THE DIVISOR CLASS GROUPS AND THE GRADED CANONICAL MODULES OF MULTI-SECTION RINGS

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Abstract. We shall describe the divisor class group and the graded canonical module of the multi-section ring $T(X; D_1, \ldots, D_s)$ defined in (1.1) below for a normal projective variety X and Weil divisors $D_1, \ldots, D_s$ on X under a mild condition. In the proof, we use the theory of Krull domain and the equivariant twisted inverse functor due to Hashimoto [4].

1. Introduction

We shall describe the divisor class groups and the graded canonical modules of multi-section rings associated with a normal projective variety.

Suppose that $\mathbb{Z}$, $\mathbb{N}_0$ and $\mathbb{N}$ are the set of integers, non-negative integers and positive integers, respectively.

Let X be a normal projective variety over a field k with the function field $k(X)$. We always assume $\dim X > 0$. We denote by $C^1(X)$ the set of closed subvarieties of X of codimension 1. For $V \in C^1(X)$ and $a \in k(X)^\times$, we define as

$$\begin{array}{rcl} \mathrm{ord}_V(a) & = & \ell_{\mathcal{O}_{X,V}}(\mathcal{O}_{X,V}/\alpha\mathcal{O}_{X,V}) - \ell_{\mathcal{O}_{X,V}}(\mathcal{O}_{X,V}/\beta\mathcal{O}_{X,V}) \\ \mathrm{div}_X(a) & = & \displaystyle\sum_{V \in C^1(X)} \mathrm{ord}_V(a) \cdot V \in \mathrm{Div}(X) = \bigoplus_{V \in C^1(X)} \mathbb{Z} \cdot V, \end{array}$$

where α and β are elements in $\mathcal{O}_{X,V}$ such that $a = \alpha/\beta$, and $\ell_{\mathcal{O}_{X,V}}(\ )$ denotes the length as an $\mathcal{O}_{X,V}$-module.

We call an element in $\mathrm{Div}(X)$ a *Weil divisor* on X. For a Weil divisor $D = \sum n_V V$, we say that D is *effective*, and write $D \geq 0$, if $n_V \geq 0$ for any $V \in C^1(X)$. For a Weil divisor D on X, we put

$$H^0(X, \mathcal{O}_X(D)) = \{a \in k(X)^\times \mid \mathrm{div}_X(a) + D \geq 0\} \cup \{0\}.$$

Here we note that $H^0(X, \mathcal{O}_X(D))$ is a k-vector subspace of $k(X)$.

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Let $D_1, \ldots, D_s$ be Weil divisors on X. We define the multi-section rings $T(X; D_1, \ldots, D_s)$ and $R(X; D_1, \ldots, D_s)$ associated with $D_1, \ldots, D_s$ as follows:

$$(1.1) \qquad T(X; D_1, \ldots, D_s) = \bigoplus_{(n_1,\ldots,n_s)\in \mathbb{N}_0{}^s} H^0(X, \mathcal{O}_X(\sum_i n_i D_i)) t_1^{n_1} \cdots t_s^{n_s} \subset k(X)[t_1, \ldots, t_s]$$

$$R(X; D_1, \ldots, D_s) = \bigoplus_{(n_1,\ldots,n_s)\in \mathbb{Z}^s} H^0(X, \mathcal{O}_X(\sum_i n_i D_i)) t_1^{n_1} \cdots t_s^{n_s} \subset k(X)[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$$

We want to describe the divisor class groups and the graded canonical modules of the above rings.

For a Weil divisor F on X, we set

$$M_F = \bigoplus_{(n_1,\ldots,n_s)\in \mathbb{Z}^s} H^0(X, \mathcal{O}_X(\sum_i n_i D_i + F)) t_1^{n_1} \cdots t_s^{n_s} \subset k(X)[t_1^{\pm 1}, \ldots, t_s^{\pm 1}],$$

that is, M_F is a $\mathbb{Z}^s$-graded reflexive $R(X; D_1, \ldots, D_s)$-module with

$$[M_F]_{(n_1,\ldots,n_s)} = H^0(X, \mathcal{O}_X(\sum_i n_i D_i + F)) t_1^{n_1} \cdots t_s^{n_s}.$$

We denote by $\overline{M_F}$ the isomorphism class of the reflexive module M_F in $\mathrm{Cl}(R(X; D_1, \ldots, D_s))$.

For a normal variety X, we denote by $\mathrm{Cl}(X)$ the class group of X, and for a Weil divisor F on X, we denote by $\overline{F}$ the residue class represented by the Weil divisor F in $\mathrm{Cl}(X)$.

In the case where $\mathrm{Cl}(X)$ is freely generated by $\overline{D_1}, \ldots, \overline{D_s}$, the ring $R(X; D_1, \ldots, D_s)$ is usually called the *Cox ring* of X and denoted by $\mathrm{Cox}(X)$.

Remark 1.1. Assume that D is an ample divisor on X. In this case, $T(X; D)$ coincides with $R(X; D)$, and it is a Noetherian normal domain by a famous result of Zariski (see Lemma 2.8 in [6]). It is well-known that $\mathrm{Cl}(T(X; D))$ is isomorphic to $\mathrm{Cl}(X)/\mathbb{Z}\overline{D}$. Mori [8] constructed a lot of examples of non-Cohen Macaulay factorial domains using this isomorphism.

It is well-known that the canonical module of $T(X; D)$ is isomorphic to M_{K_X}, and the canonical sheaf ω_X coincides with $\widetilde{M_{K_X}}$. Watanabe proved a more general result in Theorem (2.8) in [12].

We want to establish the same type of the above results for multi-section rings.

