HOCHSTER'S THETA PAIRING AND NUMERICAL EQUIVALENCE

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Abstract. Let $(A, \mathfrak{m})$ be a local hypersurface with an isolated singularity. We show that Hochster's theta pairing θ^A vanishes on elements that are numerically equivalent to zero in the Grothendieck group of A under the mild assumption that $\operatorname{Spec} A$ admits a resolution of singularities. This extends a result by Celikbas-Walker. We also prove that when $\dim A = 3$, Hochster's theta pairing is positive semi-definite. These results combine to show that the counter-example of Dutta-Hochster-McLaughlin to the general vanishing of Serre's intersection multiplicity exists for any three dimensional isolated hypersurface singularity that is not a UFD and has a desingularization. We also show that, if A is three dimensional isolated hypersurface singularity that has a desingularization, the divisor class group is finitely generated torsion-free. Our method involves showing that θ^A gives a bivariant class for the morphism $\operatorname{Spec}(A/\mathfrak{m}) \to \operatorname{Spec} A$.

1. Introduction

Let A be a local hypersurface with an isolated singularity (so $A_{\mathfrak{p}}$ is regular for each non-maximal prime ideal $\mathfrak{p}$). For any pair of finitely generated A-modules M and N, one has $\ell(\operatorname{Tor}_i^A(M, N)) < \infty$ for $i > \dim A$, where $\ell(-)$ denotes length. The function $\theta^A(M, N)$ was introduced by Hochster ([16]) to be:

$$\theta^A(M, N) = \ell(\operatorname{Tor}_{2e+2}^A(M, N)) - \ell(\operatorname{Tor}_{2e+1}^A(M, N))$$

where e is any integer such that $2e \geq \dim A$. The function $\theta^A(M, N)$ is additive on short exact sequences and thus defines a pairing on the Grothendieck group of finitely generated modules $G_0(A)$ or the reduced group $\underline{G_0}(A) := G_0(A)/\langle [A] \rangle$.

The theta pairing has attracted quite a bit of attention lately due to its recently discovered connections to a number of diverse areas and problems ([2, 3, 5, 6, 29,

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2010 *Mathematics Subject Classification.* 13D07, 13D15, 13D22, 14C17, 14C35.

Key words and phrases. local hypersurface, isolated singularity, Hochster's theta invariant, intersection multiplicity, divisor class group, Grothendieck group, numerical equivalences.

The first author is partially supported by NSF grants DMS 0834050 and DMS 1104017. The second author is partially supported by JSPS KAKENHI Grant 21540050.

31]). For a brief history of these recent developments and how they connect to our work, we refer to Section 3.

The first main result of the present paper is the following:

Theorem 1.1. *Assume that* $\operatorname{Spec} A$ *admits a resolution of singularities. Then* θ^A *vanishes on all pairs* (M, N) *as long as one of the modules in the pair represents an element in the Grothendieck group* $G_0(A)$ *that is numerically equivalent to zero.*

The concept of numerical equivalence over local rings was introduced by the second author in parallel with intersection theory on projective varieties. An element $[M]$ of the Grothendieck group $G_0(A)$ is numerically equivalent to zero if for any module N of finite length and finite projective dimension,

$$\chi^A(M, N) := \sum_{i \geq 0} (-1)^i \ell(\operatorname{Tor}_i^A(M, N)) = 0$$

(in general this should be defined using the category of perfect complexes with finite length homologies but it makes no difference in our case, see Section 2).

Since numerical equivalence follows from algebraic equivalence even in our setting (Remark 6.4), our main result recovers [3, Theorem 2.7] and [29, Theorem 3.2]. Several corollaries follow. It gives a new proof on the vanishing of θ^A when $\dim A$ is even (Conjecture 3.1, part (1)) in the graded case (proved in [29]).

The proof of Theorem 1.1 contains some ingredients which we believe are of independent interest. In Theorem 4.4, we give a criterion for a map from the Grothendieck group $G_0(A)$ to $\mathbb{Z}$ to arise from a perfect complex with finite length homologies. In addition, a key technical result, Theorem 5.2, shows that θ^A gives a bivariant class on the Grothendieck groups (as well as the Chow groups). Recall that a bivariant class for a morphism of schemes $f : X \to Y$ gives a homomorphism for the Grothendieck groups $G_0(Y') \to G_0(X')$ for any fibre square that commutes with pushforwards, pullbacks and intersection products (for a precise definition see Section 4).

The second main result is that in dimension three, θ^A is positive semi-definite, which confirms in this special case Conjecture 3.6 of [29].

Theorem 1.2. *Let* A *be a local hypersurface with an isolated singularity of dimension three. Then* $\theta^A(M, M) \geq 0$ *for any module* M*. Furthermore, if* M *is reflexive of rank one, then equality holds if and only if* M *is free.*

By combining Theorems 1.1 and 1.2 one obtains (see Corollary 8.1) a vast generalization of the famous original example by Dutta-Hochster-McLaughlin of the non-vanishing of Serre's intersection multiplicity, which was constructed for $A = k[[x, y, u, v]]/(xy - uv)$:

Corollary 1.3. *Let A be a local hypersurface of dimension three with an isolated singularity. Assume that* $\operatorname{Spec} A$ *admits a resolution of singularities. Then for any torsion A-module M, the following are equivalent:*

(1) There exists a module N of finite length and finite projective dimension such that $\chi^A(M, N) \neq 0$.
(2) The divisor class of M is non-trivial (for example, if $M = A/I$ for a non-free reflexive ideal I).

As an added bonus, it also follows that in the situation above, the class group of A is always finitely generated torsion-free (Corollary 8.2).

The paper is organized as follows. In Section 2 we recall basic definitions and notations. Section 3 briefly recalls the recent history of Hochster's theta pairing as well as a group of motivating conjectures (see Conjectures 3.1, 3.2 and 3.4). Section 4 gives the definition of a bivariant class for Grothendieck groups and a criterion for a map from $G_0(A)$ to $\mathbb{Z}$ to arise from the Euler characteristic of perfect complexes with finite length homologies. Theorem 5.2 proves that θ^A gives a bivariant class on Grothendieck groups. The proof of Theorem 1.1 is contained in Section 6. Section 7 contains the proofs of Theorem 1.2 as well as partial results in small dimensions. Finally, Section 8 discusses some applications such as Corollary 1.3 and Corollary 8.2.

2. Notations and preliminaries

Unless otherwise noted, all schemes in this paper are of finite type over some regular base scheme S and all morphisms are of finite type. When dealing with Chow groups of schemes we always consider the *relative dimension* as in Chapter 20 in Fulton [11].

For a scheme X, $G_0(X)$ denotes the Grothendieck group of coherent sheaves on X. For each $i \geq 0$, $A_i(X)$ denotes the Chow group of i-dimensional cycles in X modulo rational equivalence. Let $Cl(X)$ denote the divisor class group of X. When $X = \operatorname{Spec} A$ is affine we shall write $G_0(A)$, $A_i(A)$ and $Cl(A)$. Let $\underline{G_0}(A) := G_0(A)/\mathbb{Z}[A]$ be the reduced Grothendieck group. For an additive group G, $G_{\mathbb{Q}}$ denotes the tensor product $G \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let A be a local Noetherian ring. We recall the Riemann-Roch theory and the concept of numerical equivalence for elements in Chow or Grothendieck groups. If A is a homomorphic image of a regular local ring, then we have an isomorphism of $\mathbb{Q}$-vector spaces (see [11]):

$$\tau : G_0(A)_{\mathbb{Q}} \to A_*(A)_{\mathbb{Q}}.$$

We denote the i-th component of τ by τ_i, that is, the maps

$$\tau_i : G_0(A)_{\mathbb{Q}} \longrightarrow A_i(A)_{\mathbb{Q}}$$

for $i = 0, 1, \dots, d$ satisfying $\tau = \tau_d + \tau_{d-1} + \cdots + \tau_0$. These are defined by localized Chern characters. There is a map (see [4])

$$c_1 : G_0(A) \to A_{d-1}(A)$$

which satisfies

- $c_1([A]) = 0$,
- $c_1([A/P]) = -c_1([P]) = [\mathrm{Spec}(A/P)]$ for each prime ideal P with $\dim A/P = d-1$,
- $c_1([M]) = 0$ if $\dim M \le d-2$.

When A is normal, $c_1([A/I])$ is equal to $[\mathrm{Spec}(A/I)]$ in $A_{d-1}(A)$ for each reflexive ideal I of A. Furthermore, we obtain an isomorphism

$$A_{d-1}(A) \to Cl(A)$$

sending $-[\mathrm{Spec}(A/I)]$ to the isomorphism class $[I]$ of a reflexive ideal I of A. For an A-module M, we have

$$\tau_{d-1}([M]) = c_1([M]) - \frac{\mathrm{rank}_A M}{2} K_A \tag{2.1}$$

in $A_{d-1}(A)_{\mathbb{Q}}$, where K_A is the canonical divisor of A, that is, $K_A = c_1([\omega_A])$.

For a bounded finite A-free complex $\mathbb{F}.$ with finite length homologies, we define

$$\chi_{\mathbb{F}.} : G_0(A) \longrightarrow \mathbb{Z}$$

to be

$$\chi_{\mathbb{F}.}([M]) = \sum_i (-1)^i \ell(H_i(\mathbb{F}. \otimes_A M)).$$

We say that a cycle α in $G_0(A)$ is *numerically equivalent to* 0 if $\chi_{\mathbb{F}.}(\alpha) = 0$ for any bounded finite A-free complex $\mathbb{F}.$ with finite length homologies. In the same way, we say that a cycle β in $A_*(A)$ is *numerically equivalent to* 0 if $\mathrm{ch}(\mathbb{F}.)(\beta) = 0$ for any above $\mathbb{F}.$, where $\mathrm{ch}(\mathbb{F}.)$ is the localized Chern character which appears in Chapter 18 in [11].

