that

$$\epsilon_i = \begin{cases} 1 & i = d, \\ -1, 0 \text{ or } 1 & d/2 < i < d, \\ 0 & i \leq d/2. \end{cases}$$

Then, there exists a $d$-dimensional Cohen-Macaulay local ring $R$ of characteristic $p$, a maximal primary ideal $I$ of $R$ of finite projective dimension, and positive rational numbers $\alpha, \beta_{d-1}, \beta_{d-2}, \ldots, \beta_0$ such that

$$\ell_R(R/I[p^n]) = \epsilon_d \alpha p^{dn} + \sum_{i=0}^{d-1} \epsilon_i \beta_i p^{in}$$

for any $n > 0$. 

12
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