For $R(X; D_1, \ldots, D_s)$, we had already proven the following:

Theorem 1.2 (Elizondo-Kurano-Watanabe [2], Hashimoto-Kurano [5])**.** *Let X be a normal projective variety over a field such that $\dim X > 0$. Assume that $D_1, \ldots, D_s$ are Weil divisors on X such that $\mathbb{Z}D_1 + \cdots + \mathbb{Z}D_s$ contains an ample Cartier divisor. Then, we have the following:*

(1) $R(X; D_1, \ldots, D_s)$ is a Krull domain.

(2) The set $\{P_V \mid V \in C^1(X)\}$ coincides with the set of homogeneous prime ideals of $R(X; D_1, \ldots, D_s)$ of height 1, where $P_V = M_{-V}$.

(3) *We have an exact sequence*

$$0 \longrightarrow \sum_i \mathbb{Z}\overline{D_i} \longrightarrow \mathrm{Cl}(X) \xrightarrow{p} \mathrm{Cl}(R(X; D_1, \dots, D_s)) \longrightarrow 0$$

such that $p(\overline{F}) = \overline{M_F}$.

(4) *Assume that* $R(X; D_1, \dots, D_s)$ *is Noetherian. Then* $\omega_{R(X;D_1,\dots,D_s)}$ *is isomorphic to* M_{K_X} *as a* $\mathbb{Z}^s$*-graded module. Therefore,* $\omega_{R(X;D_1,\dots,D_s)}$ *is* $R(X; D_1, \dots, D_s)$*-free if and only if* $\overline{K_X} \in \sum_i \mathbb{Z}\overline{D_i}$ *in* $\mathrm{Cl}(X)$.

Suppose that $\mathrm{Cl}(X)$ is finitely generated free $\mathbb{Z}$-module generated by $\overline{D_1}, \dots, \overline{D_s}$. By the above theorem, the Cox ring $\mathrm{Cox}(X)$ is factorial and

$$\omega_{\mathrm{Cox}(X)} = M_{K_X} = \mathrm{Cox}(X)(\overline{K_X}),$$

where we regard $\mathrm{Cox}(X)$ as a $\mathrm{Cl}(X)$-graded ring.

The main result of this paper is the following:

Theorem 1.3. *Let X be a normal projective variety over a field k such that $d = \dim X > 0$. Assume that $D_1, \dots, D_s$ are Weil divisors on X such that $\mathbb{N}D_1 + \cdots + \mathbb{N}D_s$ contains an ample Cartier divisor. Put*

$$U = \{j \mid \mathrm{tr.deg}_k T(X; D_1, \dots, D_{j-1}, D_{j+1}, \dots, D_s) = d + s - 1\}.$$

Then, we have the following:

(1) $T(X; D_1, \dots, D_s)$ *is a Krull domain.*

(2) *The set*

$$\{Q_V \mid V \in C^1(X)\} \cup \{Q_j \mid j \in U\}$$

coincides with the set of homogeneous prime ideals of $T(X; D_1, \dots, D_s)$ *of height* 1, *where*

$$Q_V = P_V \cap T(X; D_1, \dots, D_s)$$

and

$$Q_j = \bigoplus_{\substack{n_1,\dots,n_s \in \mathbb{N}_0 \\ n_j > 0}} T(X; D_1, \dots, D_s)_{(n_1,\dots,n_s)}.$$

(3) *We have an exact sequence*

$$0 \longrightarrow \sum_{j \notin U} \mathbb{Z}\overline{D_j} \longrightarrow \mathrm{Cl}(X) \xrightarrow{q} \mathrm{Cl}(T(X; D_1, \dots, D_s)) \longrightarrow 0$$

such that $q(\overline{F}) = \overline{M_F \cap k(X)[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}]}$.

(4) *Assume that* $T(X; D_1, \dots, D_s)$ *is Noetherian. Then* $\omega_{T(X;D_1,\dots,D_s)}$ *is isomorphic to*

$$M_{K_X} \cap t_1 \cdots t_s k(X)[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}]$$

as a $\mathbb{Z}^s$*-graded module. Further, we have*

$$q(\overline{K_X + \sum_i D_i}) = \overline{\omega_{T(X;D_1,\dots,D_s)}}.$$

Therefore, $\omega_{T(X;D_1,\ldots,D_s)}$ *is* $T(X;D_1,\ldots,D_s)$*-free if and only if*

$$\overline{K_X + \sum_i D_i} \in \sum_{j\notin U} \mathbb{Z}\overline{D_j}$$

in $\mathrm{Cl}(X)$*.*

Here, $\mathrm{tr.deg}_k T$ denotes the transcendence degree of the fractional field of T over a field k.

Remark 1.4. With notation as in the previous theorem, $\mathrm{ht}(Q_j) = 1$ if and only if $j \in U$. This will be proven in Lemma 3.3. Since $\mathbb{N}D_1 + \cdots + \mathbb{N}D_s$ contains an ample Cartier divisor, $Q_j \neq (0)$ for any j. Therefore, $\mathrm{ht}(Q_j) \geq 2$ if and only if $j \notin U$.

2. Examples

Example 2.1. Let X be a normal projective variety with $\dim X > 0$. Assume that all of D_i's are ample Cartier divisors on X. Then, $T(X;D_1,\ldots,D_s)$ is Noetherian by a famous result of Zariski (see Lemma 2.8 in [6]).