Definition 2.1. One denotes by $\overline{G_0(A)}$ and $\overline{A_*(A)}$ the groups modulo numerical equivalence.

By Theorem 3.1 and Remark 3.5 in [23], both of $\overline{G_0(A)}$ and $\overline{A_*(A)}$ are finitely generated torsion-free abelian group under a mild condition. It is proved in Proposition 2.4 in [23] that numerical equivalence is consistent with the dimension of cycles in $A_*(A)$, so we have

$$\overline{A_*(A)} = \oplus_{i=0}^{d} \overline{A_i(A)}.$$

The Riemann-Roch map τ preserves numerical equivalence as in [23], that is, it induces a map $\overline{\tau}$ that makes the following diagram commutative:

$$\begin{array}{ccc} G_0(A)_{\mathbb{Q}} & \xrightarrow{\tau} & A_*(A)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \overline{G_0(A)}_{\mathbb{Q}} & \xrightarrow{\overline{\tau}} & \overline{A_*(A)}_{\mathbb{Q}} \end{array} \tag{2.2}$$

If A is Cohen-Macaulay, the Grothendieck group of bounded A-free complexes with support in $\{\mathfrak{m}\}$ is generated by finite free resolutions of modules of finite length and finite projective dimension (see Proposition 2 in [34]). Therefore, in this case, α in $G_0(A)$ is numerically equivalent to 0 if and only if $\chi_{\mathbb{F}.}(\alpha) = 0$ for any free resolution $\mathbb{F}.$ of a module of finite length and finite projective dimension.

3. A brief history of theta pairing and some motivating open questions

In this section we briefly recall the (recent) history of the theta functions. Let A be a local hypersurface with isolated singularity (so $A_{\mathfrak{p}}$ is regular for each non-maximal prime ideal $\mathfrak{p}$). Then, for any pair of finitely generated A-modules M and N, one has $\ell(\mathrm{Tor}_i^A(M,N)) < \infty$ for $i > \dim A$, where $\ell(-)$ denotes length.

The function $\theta^A(M,N)$ was introduced by Hochster ([16]) to be:

$$\theta^A(M,N) = \ell(\mathrm{Tor}_{2e+2}^A(M,N)) - \ell(\mathrm{Tor}_{2e+1}^A(M,N))$$

where e is any integer such that $2e \geq \dim A$. It is well known (see [10]) that the sequence of modules $\{\mathrm{Tor}_i^A(M,N)\}_i$ is periodic of period 2 for $i > \operatorname{depth} A - \operatorname{depth} M$, so this function is well-defined. Note that if M or N is maximal Cohen-Macaulay, one simply gets:

$$\theta^A(M,N) = \ell(\mathrm{Tor}_2^A(M,N)) - \ell(\mathrm{Tor}_1^A(M,N)).$$

A key point here is that the function $\theta^A(-,-)$ is additive on short exact sequences and thus defines a pairing on the Grothendieck group of finitely generated modules $G_0(A)$ or the reduced group $\underline{G_0}(A)$. This function was originally introduced in [16] as a possible means to attack the Direct Summand Conjecture (where it was necessary to study θ^A for some non-isolated singularity A). However it has gained independent interest recently. In particular, the behavior of θ^A over isolated singularities has been linked to some interesting topics in algebraic geometry and K-theory.

To be more specific, the main motivation of this note is the following group of open questions:

Conjecture 3.1. *Let A be a local hypersurface with an isolated singularity and $d = \dim A$. Then for any finitely generated modules M, N:*

(1) If d is even, then $\theta^A(M,N) = 0$.
(2) If $\dim M + \dim N \le d$, then $\theta^A(M,N) = 0$.
(3) If $\dim M \le d/2$, then $\theta^A(M,N) = 0$.
(4) If M, N are maximal Cohen-Macaulay, then $\theta^A(M,N) + (-1)^{\frac{d+1}{2}}\theta^A(M^*,N) = 0$. Note that when d is even this implies (1)*.*
(5) ([29, Conjecture 3.6]*)If d is odd, then $(-1)^{\frac{d+1}{2}}\theta^A(M,M) \ge 0$. In other words, $(-1)^{\frac{d+1}{2}}\theta^A(-,-)$ defines a positive semi-definite form on $\underline{G_0}(A)_{\mathbb{Q}}$.*

Most of the statements above have appeared or been hinted at in the literature in one form or another. Conjecture 3.1 (1) was made and established in several cases by the first author in [5, 8]. Since then, it has captured the attention of many researchers and has now been established in characteristic 0 via three different approaches: intersection theory on smooth hypersurfaces (for the graded case), topological K-theory and Hochschild cohomology ([29], [2], [31]). These results suggest much deeper facts about θ^A, namely that it should be thought of as a Riemann-Roch form on the category of maximal Cohen-Macaulay modules (or matrix factorizations) over A. Thus they motivate the rest of Conjecture 3.1, which can be viewed as an analogue of the Lefschetz hyperplane theorem and properties of Hodge-Riemann bilinear relations. We note that (2) is proved for the excellent and equicharateristic case in [8]. The statement (4) can be viewed as a strengthening of a result by Buchweitz which implies that $\theta^A(M,N) + (-1)^d\theta^A(M^*,N^*) = 0$ in any dimension, see [5, Proposition 4.3].

In view of the above conjecture and the main results of this work, it is reasonable to make the following conjecture, which would explain some parts of Conjecture 3.1. Precisely speaking, by Theorem 1.1, we know that Conjecture 3.1 (1) follows from Conjecture 3.2 (1) (a). Also Conjecture 3.1 (3) follows from Conjecture 3.2 (1) (b) or (2).

Conjecture 3.2. *Let $(A, \mathfrak{m})$ be a d-dimensional local domain with an isolated singularity.*

(1) Assume that A is a complete intersection.

(a) If d is even, then $\overline{G_0(A)_{\mathbb{Q}}} = \mathbb{Q}[A]$ *and, equivalently,* $\overline{A_i(A)_{\mathbb{Q}}} = 0$ *for* $i < d$.
(b) If d is odd, then $\overline{A_i(A)_{\mathbb{Q}}} = 0$ *for* $i \neq \frac{d+1}{2}, d$.

(2) If $i \leq d/2$, *then* $\overline{A_i(A)_{\mathbb{Q}}} = 0$.

This conjecture is true in some special cases, as seen below.

Proposition 3.3. *Assume that R is a homogeneous ring over a field k such that* $\operatorname{Proj} R$ *is smooth over k. Put* $A = R_{R_+}$, *where R_+ is the unique homogeneous maximal ideal.*

Then, Conjecture 3.2 (1) holds in this case. Furthermore, if Grothendieck's standard conjecture (see [22]*) is true, then Conjecture 3.2 (2) holds.*

Proof. Put $X = \operatorname{Proj} R$. Suppose $n = \dim X$ and $d = \dim A$. Note that $d = n + 1$.

We may assume that k is algebraically closed (see Lemma 4.2 in [23]).

Suppose that the characteristic of k is 0. We may assume that k is the complex number field $\mathbb{C}$ by the Lefschetz principle.

Assume that X is a complete intersection smooth projective variety over $\mathbb{C}$. Then, it is well known (e.g. see [24, Example 11.20]) that

$$H^j(X(\mathbb{C}), \mathbb{Q}) = \begin{cases} \mathbb{Q} & (0 \leq j \leq 2n,\ j \neq n \text{ and } j \text{ is even}) \\ ? & (j = n) \\ 0 & (\text{otherwise}). \end{cases}$$

Then, we have

$$\begin{array}{ccccc} \mathrm{CH}^j(X)_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^j_{hom}(X)_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^j_{num}(X)_{\mathbb{Q}} \\ & \searrow & \downarrow & & \\ & & H^{2j}(X(\mathbb{C}), \mathbb{Q}) & & \end{array}$$

where $\mathrm{CH}^j_{hom}(X)_{\mathbb{Q}}$ (resp. $\mathrm{CH}^j_{num}(X)_{\mathbb{Q}}$) is the Chow group divided by homological equivalence (resp. numerical equivalence). Here, the map $\mathrm{CH}^j_{hom}(X)_{\mathbb{Q}} \to H^{2j}(X(\mathbb{C}), \mathbb{Q})$ is injective, and $\mathrm{CH}^j_{hom}(X)_{\mathbb{Q}} \to \mathrm{CH}^j_{num}(X)_{\mathbb{Q}}$ is surjective. Remember that $\mathrm{CH}^j_{num}(X)_{\mathbb{Q}} \neq 0$ for $j = 0, 1, \ldots, n$. Therefore, if n is odd, $\mathrm{CH}^j_{num}(X)_{\mathbb{Q}} = \mathbb{Q}$ for $j = 0, 1, \ldots, n$. If n is even, $\mathrm{CH}^j_{num}(X)_{\mathbb{Q}} = \mathbb{Q}$ for $j = 0, 1, \ldots, n$ except for $j = n/2$. On the other hand, we have the natural surjections

$$\mathrm{CH}^j_{num}(X)_{\mathbb{Q}}/h\mathrm{CH}^{j-1}_{num}(X)_{\mathbb{Q}} \longrightarrow \overline{A_{d-j}(A)_{\mathbb{Q}}}$$

for $j = 0, 1, \ldots, n$ by (7.5) in [23], where h is the very ample divisor corresponding to the embedding $\operatorname{Proj} R$. Note that $\overline{A_0(A)_{\mathbb{Q}}} = 0$ if $\dim A > 0$.

- If n is odd, then $\overline{A_i(A)_{\mathbb{Q}}} = 0$ for $i < d$. Conjecture 3.2 (1) (a) is proved.

- If n is even, then $\overline{A_i(A)_{\mathbb{Q}}} = 0$ for $i \neq (d+1)/2, d$. Conjecture 3.2 (1) (b) is proved.

Conjecture 3.2 (2) is true by Remark 7.12 in [23].

