On finite generation of symbolic Rees rings of space monomial curves and existence of negative curves

Kazuhiko Kurano and Naoyuki Matsuoka

Dedicated to Professor Paul C. Roberts
on the occasion of his 60th birthday

Abstract

In this paper, we shall study finite generation of symbolic Rees rings of the defining ideal of the space monomial curves (t^a, t^b, t^c) for pairwise coprime integers a, b, c such that $(a, b, c) \neq (1, 1, 1)$. If such a ring is not finitely generated over a base field, then it is a counterexample to the Hilbert's fourteenth problem. Finite generation of such rings is deeply related to existence of negative curves on certain normal projective surfaces. We study a sufficient condition (Definition 3.6) for existence of a negative curve. Using it, we prove that, in the case of $(a+b+c)^2 > abc$, a negative curve exists. Using a computer, we shall show that there exist examples in which this sufficient condition is not satisfied.

1 Introduction

Let k be a field. Let R be a polynomial ring over k with finitely many variables. For a field L satisfying $k \subset L \subset Q(R)$, Hilbert asked in 1900 whether the ring $L \cap R$ is finitely generated as a k-algebra or not. It is called the *Hilbert's fourteenth problem.*

The first counterexample to this problem was discovered by Nagata [14] in 1958. An easier counterexample was found by Paul C. Roberts [16] in 1990. Further counterexamples were given by Kuroda, Mukai, etc.

The Hilbert's fourteenth problem is deeply related to the following question of Cowsik [2]. Let R be a regular local ring (or a polynomial ring over a field). Let P be a prime ideal of R. Cowsik asked whether the symbolic Rees ring

$$R_s(P) = \bigoplus_{r \geq 0} P^{(r)} T^r$$

of P is a Noetherian ring or not. His aim is to give a new approach to the Kronecker's problem, that asks whether affine algebraic curves are set theoretic complete intersection or not. Kronecker's problem is still open, however, Roberts [15] gave a counterexample to Cowsik's question in 1985. Roberts constructed a regular local ring and a prime ideal such that the completion coincides with Nagata's counterexample to the Hilbert's fourteenth problem. In Roberts' example, the regular local ring contains a field of characteristic zero, and the prime ideal splits after completion. Later, Roberts [16] gave a new easier counterexample to both Hilbert's fourteenth problem and Cowsik's question. In his new example, the prime ideal does not split after completion, however, the ring still contains a field of characteristic zero. It was proved that analogous rings of characteristic positive are finitely generated ([9], [10]).

On the other hand, let $\mathfrak{p}_k(a,b,c)$ be the defining ideal of the space monomial curves (t^a, t^b, t^c) in k^3. Then, $\mathfrak{p}_k(a,b,c)$ is generated by at most three binomials in $k[x,y,z]$. The symbolic Rees rings are deeply studied by many authors. Huneke [7] and Cutkosky [3] developed criterions for finite generation of such rings. In 1994, Goto, Nishida and Watanabe [4] proved that $R_s(\mathfrak{p}_k(7n-3,(5n-2)n,8n-3))$ is not finitely generated over k if the characteristic of k is zero, $n \geq 4$ and $n \not\equiv 0$ (3). In their proof of infinite generation, they proved the finite generation of $R_s(\mathfrak{p}_k(7n-3,(5n-2)n,8n-3))$ in the case where k is of characteristic positive. Goto and Watanabe conjectured that, for any a, b and c, $R_s(\mathfrak{p}_k(a,b,c))$ is always finitely generated over k if the characteristic of k is positive.