Assume that $s = 1$. By definition, $U = \emptyset$ since $\dim X > 0$. By Theorem 1.3 (3), $\mathrm{Cl}(T(X;D_1))$ is isomorphic to $\mathrm{Cl}(X)/\mathbb{Z}\overline{D_1}$. By Theorem 1.3 (4), $\omega_{T(X;D_1)}$ is $T(X;D_1)$-free module if and only if

$$\overline{K_X} \in \mathbb{Z}\overline{D_1}$$

in $\mathrm{Cl}(X)$ (see Remark 1.1).

Next, assume that $s \geq 2$. In this case, $U = \{1, 2, \ldots, s\}$. By Theorem 1.3 (3), $\mathrm{Cl}(X)$ is isomorphic to $\mathrm{Cl}(T(X;D_1,\ldots,D_s))$. By Theorem 1.3 (4), $\omega_{T(X;D_1,\ldots,D_s)}$ is $T(X;D_1,\ldots,D_s)$-free module if and only if

$$\overline{K_X} = \overline{-D_1 - \cdots - D_s}$$

in $\mathrm{Cl}(X)$. When this is the case, $-K_X$ is ample, that is, X is a Fano variety.

Example 2.2. Set $X = \mathbb{P}^m \times \mathbb{P}^n$. Let p_1 (resp. p_2) be the first (resp. second) projection.

Let H_1 be a hyperplane of $\mathbb{P}^m$, and H_2 a hyperplane of $\mathbb{P}^n$. Put $A_i = p_i^{-1}(H_i)$ for $i = 1, 2$. In this case, $\mathrm{Cl}(X) = \mathbb{Z}\overline{A_1} + \mathbb{Z}\overline{A_2} \simeq \mathbb{Z}^2$, and $K_X = -(m+1)A_1 - (n+1)A_2$.

We have

$$\mathrm{Cox}(X) = R(X;A_1,A_2) = k[x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n].$$

$\mathrm{Cox}(X)$ is a $\mathbb{Z}^2$-graded ring such that x_i's (resp. y_j's) are of degree $(1,0)$ (resp. $(0,1)$).

Let a, b, c, d be positive integers such that $ad - bc \neq 0$. Put $D_1 = aA_1 + bA_2$ and $D_2 = cA_1 + dA_2$. Then, both D_1 and D_2 are ample divisors. Consider the multi-section rings:

$$\begin{aligned} R(X;D_1,D_2) &= \oplus_{p,q\in\mathbb{Z}} \mathrm{Cox}(X)_{p(a,b)+q(c,d)} \\ T(X;D_1,D_2) &= \oplus_{p,q\geq 0} \mathrm{Cox}(X)_{p(a,b)+q(c,d)} \end{aligned}$$

Here, both $R(X;D_1,D_2)$ and $T(X;D_1,D_2)$ are Cohen-Macaulay rings.

By Theorem 1.2 (4), we know

$$\begin{array}{rcl} R(X;D_1,D_2) \text{ is a Gorenstein ring} & \iff & \overline{K_X} \in \mathbb{Z}\overline{D_1} + \mathbb{Z}\overline{D_2} \text{ in } \mathrm{Cl}(X) \\ & \iff & (m+1,n+1) \in \mathbb{Z}(a,b) + \mathbb{Z}(c,d). \end{array}$$

In this case, we have $U = \{1,2\}$ since all of a, b, c and d are positive. By Theorem 1.3 (4), we have

$$\begin{array}{rcl} T(X;D_1,D_2) \text{ is a Gorenstein ring} & \iff & \overline{K_X + D_1 + D_2} = 0 \text{ in } \mathrm{Cl}(X) \\ & \iff & m+1 = a+c \text{ and } n+1 = b+d. \end{array}$$

Example 2.3. Let a, b, c be pairwise coprime positive integers. Let $\mathfrak{p}$ be the kernel of the k-algebra map $S = k[x,y,z] \to k[T]$ given by $x \mapsto T^a$, $y \mapsto T^b$, $z \mapsto T^c$.

Let $\pi : X \to \mathbb{P} = \mathrm{Proj}(k[x,y,z])$ be the blow-up at $V_+(\mathfrak{p})$, where $a = \deg(x)$, $b = \deg(y)$, $c = \deg(z)$. Put $E = \pi^{-1}(V_+(\mathfrak{p}))$. Let A be a Weil divisor on X satisfying $\pi^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_X(A)$. In this case, we have $\mathrm{Cl}(X) = \mathbb{Z}\overline{E} + \mathbb{Z}\overline{A} \simeq \mathbb{Z}^2$, and $K_X = E - (a+b+c)A$.

Then, we have

$$\begin{array}{rcl} \mathrm{Cox}(X) = R(X;-E,A) & = & R'_s(\mathfrak{p}) := S[t^{-1}, \mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3, \ldots] \subset S[t^{\pm 1}], \\ T(X;-E,A) & = & R_s(\mathfrak{p}) := S[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3, \ldots] \subset S[t]. \end{array}$$

By Theorem 1.2 (4), we have

$$\omega_{R'_s(\mathfrak{p})} = M_{K_X} = R'_s(\mathfrak{p})(\overline{K_X}) = R'_s(\mathfrak{p})(-1, -a-b-c).$$

In this case, $U = \{1\}$. By Theorem 1.3 (4),

$$\begin{array}{rcl} \omega_{R_s(\mathfrak{p})} & = & M_{K_X} \cap t_1t_2k(X)[t_1, t_2^{\pm 1}] \\ & = & \omega_{R'_s(\mathfrak{p})} \cap t_1t_2k(X)[t_1, t_2^{\pm 1}] \\ & = & R'_s(\mathfrak{p})(-1, -a-b-c) \cap t_1t_2k(X)[t_1, t_2^{\pm 1}] \\ & = & R_s(\mathfrak{p})(-1, -a-b-c). \end{array}$$

Therefore, both of $R'_s(\mathfrak{p})$ and $R_s(\mathfrak{p})$ are quasi-Gorenstein rings, that were first proven by Simis-Trung (Corollary 3.4 in [11]). Cohen-Macaulayness of such rings are deeply studied by Goto-Nishida-Shimoda [3].