In the case where the characteristic of k is positive, we use the étale cohomology instead of the Betti cohomology. The proof is the same as the case of characteristic zero, so we omit it. ☐

The localized Chern characters induce a map $\theta^A_{ch} : A_*(A)_{\mathbb{Q}} \times A_*(A)_{\mathbb{Q}} \to \mathbb{Q}$. Thus, we can also state the following conjecture for Chow groups.

Conjecture 3.4. *For $\alpha \in A_*(A)$ a homogenous element, we have $\theta^A_{ch}(\alpha, \beta) = 0$ if $\alpha \notin A_{\frac{d+1}{2}}(A)$(in particular, it is always 0 when d is even). Also, $(-1)^{\frac{d+1}{2}}\theta^A_{ch}(-,-)$ defines a positive semi-definite form on $A_{\frac{d+1}{2}}(A)$.*

4. A bivariant class on K-groups

We assume that all schemes in this paper are of finite type over some regular base scheme S and all morphisms are of finite type. When dealing with Chow groups of schemes we always consider the *relative dimension* as in Chapter 20 in Fulton [11].

Definition 4.1. Let Y be a scheme and X a closed subscheme of Y. Consider the following fibre square of schemes:

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & \square & \downarrow g \\ X & \longrightarrow & Y \end{array} \tag{4.1}$$

Suppose that

$$\varphi_g : G_0(Y') \longrightarrow G_0(X')$$

is a homomorphism between K-groups, where $G_0(\ )$ denotes the Grothendieck group of coherent sheaves.

We say that a collection of homomorphisms

$$\{\varphi_g : G_0(Y') \longrightarrow G_0(X') \mid g : Y' \to Y \text{ is a morphism of schemes}\}$$

is a *bivariant class* on K-groups for $X \hookrightarrow Y$ if the following three conditions are satisfied.

(B$_1$) If $h : Y'' \to Y'$ is proper, $g : Y' \to Y$ arbitrary, and one forms the fibre diagram

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ h' \downarrow & \square & \downarrow h \\ X' & \longrightarrow & Y' \\ \downarrow & \square & \downarrow g \\ X & \longrightarrow & Y \end{array} , \tag{4.2}$$

then the diagram

$$\begin{array}{ccc} G_0(Y'') & \stackrel{\varphi_{gh}}{\longrightarrow} & G_0(X'') \\ h_* \downarrow & & \downarrow h'_* \\ G_0(Y') & \stackrel{\varphi_g}{\longrightarrow} & G_0(X') \end{array}$$

is commutative, where h_* is the push-forward map induced by the proper morphism h (e.g. [11]).

(B$_2$) If $h : Y'' \to Y'$ is flat, $g : Y' \to Y$ arbitrary, and one forms the fibre diagram (4.2), then the diagram

$$\begin{array}{ccc} G_0(Y') & \stackrel{\varphi_g}{\longrightarrow} & G_0(X') \\ h^* \downarrow & & \downarrow h'^* \\ G_0(Y'') & \stackrel{\varphi_{gh}}{\longrightarrow} & G_0(X'') \end{array}$$

is commutative, where h^* is the pull-back map induced by the flat morphism h (e.g. [11]).

(B$_3$) Let Z' be a scheme and Z'' be a closed subscheme of Z'. Let $\mathbb{G}.$ be a bounded locally free complex on Z' which is exact on $Z' \backslash Z''$. If $g : Y' \to Y$, $h : Y' \to Z'$ are morphisms, and one forms the fibre diagram

$$\begin{array}{ccccc} X'' & \longrightarrow & Y'' & \longrightarrow & Z'' \\ i'' \downarrow & \square & \downarrow i' & \square & \downarrow i \\ X' & \stackrel{f}{\longrightarrow} & Y' & \stackrel{h}{\longrightarrow} & Z' \\ \downarrow & \square & \downarrow g & & \\ X & \longrightarrow & Y & & \end{array} , \tag{4.3}$$

then the diagram

$$\begin{array}{ccc} G_0(Y') & \stackrel{\varphi_g}{\longrightarrow} & G_0(X') \\ \chi_{h^*(\mathbb{G}.)} \downarrow & & \downarrow \chi_{(hf)^*(\mathbb{G}.)} \\ G_0(Y'') & \stackrel{\varphi_{gi'}}{\longrightarrow} & G_0(X'') \end{array}$$

is commutative, where $\chi_{h^*(\mathbb{G}.)}$ is the map taking the alternating sum of the homologies of the complex $h^*(\mathbb{G}.) \otimes_{\mathcal{O}_{Y'}} \mathcal{F}$ for a coherent $\mathcal{O}_{Y'}$-module $\mathcal{F}$.

Example 4.2. Let X be a closed subscheme of a scheme Y. Let $\mathbb{F}.$ be a bounded locally free $\mathcal{O}_X$-complex which is exact on $Y \setminus X$. For a fibre square as in (4.1), we define

$$\chi(\mathbb{F}.)_g : G_0(Y') \longrightarrow G_0(X')$$

by

$$\chi(\mathbb{F}.)_g([\mathcal{F}]) = \sum_i (-1)^i [H_i(g^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y'}} \mathcal{F})].$$

Then,

$$\{\chi(\mathbb{F}.)_g : G_0(Y') \longrightarrow G_0(X') \mid g : Y' \to Y \text{ is a morphism of schemes}\}$$

is a bivariant class on K-groups for $X \hookrightarrow Y$.

It is easy to check that (B_2) and (B_3) are satisfied. The condition (B_1) is proved using a spectral sequence. For a coherent $\mathcal{O}_{Y''}$-module $\mathcal{F}$, we show that both of $h'_* \chi(\mathbb{F}.)_{gh}([\mathcal{F}])$ and $\chi(\mathbb{F}.)_g h_*([\mathcal{F}])$ coincide with

$$\sum_i (-1)^i [Rh^i_*((gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F})] \in G_0(X').$$

For a local complete inersection morphism $f : X \to Y$ and the fibre square (4.1), the map $f^! : A_*(Y') \to A_*(X')$ is defined as in 6.6 in [11]. A bivariant class on Chow groups is a generalization of

$$\{f^! : A_*(Y') \to A_*(X') \mid g : Y' \to Y \text{ is a morphism of schemes}\}.$$

In the same way, a bivariant class on Grothendieck groups is a generalization of

$$\{\chi(\mathbb{F}.)_g : G_0(Y') \longrightarrow G_0(X') \mid g : Y' \to Y \text{ is a morphism of schemes}\}.$$

Under a mild condition, we can naturally define a bivariant class on Chow groups in the sense of Fulton ([11], Section 17) corresponding to a given bivariant class on K-groups in Definition 4.1 (see Remark 4.3).

Remark 4.3. Let Y be a scheme and X a closed subscheme of Y. Assume that there exists a proper surjective morphism $Z \to Y$ such that Z is a regular scheme.

If we consider Grothendieck groups and Chow groups with rational coefficients, there exists the natural one-to-one corresponding between bivariant classes on K-groups for $X \hookrightarrow Y$ and bivariant classes on Chow groups for $X \hookrightarrow Y$ as follows.

(1) Let

$$\{\varphi_g : G_0(Y')_{\mathbb{Q}} \longrightarrow G_0(X')_{\mathbb{Q}} \mid g : Y' \to Y \text{ is a morphism of schemes}\}$$

be a bivariant class on K-groups for $X \hookrightarrow Y$ as in Definition 4.1.

We denote the composite map of

$$A_*(Y')_{\mathbb{Q}} \xrightarrow{\tau_{Y'}^{-1}} G_0(Y')_{\mathbb{Q}} \xrightarrow{\varphi_g} G_0(X')_{\mathbb{Q}} \xrightarrow{\tau_{X'}} A_*(X')_{\mathbb{Q}}$$

by $c_g : A_*(Y')_{\mathbb{Q}} \to A_*(X')_{\mathbb{Q}}$, where $\tau_{Y'}$ and $\tau_{X'}$ are the isomorphisms given by Riemann-Roch theorem (Chapter 18 and 20 in Fulton [11]) in the category of S-schemes.

Then, the collection of homomorphisms

$$\{c_g : A_*(Y')_{\mathbb{Q}} \to A_*(X')_{\mathbb{Q}} \mid g : Y' \to Y \text{ is a morphism of schemes}\}$$

satisfies the conditions (C_1), (C_2) and (C_3) in Definition 17.1 in Fulton [11], i.e., it is a bivariant class on Chow groups for $X \hookrightarrow Y$.

The condition (C_1) is easily checked by definition. The condition (C_3) is proved using Theorem 17.1 in Fulton [11]. It is a troublesome task to prove (C_2), since the Riemann-Roch map τ can not commute with flat pull-back maps. In order to prove (C_2), we need the assumption that there exists a proper surjective morphism $\pi : Z \to Y$ such that Z is a regular scheme. We just give a sketch of a proof here. (We do not use this result in this paper.) Suppose that $\mathbb{F}.$ is a bounded locally free $\mathcal{O}_Z$-complex such that $n\varphi_\pi([\mathcal{O}_Z]) = \chi(\mathbb{F}.)_{i_Z}([\mathcal{O}_Z])$, where $i_Z : Z \to Z$ is the identity map and n is a positive integer. First, one show that, for any Z-scheme $h : Z' \to Z$, $n\varphi_{\pi h}$ coincides with $\chi(\mathbb{F}.)_h$ using Lemma 4.5. Next, using the fact that localized Chern characters are bivariant classes on Chow groups (in particular, compatible with flat pull-back maps), we can prove (C_2).

(2) Conversely, let

$$\{c_g : A_*(Y')_{\mathbb{Q}} \to A_*(X')_{\mathbb{Q}} \mid g : Y' \to Y \text{ is a morphism of schemes}\}$$

be a bivariant class on Chow groups for $X \hookrightarrow Y$.