On the other hand, Cutkosky [3] gave a geometric meaning to the symbolic Rees ring $R_s(\mathfrak{p}_k(a,b,c))$. Let X be the blow-up of the weighted projective space $\mathrm{Proj}(k[x,y,z])$ at the smooth point $V_+(\mathfrak{p}_k(a,b,c))$. Let E be the exceptional curve of the blow-up. Finite generation of $R_s(\mathfrak{p}_k(a,b,c))$ is equivalent to that of the total coordinate ring

$$TC(X) = \bigoplus_{D \in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D))$$

of X. If $-K_X$ is ample, one can prove that $TC(X)$ is finitely generated using the cone theorem (cf. [8]) as in [6]. Cutkosky proved that $TC(X)$ is finitely generated if $(-K_X)^2 > 0$, or equivalently $(a+b+c)^2 > abc$. Finite generation of $TC(X)$ is deeply related to existence of a negative curve C, i.e., a curve C on X satisfying $C^2 < 0$ and $C \neq E$. In fact, in the case where $\sqrt{abc} \notin \mathbb{Z}$, a negative exists if $TC(X)$ is finitely generated. If a negative exists in the case where the characteristic of k is positive, then $TC(X)$ is finitely generated by a result of M. Artin [1].

By a standard method (mod p reduction), if there exists a negative curve in the case of characteristic zero, then one can prove that a negative curve exists in the case of characteristic positive, therefore, $R_s(\mathfrak{p}_k(a,b,c))$ is finitely generated in the case of characteristic positive (cf. Lemma 3.4). In the examples of Goto-Nishida-Watanabe [4], a negative curve exists, however, $R_s(\mathfrak{p}_k(a,b,c))$ is not finitely generated over k in the case where k is of characteristic zero (cf. Remark 3.5 below).

In Section 2, we shall prove that if $R_s(\mathfrak{p}_k(a,b,c))$ is not finitely generated, then it is a counterexample to the Hilbert's fourteenth problem (cf. Theorem 2.1 and Remark 2.2).

In Section 3, we review some basic facts on finite generation of $R_s(\mathfrak{p}_k(a,b,c))$. We define sufficient conditions for X to have a negative curve (cf. Definition 3.6).

In Section 4, we shall prove that there exists a negative curve in the case where $(a+b+c)^2 > abc$ (cf. Theorem 4.3). We should mention that if $(a+b+c)^2 > abc$, then Cutkosky [3] proved that $R_s(\mathfrak{p}_k(a,b,c))$ is finitely generated. Moreover if we assume $\sqrt{abc} \notin \mathbb{Z}$, existence of a negative curve follows from finite generation. Existence of negative curves in these cases is an immediate conclusion of the cone theorem. Our proof of existence of a negative curve is very simple, purely algebraic, and do not need the cone theorem as Cutkosky's proof.

In Section 5, we discuss the degree of a negative curve (cf. Theorem 5.4). It is used in a computer programming in Section 6.1.

In Section 6.1, we prove that there exist examples in which a sufficient condition ((C2) in Definition 3.6) is not satisfied using a computer. In Section 6.2, we give a computer programming to check whether a negative curve exist or not.

2 Symbolic Rees rings of monomial curves and Hilbert's fourteenth problem

Throughout of this paper, we assume that rings are commutative with unit.

For a prime ideal P of a ring A, $P^{(r)}$ denotes the r-th symbolic power of P, i.e.,

$$P^{(r)} = P^r A_P \cap A.$$

By definition, it is easily seen that $P^{(r)}P^{(r')} \subset P^{(r+r')}$ for any $r,\ r' \geq 0$, therefore,

$$\bigoplus_{r\geq 0} P^{(r)}T^r$$

is a subring of the polynomial ring $A[T]$. This subring is called the *symbolic Rees ring* of P, and denoted by $R_s(P)$.

Let k be a field and m be a positive integer. Let $a_1, \ldots, a_m$ be positive integers. Consider the k-algebra homomorphism

$$\phi_k : k[x_1, \ldots, x_m] \longrightarrow k[t]$$

given by $\phi_k(x_i) = t^{a_i}$ for $i = 1, \ldots, m$, where $x_1, \ldots, x_m$, t are indeterminates over k. Let $\mathfrak{p}_k(a_1, \ldots, a_m)$ be the kernel of ϕ_k. We sometimes denote $\mathfrak{p}_k(a_1, \ldots, a_m)$ simply by $\mathfrak{p}$ or $\mathfrak{p}_k$ if no confusion is possible.