Divisor class groups of ordinary and symbolic Rees rings were studied by Shimoda [10], Simis-Trung [11], etc.

3. Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3.

Throughout this section, we assume that X is a normal projective variety over a field k such that $d = \dim X > 0$, and $D_1, \ldots, D_s$ are Weil divisors on X such that $\mathbb{N}D_1 + \cdots + \mathbb{N}D_s$ contains an ample Cartier divisor.

We need the following lemmta. They are well-known, but the author could not find a reference.

Lemma 3.1. *Let G be an integral domain containing a field k. Let P be a prime ideal of G. Assume that both* $\mathrm{tr.deg}_k G$ *and* $\mathrm{tr.deg}_k G/P$ *are finite.*

Then, the height of P is less than or equal to

$$\mathrm{tr.deg}_k G - \mathrm{tr.deg}_k G/P.$$

Proof. Assume the contrary. Then there exists a ring G' which satisfies the following five conditions:

- $k \subset G' \subset G$.
- G' is finitely generated (as a ring) over k.
- $\mathrm{tr.deg}_k G = \mathrm{tr.deg}_k G'$.
- $\mathrm{tr.deg}_k G/P = \mathrm{tr.deg}_k G'/(G' \cap P)$.
- $\mathrm{tr.deg}_k G - \mathrm{tr.deg}_k G/P < \mathrm{ht}(G' \cap P)$.

However, using the dimension formula (e.g. 119p in [7]), we have

$$\mathrm{ht}(G' \cap P) = \mathrm{tr.deg}_k G' - \mathrm{tr.deg}_k G'/(G' \cap P) = \mathrm{tr.deg}_k G - \mathrm{tr.deg}_k G/P.$$

It is a contradiction. **q.e.d.**

Lemma 3.2. *Let r be a positive integer. Let $F_1, \ldots, F_r$ be Weil divisors on X. Let S be the set of all non-zero homogeneous elements of $T(X; F_1, \ldots, F_r)$. Then the following conditions are equivalent:*

(1) There exist non-negative integers $q_1, \ldots, q_r$ such that $\sum_{i=1}^r q_i F_i$ is linearly equivalent to a sum of an ample Cartier divisor and an effective Weil divisor.

(2) There exist positive integers $q_1, \ldots, q_r$ such that $\sum_{i=1}^r q_i F_i$ is linearly equivalent to a sum of an ample Cartier divisor and an effective Weil divisor.

(3) $S^{-1}(T(X; F_1, \ldots, F_r)) = k(X)[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$.

(4) $Q(T(X; F_1, \ldots, F_r)) = k(X)(t_1, \ldots, t_r)$, where $Q(\ )$ denotes the field of fractions.

(5) $\mathrm{tr.deg}_k T(X; F_1, \ldots, F_r) = \dim X + r$.

Using Theorem 1.5.5 in [1], it is easy to see that $T(X; F_1, \ldots, F_r)$ is Noetherian if and only if $T(X; F_1, \ldots, F_r)$ is finitely generated (as a ring) over the field $H^0(X, \mathcal{O}_X)$. Therefore, if $T(X; F_1, \ldots, F_r)$ is Noetherian, then the condition (5) is equivalent to that the Krull dimension of $T(X; F_1, \ldots, F_r)$ is $\dim X + r$.

Proof. (2) $\Rightarrow$ (1), and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial.

First we shall prove (1) $\Rightarrow$ (3). Suppose

$$\sum_{i=1}^{r} q_i F_i \sim D + F,$$

where q_i's are non-negative integers, D is a very ample Cartier divisor and F is an effective divisor. We put

$$(3.1) \qquad C = \bigoplus_{m \in \mathbb{Z}} \bigoplus_{(n_1, \ldots, n_r) \in \mathbb{N}_0{}^r} H^0(X, \mathcal{O}_X(\sum_i n_i F_i + mD)) t_1^{n_1} \cdots t_r^{n_r} t_{r+1}^m \subset k(X)[t_1, \ldots, t_r, t_{r+1}^{\pm 1}].$$

We regard C as a $\mathbb{Z}^{r+1}$-graded ring with

$$C_{(n_1,\ldots,n_r,m)} = H^0(X, \mathcal{O}_X(\sum_i n_i F_i + mD))t_1^{n_1}\cdots t_r^{n_r}t_{r+1}^m.$$

Then, we have

$$T(X; F_1, \ldots, F_r) = \bigoplus_{(n_1,\ldots,n_r)\in\mathbb{N}_0{}^r} C_{(n_1,\ldots,n_r,0)},$$

so $T(X; F_1, \ldots, F_r)$ is a subring of C. Thus, $S^{-1}C$ is a $\mathbb{Z}^{r+1}$-graded ring such that

$$S^{-1}T(X; F_1, \ldots, F_r) = \bigoplus_{(n_1,\ldots,n_r)\in\mathbb{N}_0{}^r} (S^{-1}C)_{(n_1,\ldots,n_r,0)}.$$

Since $\sum_{i=1}^r q_i F_i - D$ is linearly equivalent to an effective divisor F, there exists a non-zero element a in

$$H^0(X, \mathcal{O}_X(\sum_i q_i F_i - D)).$$

For any $0 \neq b \in H^0(X, \mathcal{O}_X(D))$,

$$(at_1^{q_1}\cdots t_r^{q_r}t_{r+1}^{-1})(bt_{r+1})$$

is contained in S. Therefore, $S^{-1}C$ contains $(bt_{r+1})^{-1}$. Hence, $k(X)$ is contained in $S^{-1}C$. Since $k(X) = (S^{-1}C)_{(0,\ldots,0)}$, $k(X)$ is contained in $S^{-1}T(X; F_1, \ldots, F_r)$.