We denote the composite map of

$$G_0(Y')_{\mathbb{Q}} \xrightarrow{\tau_{Y'}} A_*(Y')_{\mathbb{Q}} \xrightarrow{c_g} A_*(X')_{\mathbb{Q}} \xrightarrow{\tau_{X'}^{-1}} G_0(X')_{\mathbb{Q}}$$

by $\varphi_g : G_0(Y')_{\mathbb{Q}} \to G_0(X')_{\mathbb{Q}}$.

Then, the collection of homomorphisms

$$\{\varphi_g : G_0(Y')_{\mathbb{Q}} \longrightarrow G_0(X')_{\mathbb{Q}} \mid g : Y' \to Y \text{ is a morphism of schemes}\}$$

is a bivariant class on K-groups for $X \hookrightarrow Y$.

The condition (B_1) is easily checked by definition. The condition (B_3) is proved using the fact that a localized Chern character is commutative with any bivariant class on Chow groups as in Roberts [33]. It is a delicate task to prove (B_2), since the Riemann-Roch map τ can not commute with flat pull-back maps. In order to prove (B_2), we need the assumption that there exists a proper surjective morphism $Z \to Y$ such that Z is a regular scheme. Then one can prove (B_2) in the same way as (C_2) in (1).

In the rest of this section, we give a sufficient condition for a bivariant class of K-groups to coincide with $\{\chi(\mathbb{F}.)_g\}$ for some bounded finite A-free complex $\mathbb{F}.$.

Theorem 4.4. *Let A be a Noetherian local domain that is a homomorphic image of a regular local ring. Assume that $\dim A > 0$. Let I be a non-zero ideal of A. Put $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec}(A/I)$.*

We assume that there exists a resolution of singularity of Y, i.e., a proper birational morphism $\pi : Z \to Y$ such that Z is regular. Put $W = \pi^{-1}(\operatorname{Spec}(A/I))$ and $U = Z \setminus W$. Assume that U is isomorphic to $Y \setminus X$. Let $i : W \to Z$ be the closed immersion. Let $i_* : G_0(W) \to G_0(Z)$ be the induced map by i.

Let

$$\{\varphi_g : G_0(Y') \longrightarrow G_0(X') \mid g : Y' \to Y \text{ is a morphism of schemes}\}$$

be a bivariant class on K-groups for $X \hookrightarrow Y$.

If $i_*\varphi_\pi([\mathcal{O}_Z]) = 0$ in $G_0(Z)$, then there exists a bounded A-free complex $\mathbb{F}.$ satisfing the following conditions:

(1) The complex $\mathbb{F}.$ is exact on $Y \setminus X$.

(2) For any morphism of schemes $g : Y' \to Y$, the map

$$\varphi_g \otimes 1 : G_0(Y')_{\mathbb{Q}} \longrightarrow G_0(X')_{\mathbb{Q}}$$

coincides with

$$\chi(\mathbb{F}.)_g \otimes 1 : G_0(Y')_{\mathbb{Q}} \longrightarrow G_0(X')_{\mathbb{Q}}.$$

Remark that, if A is an excellent local domain containing $\mathbb{Q}$, and regular on $\operatorname{Spec} A \setminus \operatorname{Spec}(A/I)$, then there exists a resolution of singularity of $\operatorname{Spec} A$ satisfying the condition in Theorem 4.4. For an excellent local domain of any characteristic, it is expected such a resolution of singularities exists.

Proof. By Thomason-Trobaugh [35], we have the following commutative diagram

$$\begin{array}{ccccc} & & G_0(W) & \xrightarrow{i_*} & G_0(Z) \\ & & \|\eta & & \| \\ K_1(U) & \longrightarrow & K_0^W(Z) & \longrightarrow & K_0(Z) \\ \| & & \uparrow \pi^* & & \uparrow \\ K_1(U) & \longrightarrow & K_0^I(A) & \longrightarrow & K_0(A) \end{array},$$

where $K_0^I(A)$ is the Grothendieck group of perfect A-complexes which are exact outside of $X = \mathrm{Spec}(A/I)$. The map η takes the alternating sum of homologies of complexes.

Since $I \neq 0$, the homomorphism $K_0^I(A) \to K_0(A)$ is zero.

Therefore, we have an exact sequence

$$K_0^I(A) \xrightarrow{\eta\pi^*} G_0(W) \xrightarrow{i_*} G_0(Z). \tag{4.4}$$

Consider the cycle $\varphi_\pi([\mathcal{O}_Z]) \in G_0(W)$. By assumption, we have

$$i_*\varphi_\pi([\mathcal{O}_Z]) = 0$$

in $G_0(Z)$. By the exact sequence (4.4), we have a bounded A-free complex $\mathbb{F}.$ which is exact on $Y \setminus X$ such that

$$\varphi_\pi([\mathcal{O}_Z]) = \eta\pi^*([\mathbb{F}.]) = \chi(\mathbb{F}.)_\pi([\mathcal{O}_Z]) \tag{4.5}$$

in $G_0(W)$. Here, remark that, for a complex $\mathbb{G}.$, $-[\mathbb{G}.]$ coincides with the shifted complex $[\mathbb{G}.(-1)]$ in the Grothendieck group of complexes since the mapping cone $C(\mathbb{G}. \xrightarrow{1} \mathbb{G}.)$ is exact, and the sequence of complexes

$$0 \longrightarrow \mathbb{G}. \longrightarrow C(\mathbb{G}. \xrightarrow{1} \mathbb{G}.) \longrightarrow \mathbb{G}.(-1) \longrightarrow 0$$

is exact. Therefore, we can choose a complex $\mathbb{F}.$ satisfying (4.5).

We shall show that the map $\varphi_g \otimes 1$ coincides with $\chi(\mathbb{F}.)_g \otimes 1$ for any g.

Remark that

$$\{\varphi_g - \chi(\mathbb{F}.)_g : G_0(Y') \to G_0(X') \mid g : Y' \to Y \text{ is a morphism of schemes}\}$$

is also a bivariant class on K-groups for $X \hookrightarrow Y$. By (4.5), one has that

$$(\varphi_\pi - \chi(\mathbb{F}.)_\pi)([\mathcal{O}_Z]) = 0 \in G_0(W).$$

We have only to prove the following lemma.

Lemma 4.5. *Let Y be a scheme and X be a closed subscheme of Y. Let $\pi : Z \to Y$ be a proper surjective morphism such that Z is a regular scheme. Let*

$$\{\phi_g : G_0(Y') \to G_0(X') \mid g : Y' \to Y \text{ is a morphism of schemes}\}$$

be a bivariant class on K-groups for $X \hookrightarrow Y$. Let $W = \pi^{-1}(X)$.

If $\phi_\pi([\mathcal{O}_Z]) = 0$ in $G_0(W)$, then

$$\phi_g \otimes 1 : G_0(Y')_{\mathbb{Q}} \longrightarrow G_0(X')_{\mathbb{Q}}$$

is zero for any morphism $g : Y' \to Y$.

Proof. We shall prove it in two steps.

Step 1. Let $f : Z' \to Z$ be a morphism such that Z' is an integral scheme. We shall prove that $\phi_{\pi f}([\mathcal{O}_{Z'}]) = 0$.

We shall prove it by induction on $\dim Z'$. Suppose that the assertion is true if the dimension is less than $\dim Z'$. By Nagata's compactification ([26, 30]) and (B_2), we may assume that f is a proper morphism. By Chow's lemma ([14, Exercise 4.10 in p107]), there exists a proper birational morphism $h : Z'' \to Z'$ such that $fh : Z'' \to Z$ is projective. By (B_1) and induction on dimension, it is enough to show that $\phi_{\pi fh}([\mathcal{O}_{Z''}]) = 0$. So, we may assume that f is projective. Then f can be factored as

$$Z' \overset{i}{\hookrightarrow} V \overset{p}{\longrightarrow} Z$$

where $p : V \to Z$ is a smooth morphism and i is a closed immersion. By (B_2), $\phi_{\pi p}([\mathcal{O}_V]) = 0$. Let $\mathbb{G}.$ be a bounded locally free $\mathcal{O}_V$-resolution of $\mathcal{O}_{Z'}$. Then, by (B_3), we have

$$\phi_{\pi f}([\mathcal{O}_{Z'}]) = \phi_{\pi f}\chi_{\mathbb{G}.}([\mathcal{O}_V]) = \chi_{\mathbb{G}.\otimes \mathcal{O}_{V'}}\phi_{\pi p}([\mathcal{O}_V]) = 0,$$

where $V' = V \times_Y X$.

Step 2. Let $g : Y' \to Y$ be a morphism such that Y' is an integral scheme. We shall prove that $\phi_g([\mathcal{O}_{Y'}])$ is a torsion element of $G_0(X')$ by induction on $\dim Y'$. Suppose that the assertion is true if the dimension is less than $\dim Y'$. Consider the following fibre square:

$$\begin{array}{ccc} Z' & \overset{g'}{\longrightarrow} & Z \\ \pi' \downarrow & \square & \downarrow \pi \\ Y' & \overset{g}{\longrightarrow} & Y \end{array}$$

Let $i : Z'' \to Z'$ be a closed immersion such that Z'' is integral and $\pi' i : Z'' \to Y'$ is generically finite and surjective. Then, by Step 1 and (B_1), $\phi_g((\pi' i)_*([\mathcal{O}_{Z''}])) = 0$. That $\phi_g([\mathcal{O}_{Y'}])$ is a torsion element now follows by induction on dimension. □

5. Theta pairing as a bivariant class on K-groups

Definition 5.1. Let $(A, \mathfrak{m})$ be a hypersurface. Let N be a finitely generated A-module, and $\mathbb{F}.$ be an A-free resolution of N. Let I be an ideal of A. Assume that N_P is an A_P-module of finite projective dimension for $P \in \operatorname{Spec} A \setminus V(I)$. Put $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec}(A/I)$.