Theorem 2.1 *Let k be a field and m be a positive integer. Let $a_1, \ldots, a_m$ be positive integers. Consider the prime ideal $\mathfrak{p}_k(a_1, \ldots, a_m)$ of the polynomial ring $k[x_1, \ldots, x_m]$.*

Let α_1, α_2, β_1, $\ldots$, β_m, t, T be indeterminates over k. Consider the following injective k-homomorphism

$$\xi : k[x_1, \ldots, x_m, T] \longrightarrow k(\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t)$$

given by $\xi(T) = \alpha_2/\alpha_1$ *and* $\xi(x_i) = \alpha_1\beta_i + t^{a_i}$ *for* $i = 1, \ldots, m$.

Then,

$$k(\alpha_1\beta_1+t^{a_1}, \alpha_1\beta_2+t^{a_2}, \ldots, \alpha_1\beta_m+t^{a_m}, \alpha_2/\alpha_1) \cap k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t] = \xi(R_s(\mathfrak{p}_k(a_1, \ldots, a_m)))$$

holds true.

Proof. Set $L = k(\alpha_1\beta_1 + t^{a_1}, \ldots, \alpha_1\beta_m + t^{a_m}, \alpha_2/\alpha_1)$. Set $d = GCD(a_1, \ldots, a_m)$. Then, L is included in $k(\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t^d)$. Since

$$k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t] \cap k(\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t^d) = k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t^d],$$

we obtain the equality

$$L \cap k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t] = L \cap k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t^d].$$

By the commutativity of the diagram

$$\begin{array}{ccccc} & & L & & \\ & & \downarrow & & \\ k[x_1, \ldots, x_m, T] & \longrightarrow & k(\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t^d) & \supset & k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t^d] \\ & {\scriptstyle\xi} \searrow & \downarrow & & \downarrow \\ & & k(\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t) & \supset & k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t] \end{array}$$

it is enough to prove this theorem in the case where $GCD(a_1, \ldots, a_m) = 1$.

In the rest of this proof, we assume $GCD(a_1, \ldots, a_m) = 1$.

Consider the following injective k-homomorphism

$$\tilde{\xi} : k[x_1, \ldots, x_m, T, t] \longrightarrow k(\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t)$$

given by $\tilde{\xi}(T) = \alpha_2/\alpha_1$, $\tilde{\xi}(t) = t$ and $\tilde{\xi}(x_i) = \alpha_1\beta_i + t^{a_i}$ for $i = 1, \ldots, m$. Here, remark that α_2/α_1, $\alpha_1\beta_1 + t^{a_1}$, $\alpha_1\beta_2 + t^{a_2}$, $\ldots$, $\alpha_1\beta_m + t^{a_m}$, t are algebraically independent over k. By definition, the map ξ is the restriction of $\tilde{\xi}$ to $k[x_1, \ldots, x_m, T]$.

We set $S = k[x_1, \ldots, x_m]$ and $A = k[x_1, \ldots, x_m, t]$. Let $\mathfrak{q}$ be the ideal of A generated by $x_1 - t^{a_1}$, $\ldots$, $x_m - t^{a_m}$. Then $\mathfrak{q}$ is the kernel of the map $\tilde{\phi_k} : A \to k[t]$ given by $\tilde{\phi_k}(t) = t$ and $\tilde{\phi_k}(x_i) = t^{a_i}$ for each i. Since ϕ_k is the restriction of $\tilde{\phi_k}$ to S, $\mathfrak{q} \cap S = \mathfrak{p}$ holds.