By the assumption of (1), there exists a positive integer ℓ such that

$$(S^{-1}C)_{(\ell q_1,\ldots,\ell q_r,0)} \neq 0$$

and

$$(S^{-1}C)_{(\ell q_1+1,\ell q_2,\ldots,\ell q_r,0)} \neq 0.$$

Then, it is easy to see that $t_1 \in S^{-1}C$. Therefore, $S^{-1}C$ contains $k(X)[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$. Hence $S^{-1}T(X; F_1, \ldots, F_r)$ coincides with $k(X)[t_1^{\pm}, \ldots, t_r^{\pm}]$.

Next, we shall prove (5) $\Rightarrow$ (2). Let D be a very ample divisor. Consider the ring

$$R(X; F_1, \ldots, F_r, D).$$

First, assume that

$$H^0(X, \mathcal{O}_X(\sum_i u_i F_i - vD)) \neq 0$$

for some integers u_1, ..., u_r, v such that $v > 0$. By the assumption (5), there exists positive integers u'_1, ..., u'_r such that

$$H^0(X, \mathcal{O}_X(\sum_i u'_i F_i)) \neq 0.$$

Therefore we may assume that there exists positive integers u_1, ..., u_r and v such that

$$H^0(X, \mathcal{O}_X(\sum_i u_i F_i - vD)) \neq 0.$$

Here, we have

$$\sum_i u_i F_i = vD + (\sum_i u_i F_i - vD).$$

Therefore $\sum_i u_iF_i$ is the sum of an ample divisor vD and the divisor $\sum_i u_iF_i - vD$ which is linearly equivalent to an effective divisor.

Next, assume that for any integers u_1, ..., u_r and v,

$$H^0(X, \mathcal{O}_X(\sum_i u_iF_i - vD)) = 0 \tag{3.2}$$

if $v > 0$. We put

$$P = \bigoplus_{\substack{(n_1,\dots,n_r,m)\in\mathbb{Z}^{r+1} \\ m>0}} R(X; F_1, \dots, F_r, D)_{(n_1,\dots,n_r,m)}.$$

By the assumption (5), P is a prime ideal of $R(X; F_1, \dots, F_r, D)$ of height 1 by Lemma 3.1. (Here, since D is an ample divisor, $\mathrm{tr.deg}_k R(X; F_1, \dots, F_r, D) = \dim X + r + 1$. Note that P is an ideal of $R(X; F_1, \dots, F_r, D)$ by (3.2) above. By (5), $\mathrm{tr.deg}_k R(X; F_1, \dots, F_r, D)/P = \dim X + r$.) However $R(X; F_1, \dots, F_r, D)$ has no homogeneous prime ideal of height 1 that contains

$$H^0(X, \mathcal{O}_X(D))t_{r+1}$$

by Theorem 1.2 (2). This is a contradiction. **q.e.d.**

Put $A = k(X)[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ and $B = k(X)[t_1, \dots, t_s]$. Recall that D_1, ..., D_s are Weil divisors on a normal projective variety X such that $\mathbb{N}D_1 + \cdots + \mathbb{N}D_s$ contains an ample Cartier divisor. We denote $T(X; D_1, \dots, D_s)$ and $R(X; D_1, \dots, D_s)$ simply by T and R, respectively.

Since

$$T = R \cap B,$$

T is a Krull domain. We have proven Theorem 1.3 (1).

By Theorem 1.2 (2), we have

$$\begin{aligned} R &= \left(\bigcap_{V \in C^1(X)} R_{P_V} \right) \cap A \\ A &= \bigcap_{P \in \mathrm{NHP}^1(R)} R_P, \end{aligned}$$

where $\mathrm{NHP}^1(R)$ is the set of non-homogeneous prime ideals of R of height 1.

It is easy to see $R_P = T_{P\cap T}$ for $P \in \mathrm{NHP}^1(R)$. Therefore, we have

$$A = \bigcap_{P \in \mathrm{NHP}^1(R)} T_{P\cap T}.$$

Since $T_{P\cap T}$ is a discrete valuation ring, $P \cap T$ is a non-homogeneous prime ideal of T of height 1.

For $V \in C^1(X)$, put $Q_V = P_V \cap T$. Then, $R_{P_V} = T_{Q_V}$, since $\sum_i \mathbb{N}D_i$ contains an ample divisor. Therefore Q_V is a homogeneous prime ideal of T of height 1.

On the other hand, we have $Q_i = T \cap t_i B_{(t_i)}$ and $T_{Q_i} \subset B_{(t_i)}$. Note that

$$B = A \cap (\bigcap_{j=1}^{s} B_{(t_j)}).$$

Then, we have

$$\begin{aligned} T &= R \cap B \\ &= \left(\bigcap_{V \in C^1(X)} R_{P_V} \right) \cap A \cap B \qquad (3.3) \\ &= \left(\bigcap_{V \in C^1(X)} T_{Q_V} \right) \cap \left(\bigcap_{P \in \mathrm{NHP}^1(R)} T_{P \cap T} \right) \cap \left(\bigcap_{j=1}^{s} B_{(t_j)} \right). \end{aligned}$$

Put

$$T_j = \bigoplus_{(n_1,\ldots,n_{j-1},n_{j+1},\ldots,n_s) \in \mathbb{N}_0{}^{s-1}} H^0(X, \mathcal{O}_X(\sum_{i \neq j} n_i D_i)) t_1^{n_1} \cdots t_{j-1}^{n_{j-1}} t_{j+1}^{n_{j+1}} \cdots t_s^{n_s}.$$

We need the following lemma.