For a fibre square (4.1), consider the homomorphism

$$\theta(N)_g : G_0(Y') \longrightarrow G_0(X')$$

defined by

$$\theta(N)_g([\mathcal{F}]) = [H_{2k}(g^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y'}} \mathcal{F})] - [H_{2k-1}(g^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y'}} \mathcal{F})],$$

for a sufficiently large k, where $\mathcal{F}$ is a coherent $\mathcal{O}_{Y'}$-module. It is easy to check that $\theta(N)_g$ is well-defined.

We obtain a collection of homomorphisms

$$\{\theta(N)_g : G_0(Y') \longrightarrow G_0(X') \mid g : Y' \to Y \text{ is a morphism of schemes}\}.$$

Theorem 5.2. *With notation as in Definition 5.1, the collection of homomorphisms*

$$\{\theta(N)_g : G_0(Y') \longrightarrow G_0(X') \mid g : Y' \to Y \textit{ is a morphism of schemes}\}$$

is a bivariant class on K-groups for $X \hookrightarrow Y$.

Proof. Let $\mathbb{F}.$ be an A-free resolution of N.

First we prove (B_1). Consider the diagram (4.2). For a coherent $\mathcal{O}_{Y''}$-module $\mathcal{F}$, we shall prove

$$h'_*\theta(N)_{gh}([\mathcal{F}]) = \theta(N)_g h_*([\mathcal{F}])$$

in $G_0(X')$. It is enough to show that the both sides are equal to

$$[H_{2k}(Rh_*((gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F}))] - [H_{2k-1}(Rh_*((gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F}))]$$

for a sufficiently large k.

Let

$$0 \to \mathcal{F} \to \mathcal{I}^{\cdot}$$

be an injective resolution of $\mathcal{F}$. Consider the double complex

$$h_*((gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{I}^{\cdot}) = g^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y'}} h_*(\mathcal{I}^{\cdot}).$$

Since $(gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{I}^{\cdot}$ is an injective resolution of $(gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F}$, the kth homology of the total complex of the above double complex is

$$H_k(Rh_*((gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F})).$$

Consider the spectral sequence

$$E_2^{p,q} = H_p(g^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y'}} R^{-q}h_*(\mathcal{F})) \Longrightarrow H_{p+q}(Rh_*((gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F})).$$

Then,

$$\theta(N)_g h_*([\mathcal{F}]) = \sum_q (-1)^q [E_2^{2k,q}] - \sum_q (-1)^q [E_2^{2k-1,q}]$$

for a sufficiently large k. Here, the E_3-terms of the spectral sequence are the homologies of the complex

$$\cdots \longrightarrow E_2^{p+2,q-1} \longrightarrow E_2^{p,q} \longrightarrow E_2^{p-2,q+1} \longrightarrow \cdots .$$

If p is big enough, then the above complex coincides with

$$\cdots \longrightarrow E_2^{p,q-1} \longrightarrow E_2^{p,q} \longrightarrow E_2^{p,q+1} \longrightarrow \cdots .$$

Therefore, we have

$$\sum_q (-1)^q [E_2^{2k,q}] - \sum_q (-1)^q [E_2^{2k-1,q}] = \sum_q (-1)^q [E_3^{2k,q}] - \sum_q (-1)^q [E_3^{2k-1,q}]$$

for a sufficiently large k. Next, the E_4-terms of the spectral sequence are the homologies of the complex

$$\cdots \longrightarrow E_3^{p+3,q-2} \longrightarrow E_3^{p,q} \longrightarrow E_3^{p-3,q+2} \longrightarrow \cdots .$$

Suppose that p is big enough. Then the above complex coincides with

$$\cdots \longrightarrow E_3^{2k-1,q-4} \longrightarrow E_3^{2k,q-2} \longrightarrow E_3^{2k-1,q} \longrightarrow E_3^{2k,q+2} \longrightarrow E_3^{2k-1,q+4} \longrightarrow \cdots$$

if $p = 2k-1$, and

$$\cdots \longrightarrow E_3^{2k,q-4} \longrightarrow E_3^{2k-1,q-2} \longrightarrow E_3^{2k,q} \longrightarrow E_3^{2k-1,q+2} \longrightarrow E_3^{2k,q+4} \longrightarrow \cdots$$

if $p = 2k$. Therefore, we have

$$\sum_q (-1)^q [E_3^{2k,q}] - \sum_q (-1)^q [E_3^{2k-1,q}] = \sum_q (-1)^q [E_4^{2k,q}] - \sum_q (-1)^q [E_4^{2k-1,q}]$$

for a sufficiently large k. Repeating this argument, we obtain

$$\begin{array}{rl} & \theta(N)_g h_*([\mathcal{F}]) \\ = & \sum_q (-1)^q [E_2^{2k,q}] - \sum_q (-1)^q [E_2^{2k-1,q}] \\ = & \sum_q (-1)^q [E_\infty^{2k,q}] - \sum_q (-1)^q [E_\infty^{2k-1,q}] \\ = & \sum_q [E_\infty^{2k-q,q}] - \sum_q [E_\infty^{2k-1-q,q}] \\ = & [H_{2k}(Rh_*((gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F}))] - [H_{2k-1}(Rh_*((gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F}))] \end{array}$$

in $G_0(X')$ for a sufficiently large k.

We choose another injective resolution of the complex $(gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F}$. We take a double complex $\mathbb{H}..$ of $\mathcal{O}_{Y''}$-modules such that

- the total complex of $\mathbb{H}..$ is an injective resolution of $(gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F}$,

- $0 \to H_p(\mathbb{H}_{.0}) \to H_p(\mathbb{H}_{.-1}) \to H_p(\mathbb{H}_{.-2}) \to \cdots$ is an injective resolution of $H_p((gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F})$ for each p,
- $h_*(H_p(\mathbb{H}_{.q})) = H_p(h_*(\mathbb{H}_{.q}))$ for each p and q.

Consider the following spectral sequence.

$$'E_2^{p,q} = H_q H_p(h_*(\mathbb{H}..)) = R^{-q}h'_*(H_p((gh)^*(\mathbb{F}.)\otimes_{\mathcal{O}_{Y''}}\mathcal{F})) \Longrightarrow H_{p+q}(Rh_*((gh)^*(\mathbb{F}.)\otimes_{\mathcal{O}_{Y''}}\mathcal{F})).$$

Then,

$$h'_*\theta(N)_{gh}([\mathcal{F}]) = \sum_q (-1)^q[{}'E_2^{2k,q}] - \sum_q (-1)^q[{}'E_2^{2k-1,q}]$$

for a sufficiently large k. Here, it is proved that

$$\begin{aligned} & \sum_q (-1)^q[{}'E_2^{2k,q}] - \sum_q (-1)^q[{}'E_2^{2k-1,q}] \\ = \ & \sum_q (-1)^q[{}'E_\infty^{2k,q}] - \sum_q (-1)^q[{}'E_\infty^{2k-1,q}] \\ = \ & \sum_q [{}'E_\infty^{2k-q,q}] - \sum_q [{}'E_\infty^{2k-1-q,q}] \\ = \ & [H_{2k}(Rh_*((gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F}))] - [H_{2k-1}(Rh_*((gh)^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y''}} \mathcal{F}))] \end{aligned}$$

in $G_0(X')$ for a sufficiently large k by the same argument as in the case $E_\cdot^{p,q}$. The condition (B_1) is proved.

Condition (B_2) follows immediately from the flatness assumption.

Now we prove (B_3). Consider the diagram (4.3). For a coherent $\mathcal{O}_{Y'}$-module $\mathcal{F}$, we shall prove

$$\chi_{(hf)^*(\mathbb{G}.)}\theta(N)_g([\mathcal{F}]) = \theta(N)_{gi'}\chi_{h^*(\mathbb{G}.)}([\mathcal{F}]).$$

It is enough to show that the both sides are equal to

$$[H_{2k}(g^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y'}} h^*(\mathbb{G}.) \otimes_{\mathcal{O}_{Y'}} \mathcal{F})] - [H_{2k-1}(g^*(\mathbb{F}.) \otimes_{\mathcal{O}_{Y'}} h^*(\mathbb{G}.) \otimes_{\mathcal{O}_{Y'}} \mathcal{F})]$$

for a sufficiently large k. The proof will be done in the same way as in (B_1), so we omit it. □

6. Theta pairing for hypersurface with isolated singularity

Let $(A, \mathfrak{m})$ be a hypersurface with an isolated singularity. Then, the theta pairing induces a map

$$\theta^A : G_0(A) \otimes G_0(A) \longrightarrow \mathbb{Z}$$

by $\theta^A(\sum_i \pm[M_i] \otimes [N_i]) = \sum_i \pm\theta^A(M_i, N_i)$. We sometimes denote $\theta^A(\alpha \otimes \beta)$ by $\theta^A(\alpha, \beta)$.

The following theorem shows that $\theta^A(M, N) = 0$ if either M or N is numerically equivalent to 0. See Corollary 6.2.