Now we shall prove $\mathfrak{q}^r \cap S = \mathfrak{p}^{(r)}$ for each $r > 0$. Since $\mathfrak{q}$ is a complete intersection, $\mathfrak{q}^{(r)}$ coincides with $\mathfrak{q}^r$ for any $r > 0$. Therefore, it is easy to see $\mathfrak{q}^r \cap S \supset \mathfrak{p}^{(r)}$.

Since $GCD(a_1, \ldots, a_m) = 1$, there exists a monomial M in S such that $\phi_k(x_1^u)t = \phi_k(M)$ for some $u > 0$. Let

$$\tilde{\phi_k} \otimes 1 : k[x_1, \ldots, x_m, x_1^{-1}, t] \longrightarrow k[t, t^{-1}]$$

be the localization of $\tilde{\phi_k}$. Then, the kernel of $\tilde{\phi_k} \otimes 1$ is equal to

$$\mathfrak{q}k[x_1, \ldots, x_m, x_1^{-1}, t] = (\mathfrak{p}, t - \frac{M}{x_1^u})k[x_1, \ldots, x_m, x_1^{-1}, t].$$

Setting $t' = t - \frac{M}{x_1^u}$,

$$\mathfrak{q}A[x_1^{-1}] = (\mathfrak{p}, t')k[x_1, \ldots, x_m, x_1^{-1}, t']$$

holds. Since $x_1, \ldots, x_m, t'$ are algebraically independent over k,

$$\mathfrak{q}^r A[x_1^{-1}] \cap S[x_1^{-1}] = (\mathfrak{p}, t')^r k[x_1, \ldots, x_m, x_1^{-1}, t'] \cap k[x_1, \ldots, x_m, x_1^{-1}] = \mathfrak{p}^r S[x_1^{-1}]$$

holds. Therefore,

$$\mathfrak{q}^r \cap S \subset \mathfrak{q}^r A[x_1^{-1}] \cap S = \mathfrak{p}^r S[x_1^{-1}] \cap S \subset \mathfrak{p}^{(r)}.$$

We have completed the proof of $\mathfrak{q}^r \cap S = \mathfrak{p}^{(r)}$.

Let $R(\mathfrak{q})$ be the Rees ring of the ideal $\mathfrak{q}$, i.e.,

$$R(\mathfrak{q}) = \bigoplus_{r \geq 0} \mathfrak{q}^r T^r \subset A[T].$$

Then, since $\mathfrak{q}^r \cap S = \mathfrak{p}^{(r)}$ for $r \geq 0$,

$$R(\mathfrak{q}) \cap S[T] = R_s(\mathfrak{p})$$

holds. It is easy to verify

$$R(\mathfrak{q}) \cap Q(S[T]) = R_s(\mathfrak{p})$$

because $Q(S[T]) \cap A[T] = S[T]$, where $Q(\ )$ means the field of fractions. Here remark that $S[T] = k[x_1, \ldots, x_m, T]$ and $A[T] = k[x_1, \ldots, x_m, T, t]$. Therefore, we obtain the equality

$$\tilde{\xi}(R(\mathfrak{q})) \cap L = \xi(R_s(\mathfrak{p})). \tag{1}$$

Here, remember that L is the field of fractions of $\mathrm{Im}(\xi)$.

On the other hand, setting $x_i' = x_i - t^{a_i}$ for $i = 1, \ldots, m$, we obtain the following:

$$\begin{array}{rcl} R(\mathfrak{q}) & = & k[x_1, \ldots, x_m, x_1'T, \ldots, x_m'T, t] \\ & = & k[x_1', \ldots, x_m', x_1'T, \ldots, x_m'T, t] \end{array}$$

Here, remark that $x_1', \ldots, x_m', T, t$ are algebraically independent over k.

By definition, $\tilde{\xi}(x_i') = \alpha_1\beta_i$, and $\tilde{\xi}(x_i'T) = \alpha_2\beta_i$ for each i.