Lemma 3.3. *With notation as above, the following conditions are equivalent:*

(1) $T_{Q_j} = B_{(t_j)}$.
(2) The height of Q_j *is* 1.
(3) The height of Q_j *is less than* 2.
(4) $j \in U$*, that is,* $\mathrm{tr.deg}_k T_j = d + s - 1$.

Proof. By Lemma 3.2, we have $Q(T) = Q(B)$. It is easy to see that $B_{(t_j)}$ is a discrete valuation ring. Since Q_j is a non-zero prime ideal of a Krull domain T, the equivalence of (1), (2) and (3) are easy.

Here, we shall prove (1) $\Rightarrow$ (4). Note that $T/Q_j = T_j$. Then, we have

$$Q(T_j) = T_{Q_j}/Q_j T_{Q_j} = B_{(t_i)}/(t_i) B_{(t_i)} = k(X)(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_s).$$

The implication (4) $\Rightarrow$ (3) immediately follows from

$$\mathrm{ht}(Q_j) \le \mathrm{tr.deg}_k T - \mathrm{tr.deg}_k (T_j) = 1.$$

This inequality follows from Lemma 3.1 and the fact $T_j = T/Q_j$. **q.e.d.**

By (3.3), Lemma 3.3 and Theorem 12.3 in [7], we know that

$$\{Q_V \mid V \in C^1(X)\} \cup \{Q_j \mid j \in U\}$$

is the set of homogeneous prime ideals of T of height 1, and

$$\{P \cap T \mid P \in \mathrm{NHP}^1(R)\}$$

is the set of non-homogeneous prime ideals of T of height 1. Further we obtain

$$T = \left(\bigcap_{V \in C^1(X)} T_{Q_V} \right) \cap \left(\bigcap_{P \in \mathrm{NHP}^1(R)} T_{P \cap T} \right) \cap \left(\bigcap_{j \in U} T_{Q_j} \right).$$

The proof of Theorem 1.3 (2) is completed.

Let

$$\mathrm{Div}(X) = \bigoplus_{V \in C^1(X)} \mathbb{Z} \cdot V$$

be the set of Weil divisors on X. Let

$$\mathrm{HDiv}(T) = \left(\bigoplus_{V \in C^1(X)} \mathbb{Z} \cdot \mathrm{Spec}(T/Q_V) \right) \oplus \left(\bigoplus_{j \in U} \mathbb{Z} \cdot \mathrm{Spec}(T/Q_j) \right)$$

be the set of homogeneous Weil divisors of $\mathrm{Spec}(T)$.

Here, we define

$$\phi : \mathrm{Div}(X) \longrightarrow \mathrm{HDiv}(T)$$

by $\phi(V) = \mathrm{Spec}(T/Q_V)$ for each $V \in C^1(X)$. Then, it satisfies the following:

- For each $a \in k(X)^{\times}$, we have
$$\phi(\mathrm{div}_X(a)) = \mathrm{div}_T(a) \in \bigoplus_{V \in C^1(X)} \mathbb{Z} \cdot \mathrm{Spec}(T/Q_V) \subset \mathrm{HDiv}(T).$$
- If $j \in U$, then
$$\mathrm{div}_T(t_j) = \mathrm{Spec}(T/Q_j) + \phi(D_j).$$
- If $j \notin U$, then
$$\mathrm{div}_T(t_j) = \phi(D_j).$$

They are proven essentially in the same way as in pp631–632 in [2]. Then, we have an exact sequence

$$0 \longrightarrow \sum_{j \notin U} \mathbb{Z}\overline{D_j} \longrightarrow \mathrm{Cl}(X) \xrightarrow{q} \mathrm{Cl}(T) \longrightarrow 0$$

such that $q(\overline{F}) = \overline{\phi(F)}$ in $\mathrm{Cl}(T)$. Here, remember that $\mathrm{Cl}(T)$ coincides with $\mathrm{HDiv}(T)$ divided by the group of homogeneous principal divisors (e.g., Proposition 7.1 in Samuel [9]).

It is easy to see that the class of the Weil divisor $q(\overline{F})$ corresponds to the isomorphism class of the reflexive module

$$\begin{aligned} M_F \cap \left(\bigcap_{j \in U} T_{Q_j} \right) &= M_F \cap A \cap \left(\bigcap_{j \in U} T_{Q_j} \right) \\ &= M_F \cap k(X)[t_1, \ldots, t_s, \{t_j^{-1} \mid j \notin U\}]. \end{aligned}$$

The proof of Theorem 1.3 (3) is completed.

Remark 3.4. It is easy to see

$$t_1^{d_1} \cdots t_s^{d_s} M_{F + \sum_i d_i D_i} = M_F$$

for any integers d_1, …, d_s. Therefore, we have

$$\begin{aligned} & M_F \cap t_1^{d_1} \cdots t_s^{d_s} k(X)[t_1, \ldots, t_s, \{t_j^{-1} \mid j \notin U\}] \\ = \ & t_1^{d_1} \cdots t_s^{d_s} \left(M_{F + \sum_i d_i D_i} \cap k(X)[t_1, \ldots, t_s, \{t_j^{-1} \mid j \notin U\}] \right). \end{aligned}$$

Hence,

$$M_F \cap t_1^{d_1} \cdots t_s^{d_s} k(X)[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}]$$

is isomorphic to

$$M_{F+\sum_i d_i D_i} \cap k(X)[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}] \tag{3.4}$$

as a T-module. Note that this is not an isomorphism as $\mathbb{Z}^s$-graded modules. The isomorphism class which the module (3.4) belongs to coincides with $q(\overline{F + \sum_i d_i D_i})$.