Theorem 6.1. *Let $(A, \mathfrak{m})$ be a hypersurface with an isolated singularity. Assume* $\dim A > 0$*. Let N be a finitely generated A-module.*

Assume that there exists a resolution of singularity of $\operatorname{Spec} A$*, i.e., a proper birational morphism $\pi : Z \to \operatorname{Spec} A$ such that Z is regular. Put $W = \pi^{-1}(\operatorname{Spec}(A/\mathfrak{m}))$ and $U = Z \setminus W$. Assume that U is isomorphic to* $\operatorname{Spec} A \setminus \operatorname{Spec}(A/\mathfrak{m})$*.*

Then, there exist A-modules N_1 and N_2 such that

- $\ell_A(N_1) = \ell_A(N_2) < \infty$
- $\operatorname{pd}_A(N_1) = \operatorname{pd}_A(N_2) = \dim A$
- $\theta^A(N, M) = \chi^A(N_1, M) - \chi^A(N_2, M)$ *for any finitely generated A-module M.*

Proof. First, we shall prove

$$i_*\theta(N)_\pi([\mathcal{O}_Z]) = 0$$

in $G_0(Z)$, where $i : W \to Z$ is the closed immersion. Consider the minimal A-free resolution $\mathbb{F}.$ of N. Then, it is written as

$$\cdots \xrightarrow{\alpha} F_{2k+1} \xrightarrow{\beta} F_{2k} \xrightarrow{\alpha} F_{2k-1} \xrightarrow{\beta} \cdots$$

for $k \gg 0$. Here, F_n's are A-free modules of the same rank for $n \gg 0$. Here, consider the complex $\pi^*(\mathbb{F}.)$

$$\cdots \xrightarrow{\pi^*\alpha} \pi^*F_{2k+1} \xrightarrow{\pi^*\beta} \pi^*F_{2k} \xrightarrow{\pi^*\alpha} \pi^*F_{2k-1} \xrightarrow{\pi^*\beta} \cdots .$$

Then,

$$\theta(N)_\pi([\mathcal{O}_Z]) = [H_{2k}(\pi^*(\mathbb{F}.))] - [H_{2k-1}(\pi^*(\mathbb{F}.))] \in G_0(W)$$

for $k \gg 0$. Let K_α (resp. I_α) be the kernel (resp. image) of the map $\pi^*\alpha$. Let K_β (resp. I_β) be the kernel (resp. image) of the map $\pi^*\beta$. Then, we have the following exact sequences

$$\begin{array}{l} 0 \longrightarrow K_\alpha \longrightarrow \pi^*F_{2k} \longrightarrow I_\alpha \longrightarrow 0 \\ 0 \longrightarrow K_\beta \longrightarrow \pi^*F_{2k-1} \longrightarrow I_\beta \longrightarrow 0 \\ 0 \longrightarrow I_\beta \longrightarrow K_\alpha \longrightarrow H_{2k}(\pi^*(\mathbb{F}.)) \longrightarrow 0 \\ 0 \longrightarrow I_\alpha \longrightarrow K_\beta \longrightarrow H_{2k-1}(\pi^*(\mathbb{F}.)) \longrightarrow 0. \end{array}$$

Therefore, we obtain

$$\begin{array}{rcl} i_*\theta(N)_\pi([\mathcal{O}_Z]) & = & [H_{2k}(\pi^*(\mathbb{F}.))] - [H_{2k-1}(\pi^*(\mathbb{F}.))] \\ & = & ([K_\alpha] - [I_\beta]) - ([K_\beta] - [I_\alpha]) \\ & = & [\pi^*F_{2k}] - [\pi^*F_{2k-1}] = 0 \end{array}$$

in $G_0(Z)$.

Since A is Cohen-Macaulay, the Grothendieck group of bounded A-free complexes with support in $\{\mathfrak{m}\}$ is generated by finite free resolutions of modules of finite length and finite projective dimension by Proposition 2 in [34]. Therefore, by Theorem 6.1, there exist A-modules N_1 and N_2 of finite length and of finite projective dimension such that, letting $\mathbb{A}.$ (resp. $\mathbb{B}.$) be a finite A-free resolution of N_1 (resp. N_2), we have

$$\theta(N)_g \otimes 1 = \chi(\mathbb{A}. \oplus \mathbb{B}.(-1))_g \otimes 1 = \chi(\mathbb{A}.)_g \otimes 1 - \chi(\mathbb{B}.)_g \otimes 1$$

for any $g : Y' \to \operatorname{Spec} A$.

In particular, we have

$$\chi(\mathbb{A}.)_{\mathrm{id}_{\operatorname{Spec} A}} - \chi(\mathbb{B}.)_{\mathrm{id}_{\operatorname{Spec} A}} = \chi(\mathbb{A}. \oplus \mathbb{B}.(-1))_{\mathrm{id}_{\operatorname{Spec} A}} = \theta(N)_{\mathrm{id}_{\operatorname{Spec} A}}$$

since $G_0(\operatorname{Spec}(A/\mathfrak{m})) \simeq \mathbb{Z}$. Hence, for any finitely generated A-module M, we have

$$\begin{aligned}
\theta^A(N, M) &= \theta(N)_{\mathrm{id}_{\operatorname{Spec} A}}([M]) \\
&= \chi(\mathbb{A}.)_{\mathrm{id}_{\operatorname{Spec} A}}([M]) - \chi(\mathbb{B}.)_{\mathrm{id}_{\operatorname{Spec} A}}([M]) \\
&= \chi_{\mathbb{A}.}([M]) - \chi_{\mathbb{B}.}([M]) \\
&= \chi^A(N_1, M) - \chi^A(N_2, M).
\end{aligned}$$

Here, note that

$$0 = \theta^A(N, A) = \chi^A(N_1, A) - \chi^A(N_2, A) = \ell(N_1) - \ell(N_2).$$

□

In [3] and [29], it is proved that the theta pairing vanishes on cycles which are algebraic equivalent to 0. The following corollary recovers this fact since algebraic equivalence implies numerical equivalence. It will be discussed in Remark 6.4 in detail.

The following corollary contains Theorem 1.1.

Corollary 6.2. *Let $(A, \mathfrak{m})$ be a d-dimensional hypersurface with isolated singularity. Assume $d > 0$. Assume that there exists a resolution of singularity $\pi : Z \to \operatorname{Spec} A$ such that $Z \backslash \pi^{-1}(\operatorname{Spec}(A/\mathfrak{m}))$ is isomorphic to $\operatorname{Spec} A \backslash \operatorname{Spec}(A/\mathfrak{m})$.*

(1) Suppose that $\alpha, \beta \in G_0(A)$. If one of α and β is numerically equivalent to 0 in the sense of [23], then $\theta^A(\alpha, \beta) = 0$.

(2) If $\overline{G_0(A)}_{\mathbb{Q}} = \mathbb{Q}[A]$ (or equivalently $\overline{A_i(A)}_{\mathbb{Q}} = 0$ for $i < d$), then $\theta^A(M, N) = 0$ for any finitely generated A-modules M and N.

Proof. First, we shall prove (1). We assume that α is numerically equivalent to 0 in the sense of [23]. We shall prove that $\theta^A(N, \alpha) = 0$ for any finitely generated A-module N. By Theorem 6.1, there exist bounded finite A-free complexes $\mathbb{A}.$ and $\mathbb{B}.$ with finite length homologies such that

$$\begin{aligned} \theta^A(N, \alpha) &= \theta(N)_{\mathrm{id}_{\mathrm{Spec}\,A}}(\alpha) \\ &= \chi(\mathbb{A}.)_{\mathrm{id}_{\mathrm{Spec}\,A}}(\alpha) - \chi(\mathbb{B}.)_{\mathrm{id}_{\mathrm{Spec}\,A}}(\alpha) \\ &= \chi_{\mathbb{A}.}(\alpha) - \chi_{\mathbb{B}.}(\alpha). \end{aligned}$$

It is equal to 0 since α is numerically equivalent to 0.

Next, we shall prove (2). Recall that, for a Noetherian local domain $(A, \mathfrak{m})$, $\overline{G_0(A)_{\mathbb{Q}}} = \mathbb{Q}[A]$ if and only if $\chi_{\mathbb{G}.}([T]) = 0$ for any finitely generated A-module T with $\dim T < \dim A$ and any bounded finite free A-complex $\mathbb{G}.$ with finite length homologies. For a given A-module M, letting r be the rank of M, we have an exact sequence of the form

$$0 \longrightarrow A^r \longrightarrow M \longrightarrow T \longrightarrow 0,$$

where T is an A-module with $\dim T < \dim A$. Then, we have

$$\theta^A(N, M) = \theta^A(N, A^r) + \theta^A(N, T) = \chi_{\mathbb{A}.}([T]) - \chi_{\mathbb{B}.}([T]) = 0.$$

For Chow groups of local rings divided by numerical equivalence, see the following remark. □

Remark 6.3. For a Noetherian local ring $(A, \mathfrak{m})$, $\overline{A_i(A)_{\mathbb{Q}}} = 0$ for $i = 0, 1, \ldots, s$ if and only if $\chi_{\mathbb{G}.}([M]) = 0$ for any finitely generated A-module M with $\dim M \leq s$ and any bounded A-free complex $\mathbb{G}.$ with support in $\{\mathfrak{m}\}$, see [23] for details.

Remark 6.4. For cycles of a local ring A, algebraic equivalence in Definition 3.2 in [3] implies numerical equivalence as follows.

Let A be a local ring which is essentially of finite type over an algebraically closed field k. Assume that k is isomorphic to the residue class field of A. Let B be an integral domain that is smooth of finite type over k. Let $\mathfrak{m}_\ell$ be a maximal ideal of B and

$$i_\ell : \mathrm{Spec}(B/\mathfrak{m}_\ell) \longrightarrow \mathrm{Spec}\, B$$

be the closed immersion for $\ell = 1, 2$. We define

$$i_\ell^* : G_0(A \otimes_k B) \longrightarrow G_0(A)$$

by

$$i_\ell^*([M]) = \sum_j (-1)^j [\mathrm{Tor}_j^B(M, B/\mathfrak{m}_\ell)].$$

Cycles of the form

$$i_1^*(\gamma) - i_2^*(\gamma)$$

generates algebraic equivalence, where $\gamma \in G_0(A \otimes_k B)$.