We set

$$B = \tilde{\xi}(R(\mathfrak{q})) \tag{2}$$

and $C = k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t]$. Here,

$$B = (k[\alpha_i\beta_j \mid i = 1, 2;\ j = 1, \ldots, m])\,[t] \subset C.$$

Since B is a direct summand of C as a B-module, the equality

$$C \cap Q(B) = B \tag{3}$$

holds in $Q(C)$.

Then, since $L \subset Q(B)$, we obtain

$$C \cap L = (C \cap Q(B)) \cap L = B \cap L = \xi(R_s(\mathfrak{p}))$$

by the equations (1), (2) and (3). **q.e.d.**

Remark 2.2 Let k be a field. Let R be a polynomial ring over k with finitely many variables. For a field L satisfying $k \subset L \subset Q(R)$, Hilbert asked in 1900 whether the ring $L \cap R$ is finitely generated as a k-algebra or not. It is called the *Hilbert's fourteenth problem.*

The first counterexample to this problem was discovered by Nagata [14] in 1958. An easier counterexample was found by Paul C. Roberts [16] in 1990. Further counterexamples were given by Kuroda, Mukai, etc.

On the other hand, Goto, Nishida and Watanabe [4] proved that $R_s(\mathfrak{p}_k(7n-3, (5n-2)n, 8n-3))$ is not finitely generated over k if the characteristic of k is zero, $n \geq 4$ and $n \not\equiv 0$ (3). By Theorem 2.1, we know that they are new counterexamples to the Hilbert's fourteenth problem.

Remark 2.3 With notation as in Theorem 2.1, we set

$$\begin{aligned} D_1 &= \alpha_1 \frac{\partial}{\partial \alpha_1} + \alpha_2 \frac{\partial}{\partial \alpha_2} - \beta_1 \frac{\partial}{\partial \beta_1} - \cdots - \beta_m \frac{\partial}{\partial \beta_m} \\ D_2 &= a_1 t^{a_1 - 1} \frac{\partial}{\partial \beta_1} + \cdots + a_m t^{a_m - 1} \frac{\partial}{\partial \beta_m} - \alpha_1 \frac{\partial}{\partial t}. \end{aligned}$$

Assume that the characteristic of k is zero.

Then, one can prove that $\xi(R_s(\mathfrak{p}_k(a_1, \ldots, a_m)))$ is equal to the kernel of the derivations D_1 and D_2, i.e.,

$$\xi(R_s(\mathfrak{p}_k(a_1, \ldots, a_m))) = \{ f \in k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t] \mid D_1(f) = D_2(f) = 0 \}.$$

3 Symbolic Rees rings of space monomial curves

In the rest of this paper, we restrict ourselves to the case $m = 3$. For the simplicity of notation, we write x, y, z, a, b, c for x_1, x_2, x_3, a_1, a_2, a_3, respectively. We regard the polynomial ring $k[x, y, z]$ as a $\mathbb{Z}$-graded ring by $\deg(x) = a$, $\deg(y) = b$ and $\deg(z) = c$.

$\mathfrak{p}_k(a, b, c)$ is the kernel of the k-algebra homomorphism

$$\phi_k : k[x, y, z] \longrightarrow k[t]$$

given by $\phi_k(x) = t^a$, $\phi_k(y) = t^b$, $\phi_k(z) = t^c$.

By a result of Herzog [5], we know that $\mathfrak{p}_k(a,b,c)$ is generated by at most three elements. For example, $\mathfrak{p}_k(3,4,5)$ is minimally generated by $x^3 - yz$, $y^2 - zx$ and $z^2 - x^2y$. On the other hand, $\mathfrak{p}_k(3,5,8)$ is minimally generated by $x^5 - y^3$ and $z - xy$. We can choose a generating system of $\mathfrak{p}_k(a,b,c)$ which is independent of k.