In the rest, we assume that T is Noetherian. We shall prove that ω_T is isomorphic to

$$M_{K_X} \cap t_1 \cdots t_s k(X)[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}]$$

as a $\mathbb{Z}^s$-graded module. (Suppose that it is true. If we forget the grading, it is isomorphic to

$$M_{K_X + \sum_i D_i} \cap k(X)[t_1, \dots, t_s, \{t_j^{-1} \mid j \notin U\}]$$

by Remark 3.4, that is corresponding to $q(\overline{K_X + \sum_i D_i})$ in $\mathrm{Cl}(T)$. Therefore, we know that ω_T is T-free if and only if

$$\overline{K_X + \sum_i D_i} \in \sum_{j \notin U} \mathbb{Z}\overline{D_j}$$

in $\mathrm{Cl}(X)$.)

Put $X' = X \setminus \mathrm{Sing}(X)$. We choose positive integers $a_1, \dots, a_s$ and sections $f_1, \dots, f_t \in H^0(X, \sum_i a_i D_i)$ such that

- $\sum_i a_i D_i$ is an ample Cartier divisor,
- $X' = \cup_k D_+(f_k)$, and
- all of the D_i's are principal Cartier divisors on $D_+(f_k)$ for $k = 1, \dots, t$.

Put $W = \{\underline{n} \in \mathbb{Z}^s \mid n_i \geq 0 \text{ if } i \in U\}$. Put $D'_i = D_i|_{X'}$ for $i = 1, \dots, s$. Consider the morphism

$$Y = \mathrm{Spec}_{X'} \left(\bigoplus_{\underline{n} \in W} \mathcal{O}_{X'}(\sum_i n_i D'_i) t_1^{n_1} \cdots t_s^{n_s} \right) \xrightarrow{\pi} X'.$$

Further, we have the natural map

$$\xi : Y \longrightarrow \mathrm{Spec}(T).$$

The group $\mathbb{G}_m^s$ naturally acts on $\mathrm{Spec}(T)$ and Y, and trivially acts on X'. Both π and ξ are equivariant morphisms.

Claim 3.5. *There exists an equivariant open subscheme Z of both Y and* $\mathrm{Spec}(T)$ *such that*

- *the codimension of $Y \setminus Z$ in Y is bigger than or equal to 2, and*
- *the codimension of* $\mathrm{Spec}(T) \setminus Z$ *in* $\mathrm{Spec}(T)$ *is bigger than or equal to 2.*

Proof. For $u \in U$, there exist integers $c_{1u}, \dots, c_{su}$ such that

- $H^0(X, \mathcal{O}_X(\sum_i c_{iu} D_i)) \neq 0$,
- $c_{uu} = -a_u$, and
- $c_{iu} > 0$ if $i \neq u$.

In fact, if $u \in U$, there exist positive integers $q_1, \ldots, q_{u-1}, q_{u+1}, \ldots, q_s$ such that

$$\sum_{i \neq u} q_i D_i$$

is a sum of an ample divisor D and a Weil divisor F which is linearly equivalent to an effective divisor by Lemma 3.2. Then,

$$H^0(X, \mathcal{O}_X(q(\sum_{i \neq u} q_i D_i) - a_u D_u) = H^0(X, \mathcal{O}_X(q(D+F) - a_u D_u) \neq 0$$

for $q \gg 0$.

For each $u \in U$, we set

$$(b_{1u}, \ldots, b_{su}) = (c_{1u}, \ldots, c_{su}) + (a_1, \ldots, a_s).$$

Here, note that $b_{uu} = 0$ and $b_{iu} > 0$ if $i \neq u$.

We choose

$$0 \neq g_u \in H^0(X, \mathcal{O}_X(\sum_i c_{iu} D_i))$$

for each $u \in U$.

Consider the closed set of $\mathrm{Spec}(T)$ defined by the ideal J generated by

$$\{f_k t_1^{a_1} \cdots t_s^{a_s} \mid k = 1, \ldots, t\}$$

and

$$\left\{ g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}} \;\middle|\; k = 1, \ldots, t;\ u \in U \right\}.$$

By Theorem 1.3 (2), we know that the height of J is bigger than or equal to 2 since there is no prime ideal of T of height one which contains J.

We choose $d_{ki} \in k(X)^\times$ satisfying

$$H^0(D_+(f_k), \mathcal{O}_X(D_i)) = d_{ki} H^0(D_+(f_k), \mathcal{O}_X)$$

for each k and i. Then

$$Y = \bigcup_{k=1}^{t} \pi^{-1}(D_+(f_k)) \ \text{ and } \ \pi^{-1}(D_+(f_k)) = \mathrm{Spec}(C_k), \tag{3.5}$$

where

$$C_k = H^0(D_+(f_k), \mathcal{O}_X)[d_{k1}t_1, \ldots, d_{ks}t_s, \{(d_{kj}t_j)^{-1} \mid j \notin U\}].$$

We put

$$Z = \mathrm{Spec}(T) \setminus V(J).$$

Then we have

$$Z = \bigcup_{k=1}^{t} \left[\mathrm{Spec}\left(T[(f_k t_1^{a_1} \cdots t_s^{a_s})^{-1}]\right) \cup \left\{ \bigcup_{u \in U} \mathrm{Spec}\left(T[(g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}]\right) \right\} \right]. \tag{3.6}$$