We shall prove that the above cycle is numerically equivalent to 0. Let $\mathbb{F}.$ be a bounded finite A-free complex with finite length homologies. We have

$$\chi_{\mathbb{F}.}(i_1^*(\gamma) - i_2^*(\gamma)) = i_1^* \chi_{\mathbb{F}.\otimes_k B}(\gamma) - i_2^* \chi_{\mathbb{F}.\otimes_k B}(\gamma)$$

where $i_\ell^* : G_0(B) \to G_0(k)$ in the right-hand side is defined by

$$i_\ell^*([L]) = \sum_j (-1)^j [\mathrm{Tor}_j^B(L, B/\mathfrak{m}_\ell)].$$

It is easy to see

$$i_\ell^* \chi_{\mathbb{F}.\otimes_k B}(\gamma) = \mathrm{rank}_B\, \chi_{\mathbb{F}.\otimes_k B}(\gamma)$$

for $\ell = 1, 2$. Thus, $i_1^* \chi_{\mathbb{F}.\otimes_k B}(\gamma) = i_2 \chi_{\mathbb{F}.\otimes_k B}(\gamma)$.

7. Various cases of Conjecture 3.1

In this Section we prove various cases of Conjecture 3.1. Some of these results are already known but we provide an alternate proof. Some, such as the positive semi-definiteness of θ^A in dimension 3 (Theorem 7.9) are new.

Remark 7.1. By Theorem 6.1 and Proposition 3.3, Conjecture 3.1, part (1), (2) and (3) are true in the quasi-homogeneous case. This recovers a result of Moore-Piepmeyer-Spiroff-Walker [29]. We just need to note that the blow-up of Spec A at the maximal ideal gives a resolution of singularity in this case.

Let $(A, \mathfrak{m})$ be a d-dimensional hypersurface with an isolated singularity. Assume that there exists a resolution of a singularity $\pi : Z \to \mathrm{Spec}\, A$ such that $Z \setminus \pi^{-1}(\mathrm{Spec}(A/\mathfrak{m}))$ is isomorphic to $\mathrm{Spec}\, A \setminus \mathrm{Spec}(A/\mathfrak{m})$. Assume that d is even. If Conjecture 3.2 (1) (a) is true, then $\theta(M, N) = 0$ by Theorem 6.1 for any finitely generated A-modules M and N, that is, Conjecture 3.1(1) is true. We remark that Conjecture 3.1(1) is true in the case of characteristic zero by Buchweitz-Straten [2].

Lemma 7.2. *Let A be a local ring and N be a finitely generated module and $l \geq 0$ be an integer. Then in $\underline{G_0}(A)$ one has*

$$(-1)^l [\Omega^l(N)^*] = \sum_{i=0}^{l} (-1)^i [\mathrm{Ext}_A^i(N, A)].$$

Proof. This is an easy induction on l. □

For an element $\alpha \in \underline{G_0}(A)$, let $\dim(\alpha)$ be the smallest integer c such that α has a representation by formal sum (with coefficients) of modules of dimensions at most c.

Lemma 7.3. *Suppose that A is Gorenstein of dimension d and N is of codimension a and codepth b. Then:*

$$\dim((-1)^b[\Omega^b(N)^*] - (-1)^a[N]) < d - a$$

Proof. For each minimal prime ideal p of N with $\dim A/p = d - a$, we have

$$\ell_{A_p}(N_p) = \ell_{A_p}(\mathrm{Ext}^a_A(N, A)_p)$$

which shows $\dim([N]-[\mathrm{Ext}^a_A(N, A)]) < d-a$. Here, remark that $b \geq a$. Lemma 7.2 applies to get the desired conclusion, since the support of $\mathrm{Ext}^i_A(N, A)$ evidently has dimension less than $d - a$ for $i \neq a$. □

Proposition 7.4. *Let $\dim A = 6$. Then the statements (1), (3) of Conjecture 3.1 are equivalent.*

Proof. Clearly (1) implies (3). Clearly, it is enough to show (1) for non-free MCM modules. Let M be such a module. Then take a Bourbaki sequence ([1]):

$$0 \to F \to M \to I \to 0$$

Since $d = 6$, we know that A is a UFD by [13, Lemmata 3.16, 3.17]. We may assume that $\operatorname{height} I \geq 2$ and $\dim A/I \leq 4$. But counting depths shows that $\operatorname{depth} I \geq 5$, and thus $\operatorname{depth} A/I = \dim A/I = 4$. Hence it will be enough to prove $\theta^A(A/I, A/J) = 0$ for any height two ideals I, J whose quotients are Cohen-Macaulay.

Now let $M = \Omega^2(A/I)$ and $N = \Omega^2(A/J)$. By [5, 4.2, 4.3] we know:

$$\theta^A(M, N) + \theta^A(M^*, N^*) = 0$$

By Lemma 7.3 we know that $[M^*] = [A/I] + \alpha$ and $[N^*] = [A/J] + \beta$ in $\underline{G_0}(A)$, with $\dim \alpha$ and $\dim \beta$ at most three. Evidently, we also know $[M] = [A/I]$ and $[N] = [A/J]$ in $\underline{G_0}(A)$. Since θ^A vanishes when one of the arguments has dimension at most three, it then follows that:

$$\theta^A(A/I, A/J) + \theta^A(A/I, A/J) = 0$$

as we wanted. □

Corollary 7.5. *Let A be an excellent local hypersurface with an isolated singularity containing a field of characteristic* 0 *and suppose* $\dim A \leq 6$. *Then Conjecture 3.1(3) holds.*

Conjecture 3.1(5) is known in the standard graded case over a field of characteristic 0, see [29]. We shall establish it for $d = \dim A \leq 3$.

Proposition 7.6. *If $d = 1$, Conjecture 3.1(5) is true.*

Proof. Let $p_1, p_2, ..., p_n$ be the minimal prime ideals of A. Then $G_0(A)_{\mathbb{Q}}$ has a basis consisting of the classes $[A/p_1], ..., [A/p_n]$. In particular, since A has dimension 1 and is reduced, $[A] = \sum_1^n [A/p_i]$. Let $\alpha_{ij} = \theta^A(A/p_i, A/p_j)$. For $i \neq j$, $p_i + p_j$ is m-primary, and it is not hard (using the resolution of A/p_i, noting that each p_i is a principal ideal) to see that $\alpha_{ij} = \ell(A/(p_i + p_j)) > 0$. Since $\theta^A(A, A/p_i) = 0$, we must have $\alpha_{ii} = -\sum_{j \neq i} \alpha_{ij}$. Now, for a module M, $[M] = \sum a_i [A/p_i]$, here a_i is the rank of M_{p_i}. Then

$$\theta^A(M, M) = \sum_{i,j} \alpha_{ij} a_i a_j = -\sum_{i<j} \alpha_{ij}(a_i - a_j)^2 \leq 0.$$

Clearly, the equality happens iff $a_1 = a_2 = \cdots = a_n$. Then $[M] = a_1[A]$, so it is zero in $\underline{G_0}(A)_{\mathbb{Q}}$. In this case, $-\theta^A$ is a positive definite form on $\underline{G_0}(A)_{\mathbb{Q}}$. □

Next, we look at dimension 3. The proof here follows and improves the main result of [7]. The divisor class map as described in Section 2 will play an important role. We start with a more general result:

Theorem 7.7. *Let $(A, \mathfrak{m})$ be an abstract local hypersurface of dimension 3. Let M, N be reflexive A-modules which are locally free on $U_A = \operatorname{Spec} A - \{\mathfrak{m}\}$, the punctured spectrum of A. Suppose $\operatorname{Hom}_A(M, N)$ is a maximal Cohen-Macaulay A-module. Then $\theta^A(M^*, N) \leq 0$. Furthermore, equality happens if and only if M or N is free.*

Proof. First, there is no loss of generality by passing to the completion, so we will assume A is complete. So $A \cong S/(f)$, where S is a regular local ring of dimension 4.

Suppose that $\operatorname{Hom}_A(M, N)$ is maximal Cohen-Macaulay. Then by [6, Lemma 2.3] we have $\operatorname{Ext}^1_A(M, N) = 0$. One has the following short exact sequence (see [15, 3.6] or [21], [20]):

(7.1) $\operatorname{Tor}_2^A(M_1, N) \to \operatorname{Ext}^1_A(M, A) \otimes_A N \to \operatorname{Ext}^1_A(M, N) \to \operatorname{Tor}_1^A(M_1, N) \to 0$

Here M_1 is the cokernel of $F_1^* \to F_2^*$, where $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ is the minimal free resolution of M. Since $\operatorname{Ext}^1_A(M, N) = 0$, it follows that $\operatorname{Tor}_1^A(M_1, N) = 0$.

The change of rings long exact sequence for Tor ([17]) yields a surjection

$$\operatorname{Tor}_3^S(M_1, N) \twoheadrightarrow \operatorname{Tor}_3^A(M_1, N).$$

We claim that $\operatorname{Tor}_i^S(M_1, N) = 0$ for $i \geq 3$. As N is reflexive we have $\operatorname{depth} N \geq 2$. It follows that as an S-module, the depth of N is also at least 2, so $\operatorname{pd}_S N \leq 4 - 2 = 2$ and our assertion follows.

It follows that $\operatorname{Tor}_3^A(M_1, N) = 0$. Also, $\operatorname{Tor}_i^A(M_1, N)$ becomes periodic of period 2 after $i \geq 2$. So $\theta^A(M_1, N) = \ell(\operatorname{Tor}_2^A(M_1, N)) \geq 0$.

Finally, we have by definition a complex:

$$0 \to M^* \to F_0^* \to F_1^* \to F_2^* \to M_1 \to 0 \tag{7.2}$$

Since $\operatorname{Ext}_A^1(M, A)$ is of finite length, we have $[M^*] = -[M_1]$ in $\underline{G_0}(A)_{\mathbb{Q}}$, and as N is locally free on U_A it follows that $\theta^A(M^*, N) = -\theta^A(M_1, N) \leq 0$.