We are interested in the symbolic powers of $\mathfrak{p}_k(a,b,c)$. If $\mathfrak{p}_k(a,b,c)$ is generated by two elements, then the symbolic powers always coincide with ordinary powers because $\mathfrak{p}_k(a,b,c)$ is a complete intersection. However, it is known that, if $\mathfrak{p}_k(a,b,c)$ is minimally generated by three elements, the second symbolic power is strictly bigger than the second ordinary power. For example, the element

$$h = (x^3 - yz)^2 - (y^2 - zx)(z^2 - x^2y)$$

is contained in $\mathfrak{p}_k(3,4,5)^2$, and is divisible by x. Therefore, h/x is an element in $\mathfrak{p}_k(3,4,5)^{(2)}$ of degree 15. Since $[\mathfrak{p}_k(3,4,5)^2]_{15} = 0$, h/x is not contained in $\mathfrak{p}_k(3,4,5)^2$.

We are interested in finite generation of the symbolic Rees ring $R_s(\mathfrak{p}_k(a,b,c))$. It is known that this problem is reduced to the case where a, b and c are pairwise coprime, i.e.,

$$(a,b) = (b,c) = (c,a) = 1.$$

In the rest of this paper, we always assume that a, b and c are pairwise coprime.

Let $\mathbb{P}_k(a,b,c)$ be the weighted projective space $\mathrm{Proj}(k[x,y,z])$. Then

$$\mathbb{P}_k(a,b,c) \setminus \{V_+(x,y), \quad V_+(y,z), \quad V_+(z,x)\}$$

is a regular scheme. In particular, $\mathbb{P}_k(a,b,c)$ is smooth at the point $V_+(\mathfrak{p}_k(a,b,c))$. Let $\pi : X_k(a,b,c) \to \mathbb{P}_k(a,b,c)$ be the blow-up at $V_+(\mathfrak{p}_k(a,b,c))$. Let E be the exceptional divisor, i.e.,

$$E = \pi^{-1}(V_+(\mathfrak{p}_k(a,b,c))).$$

We sometimes denote $\mathfrak{p}_k(a,b,c)$ (resp. $\mathbb{P}_k(a,b,c)$, $X_k(a,b,c)$) simply by $\mathfrak{p}$ or $\mathfrak{p}_k$ (resp. $\mathbb{P}$ or $\mathbb{P}_k$, X or X_k) if no confusion is possible.

It is easy to see that

$$\mathrm{Cl}(\mathbb{P}) = \mathbb{Z}H \simeq \mathbb{Z},$$

where H is a Weil divisor corresponding to the reflexive sheaf $\mathcal{O}_{\mathbb{P}}(1)$. Set $H = \sum_i m_i D_i$, where D_i's are subvarieties of $\mathbb{P}$ of codimension one. We may choose D_i's such that $D_i \not\ni V_+(\mathfrak{p})$ for any i. Then, set $A = \sum_i m_i \pi^{-1}(D_i)$.

One can prove that

$$\mathrm{Cl}(X) = \mathbb{Z}A + \mathbb{Z}E \simeq \mathbb{Z}^2.$$

Since all Weil divisors on X are $\mathbb{Q}$-Cartier, we have the intersection pairing

$$\mathrm{Cl}(X) \times \mathrm{Cl}(X) \longrightarrow \mathbb{Q},$$

that satisfies

$$A^2 = \frac{1}{abc}, \quad E^2 = -1, \quad A.E = 0.$$

Therefore, we have

$$(n_1A - r_1E).(n_2A - r_2E) = \frac{n_1n_2}{abc} - r_1r_2.$$

Here, we have the following natural identification:

$$H^0(X, \mathcal{O}_X(nA - rE)) = \begin{cases} \left[\mathfrak{p}^{(r)}\right]_n & (r \geq 0) \\ S_n & (r < 0) \end{cases}$$

Therefore, the *total coordinate ring* (or *Cox ring*)

$$TC(X) = \bigoplus_{n,r \in \mathbb{Z}} H^0(X, \mathcal{O}_X(nA - rE))$$

is isomorphic to the extended symbolic Rees ring

$$R_s(\mathfrak{p})[T^{-1}] = \cdots \oplus ST^{-2} \oplus ST^{-1} \oplus S \oplus \mathfrak{p}T \oplus \mathfrak{p}^{(2)}T^2 \oplus \cdots.$$

We refer the reader to Hu-Keel [6] for finite generation of total coordinate rings. It is well-known that $R_s(\mathfrak{p})[T^{-1}]$ is Noetherian if and only if so is $R_s(\mathfrak{p})$.