Here, we have

$$\begin{aligned} T[(f_k t_1^{a_1} \cdots t_s^{a_s})^{-1}] &= H^0(D_+(f_k), \mathcal{O}_X)[(d_{k1}t_1)^{\pm 1}, \ldots, (d_{ks}t_s)^{\pm 1}] \\ &= C_k[\left(\prod_{j \in U} (d_{kj} t_j)\right)^{-1}]. \end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned}
& T[(g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}] \\
= & \bigoplus_{(\underline{n}) \in \mathbb{Z}^s} T[(g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}]_{(n_1, \ldots, n_s)} \\
= & \bigoplus_{\substack{(\underline{n}) \in \mathbb{Z}^s \\ n_u \geq 0}} R[(g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}]_{(n_1, \ldots, n_s)} \\
= & \bigoplus_{\substack{(\underline{n}) \in \mathbb{Z}^s \\ n_u \geq 0}} R[(f_k t_1^{a_1} \cdots t_s^{a_s})^{-1}, (g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}]_{(n_1, \ldots, n_s)} \\
= & C_k[\{(d_{kj} t_j)^{-1} \mid j \neq u\}, (g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}].
\end{aligned}$$

Let β_{ku} be an element in $H^0(D_+(f_k), \mathcal{O}_X)$ such that

$$g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}} = \beta_{ku} (d_{k1} t_1)^{b_{1u}} \cdots (d_{ks} t_s)^{b_{su}}$$

for $k = 1, \ldots, t$ and $u \in U$. Then,

$$C_k[\{(d_{kj} t_j)^{-1} \mid j \neq u\}, (g_u f_k t_1^{b_{1u}} \cdots t_s^{b_{su}})^{-1}] = C_k[\left(\beta_{ku} \prod_{\substack{j \in U \\ j \neq u}} (d_{kj} t_j) \right)^{-1}]. \tag{3.8}$$

By (3.5), (3.6), (3.7) and (3.8), we know that Z is an open subscheme of Y. The ideal of C_k generated by

$$\prod_{j \in U} (d_{kj} t_j) \quad \text{and} \quad \left\{ \beta_{ku} \prod_{\substack{j \in U \\ j \neq u}} (d_{kj} t_j) \;\middle|\; u \in U \right\}$$

is the unit ideal or of height two. (If $U = \emptyset$, then $Z = Y$ by the construction. If $U = \{u\}$ and if β_{ku} is a unit element, then this ideal is the unit. In other cases, this ideal is of height 2.) Therefore, the codimension of $Y \setminus Z$ in Y is bigger than or equal to two. **q.e.d.**

We can define the graded canonical module as in Definition 3.1 in [5] using the theory of the equivariant twisted inverse functor [4].

By Claim 3.5 above and Remark 3.2 in [5], we have $\omega_T = H^0(Y, \omega_Y)$. On the other hand, we have

$$\begin{aligned}
\omega_Y &= \bigwedge^s \Omega_{Y/X'} \otimes \pi^* \mathcal{O}_{X'}(K_{X'}) \\
&= \pi^* \mathcal{O}_{X'}(\sum_i D'_i)(-1, \ldots, -1) \otimes_{\mathcal{O}_Y} \pi^* \mathcal{O}_{X'}(K_{X'}) \\
&= \pi^* \mathcal{O}_{X'}(\sum_i D'_i + K_{X'})(-1, \ldots, -1),
\end{aligned}$$

where $(-1, \ldots, -1)$ denotes the shift of degree (Theorem 28.11 in [4]).

Then, we have

$$H^0(Y,\omega_Y) = H^0(X', \pi_*\pi^*\mathcal{O}_{X'}(\sum_i D'_i + K_{X'})(-1,\ldots,-1)).$$

By the projection formula (Lemma 26.4 in [4]),

$$\begin{array}{rl} & \pi_*\pi^*\mathcal{O}_{X'}(\sum_i D'_i + K_{X'})(-1,\ldots,-1) \\ = & \left(\mathcal{O}_{X'}(\sum_i D'_i + K_{X'}) \otimes \pi_*\mathcal{O}_Y\right)(-1,\ldots,-1) \\ = & \left(\mathcal{O}_{X'}(\sum_i D'_i + K_{X'}) \otimes \left[\bigoplus_{\underline{n}\in W} \mathcal{O}_{X'}(\sum_i n_i D'_i)\right]\right)(-1,\ldots,-1) \\ = & \left(\bigoplus_{\underline{n}\in W} \mathcal{O}_{X'}(\sum_i (n_i+1)D'_i + K_{X'})\right)(-1,\ldots,-1) \\ = & \bigoplus_{\underline{n}\in W+(1,\ldots,1)} \mathcal{O}_{X'}(\sum_i n_i D'_i + K_{X'}). \end{array}$$

Therefore, we have

$$\begin{array}{rcl} H^0(Y,\omega_Y) & = & H^0(X', \bigoplus_{\underline{n}\in W+(1,\ldots,1)} \mathcal{O}_{X'}(\sum_i n_i D'_i + K_{X'})) \\ & = & \bigoplus_{\underline{n}\in W+(1,\ldots,1)} H^0(X', \mathcal{O}_{X'}(\sum_i n_i D'_i + K_{X'})) \\ & = & \bigoplus_{\underline{n}\in W+(1,\ldots,1)} H^0(X, \mathcal{O}_X(\sum_i n_i D_i + K_X)) \\ & = & M_{K_X} \cap t_1\cdots t_s k(X)[t_1,\ldots,t_s,\{t_j^{-1} \mid j \notin U\}]. \end{array}$$

We have completed the proof of Theorem 1.3.

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