Now we prove the last claim. Looking at the proof, we see that equality holds if and only if $\operatorname{Tor}_2^A(M_1, N) = 0$. As $\operatorname{Tor}_1^A(M_1, N) = 0$, it now follows that $\operatorname{Tor}_i^A(M_1, N) = 0$ for all $i > 0$. The long exact sequence (7.1) implies $\operatorname{Ext}_A^1(M, A) = 0$, thus M is MCM. Since (7.2) is exact, M^* is third syzygy of M_1, so $\operatorname{Tor}_i^A(M^*, N) = 0$ for all $i > 0$. By the depth formula (Proposition 2.5 in [17]) one has

$$\operatorname{depth} M^* + \operatorname{depth} N = \operatorname{depth} A + \operatorname{depth} M^* \otimes_A N$$

Thus $\operatorname{depth} M^* \otimes_A N = \operatorname{depth} N \geq 2$. But the canonical map $M^* \otimes_A N \to \operatorname{Hom}_A(M, N)$ has finite length kernel and cokernel, so it must be an isomorphism. We conclude that $\operatorname{depth} N = 3$, i.e., N is also MCM.

As shown above, $\operatorname{Tor}_i^A(M^*, N) = 0$ for all $i > 0$, so either M^* or N has finite projective dimension by [18, Theorem 1.9] or [28, 1.1]. But since they are both MCM, one of them must be free. □

Lemma 7.8. *Let A be a local hypersurface of dimension 3 with an isolated singularity and M, N be reflexive A-modules. Let $[I], [J] \in Cl(A)$ represent $c_1([M]), c_1([N]) \in A_2(A)$ respectively. Then, we have $\theta^A(M, N) = \theta^A(I, J)$. Furthermore $\theta^A(M, N) = -\theta^A(M, N^*)$.*

Proof. It is not hard to see that in $\underline{G_0}(A)_{\mathbb{Q}}$, the reduced Grothendieck group with rational coefficients, we have an equality $[N] - [J] = \sum a_i[A/P_i]$ such that each $P_i \in \operatorname{Spec} A$ has height at least 2 (see the proof of 3.1 in [7]).

The first half will be proved by showing that $\theta^A(M, A/P) = 0$ for each $P \in \operatorname{Spec} A$ such that height $P = 2$.

By [19, Theorem 1.4] one can construct a Bourbaki sequence for M:

$$0 \to F \to M \to I \to 0$$

such that $I \not\subseteq P$. Obviously $\theta^A(M, A/P) = \theta^A(I, A/P)$. But $A/I \otimes_A A/P$ has finite length, and $\dim A/I + \dim A/P \leq 3 = \dim A$. By [16], $\theta^A(A/I, A/P) = 0$. Since $\theta^A(I, A/P) = -\theta^A(A/I, A/P)$, we have $\theta^A(M, A/P) = 0$.

The last statement follows from $[N^*]-[J^*] = \sum b_i[A/P_i]$, $[J]+[J^*] = \sum c_i[A/P_i]$ in $\underline{G_0}(A)$, where each $P_i \in \operatorname{Spec} A$ has height at least 2. □

The following contains Theorem 1.2 in the introduction.

Theorem 7.9. *Let A be a local hypersurface with an isolated singularity and $\dim A = 3$. Let M be a finitely generated A-module. Then $\theta^A(M, M) \geq 0$. Equality holds if and only if $c_1([M]) = 0$ in $A_2(A)$.*

Proof. By taking a high syzygy we can assume that M is maximal Cohen-Macaulay. Let $[I] \in Cl(A)$ represent $c_1([M]) \in A_2(A)$. Then $[I^*]$ represents $c_1([M^*])$. Since $\operatorname{Hom}_A(I, I) \cong A$ we know by Theorem 7.7 and Lemma 7.8 that $\theta^A(M, M) \geq 0$, with equality if and only if I is free. This condition implies $c_1([M]) = 0$. □

8. Some corollaries of Theorem 7.9

In this section, we give some corollaries of Theorem 7.9. The following contains Corollary 1.3 in the introduction.

Corollary 8.1. *Let A be a local hypersurface with an isolated singularity and $\dim A = 3$. Assume that the assumptions of Theorem 6.1 hold. Let M be a finitely generated, torsion A-module. The following are equivalent:*

(1) $[M]$ is numerically equivalent to 0 in $G_0(A)$.
(2) $\theta^A(M, M) = 0$.
(3) $c_1([M]) = 0$ in $A_2(A)$.

Proof. The equivalence between (2) and (3) follows from Theorem 7.9. The condition (1) implies (2) by Corollary 6.2.

We shall prove (3) ⇒ (1). Consider

$$\tau([M]) = \tau_3([M]) + \tau_2([M]) + \tau_1([M]) + \tau_0([M]),$$

where $\tau_i([M]) \in A_i(A)_{\mathbb{Q}}$. (See Section 2.) By [23, Proposition 3.7], we know $\tau_i([M]) = 0$ in $\overline{A_i(A)}_{\mathbb{Q}}$ for $i = 0, 1$. By the equation (2.1) and the assumption, we have

$$\tau_2([M]) = c_1([M]) = 0$$

in $\overline{A_2(A)}_{\mathbb{Q}}$. Since M is a torsion module, $\tau_3([M]) = 0$. Therefore, $[M]$ is 0 in $\overline{G_0(A)}_{\mathbb{Q}}$ by the commutativity of the diagram (2.2). Since $\overline{G_0(A)}$ is torsion-free, $[M]$ itself is numerically equivalent to 0. □

It is worth pointing out that one can now compute the Grothendieck group modulo numerical equivalence for most isolated hypersurface singularities in dimension three.

Corollary 8.2. *Let A be a local hypersurface with an isolated singularity and $\dim A = 3$. Assume that the assumptions of Theorem 6.1 hold. Then the following hold:*

(1) The natural map $A_2(A) \to \overline{A_2(A)}$ is an isomorphism.
(2) The class group of A is finitely generated and torsion free.
(3) $\overline{G_0(A)} \simeq \mathbb{Z} \oplus A_2(A)$.

Proof. First, we shall prove (1). Let I be a reflexive ideal of A. Assume that the cycle $[\mathrm{Spec}(A/I)]$ in $A_2(A)$ is numerically equivalent to 0, that is, $[\mathrm{Spec}(A/I)] = 0$ in $\overline{A_2(A)}$. Consider

$$\tau([A/I]) = [\mathrm{Spec}(A/I)] + \tau_1([A/I]) + \tau_0([A/I]).$$

By [23, Proposition 3.7], we know $\tau_i([A/I]) = 0$ in $\overline{A_i(A)}_{\mathbb{Q}}$ for $i = 0, 1$. Then, by the commutativity of the diagram (2.2), $[A/I]$ in $G_0(A)$ is numerically equivalent to 0. Then, by Corollary 8.1, we have

$$[\mathrm{Spec}(A/I)] = c_1([A/I]) = 0$$

in $A_2(A)$.

Statement (2) follows from (1) since $\overline{A_2(A)}$ is torsion-free.

Next, we shall prove (3). At first, we shall construct a map

$$\overline{G_0(A)} \longrightarrow \mathbb{Z} \oplus \overline{A_2(A)}.$$

The map

$$\mathrm{rank} : G_0(A) \longrightarrow \mathbb{Z}$$

which is taking the rank of a module induces

$$\mathrm{rank} : \overline{G_0(A)} \longrightarrow \mathbb{Z}.$$

(To see this, consider the Koszul complex of a system of parameters of A. Then, we know that any cycle of positive rank in $G_0(A)$ is never numerically equivalent to 0.) Since the canonical module ω_A is isomorphic to A in this case, the map

$$c_1 \otimes 1 : G_0(A)_{\mathbb{Q}} \to A_2(A)_{\mathbb{Q}}$$

coincides with τ_2 by the equation (2.2). Then, we have the following commutative diagram:

$$\begin{array}{ccc} G_0(A) & \xrightarrow{c_1} & A_2(A) \\ \downarrow & & \downarrow \\ G_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_2} & A_2(A)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \overline{G_0(A)}_{\mathbb{Q}} & \xrightarrow{\overline{\tau_2}} & \overline{A_2(A)}_{\mathbb{Q}} \end{array}$$

Since $\overline{G_0(A)}$ is torsion-free, we have the induced map $\overline{c_1}$ that makes the following diagram commutative:

$$\begin{array}{ccc} G_0(A) & \xrightarrow{c_1} & A_2(A) \\ \downarrow & & \downarrow \\ \overline{G_0(A)} & \xrightarrow{\overline{c_1}} & \overline{A_2(A)} \end{array}$$

We define a map

$$\psi : \overline{G_0(A)} \longrightarrow \mathbb{Z} \oplus \overline{A_2(A)}$$

by $\psi([M]) = (\mathrm{rank}([M]), \overline{c_1}([M]))$. Since $\psi([A]) = (1, 0)$ and $\psi([A/I]) = (0, [\mathrm{Spec}(A/I)])$ for each reflexive ideal I of A, the map ψ is surjective. Since $\psi \otimes 1$ concides with the isomorphism

$$\overline{\tau} : \overline{G_0(A)}_{\mathbb{Q}} \longrightarrow \overline{A_*(A)}_{\mathbb{Q}}$$

and $\overline{G_0(A)}$ is torsion-free, ψ is injective. □

By Theorem 7.9 and Corollary 8.2, if A is three dimensional isolated hypersurface singularity with desingularization, θ^A is a positive definite form on $\overline{G_0(A)}_{\mathbb{Q}}/\mathbb{Q}[A] = \mathrm{Cl}(A)_{\mathbb{Q}}$.

Acknowledgments

We thank the referee for a very careful reading and many helpful comments. The proof of Theorem 7.7 was essentially worked out with Mark Walker during a visit of the first author to University of Nebraska in March 2010. The first author would like to thank University of Nebraska for its hospitality and Mark Walker for allowing him to include the result here. The second author would like to thank University of Kansas for its hospitality during his visits in May 2011 and July 2012, during which most of this work was done. The first author thanks Charles Vial for several interesting conversations about the first version of this work. The second author thanks Vasudevan Srinivas for valuable comments.

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