Remark 3.1 By Huneke's criterion [7] and a result of Cutkosky [3], the following four conditions are equivalent:

(1) $R_s(\mathfrak{p})$ is a Noetherian ring, or equivalently, finitely generated over k.

(2) $TC(X)$ is a Noetherian ring, or equivalently, finitely generated over k.

(3) There exist positive integers r, s, $f \in \mathfrak{p}^{(r)}$, $g \in \mathfrak{p}^{(s)}$, and $h \in (x, y, z) \setminus \mathfrak{p}$ such that
$$\ell_{S_{(x,y,z)}}(S_{(x,y,z)}/(f, g, h)) = rs \cdot \ell_{S_{(x,y,z)}}(S_{(x,y,z)}/(\mathfrak{p}, h)),$$
where $\ell_{S_{(x,y,z)}}$ is the length as an $S_{(x,y,z)}$-module.

(4) There exist curves C and D on X such that
$$C \neq D, \quad C \neq E, \quad D \neq E, \quad C.D = 0.$$

Here, a curve means a closed irreducible reduced subvariety of dimension one.

The condition (4) as above is equivalent to that just one of the following two conditions is satisfied:

(4-1) There exist curves C and D on X such that
$$C \neq E, \quad D \neq E, \quad C^2 < 0, \quad D^2 > 0, \quad C.D = 0.$$

(4-2) There exist curves C and D on X such that
$$C \neq E, \quad D \neq E, \quad C \neq D, \quad C^2 = D^2 = 0.$$

Definition 3.2 A curve C on X is called a *negative curve* if

$$C \neq E \quad \text{and} \quad C^2 < 0.$$

Remark 3.3 Suppose that a divisor F is linearly equivalent to $nA - rE$. Then, we have

$$F^2 = \frac{n^2}{abc} - r^2.$$

If (4-2) in Remark 3.1 is satisfied, then all of a, b and c must be squares of integers because a, b, c are pairwise coprime. In the case where one of a, b and c is not square, the condition (4) is equivalent to (4-1). Therefore, in this case, a negative curve exists if $R_s(\mathfrak{p})$ is finitely generated over k.

Suppose $(a, b, c) = (1, 1, 1)$. Then $\mathfrak{p}$ coincides with $(x - y, y - z)$. Of course, $R_s(\mathfrak{p})$ is a Noetherian ring since the symbolic powers coincide with the ordinary powers. By definition it is easy to see that there is no negative curve in this case, therefore, (4-2) in Remark 3.1 happens.

The authors know no other examples in which (4-2) happens.

In the case of $(a, b, c) = (3, 4, 5)$, the proper transform of

$$V_+\left(\frac{(x^3 - yz)^2 - (y^2 - zx)(z^2 - x^2y)}{x}\right)$$

is the negative curve on X, that is linearly equivalent to $15A - 2E$.

It is proved that two distinct negative curves never exist.

In the case where the characteristic of k is positive, Cutkosky [3] proved that $R_s(\mathfrak{p})$ is finitely generated over k if there exists a negative curve on X.

We remark that there exists a negative curve on X if and only if there exists positive integers n and r such that

$$\frac{n}{r} < \sqrt{abc} \quad \text{and} \quad [\mathfrak{p}^{(r)}]_n \neq 0.$$

We are interested in existence of a negative curve. Let a, b and c be pairwise coprime positive integers. By the following lemma, if there exists a negative curve on $X_{k_0}(a, b, c)$ for a field k_0 of characteristic 0, then there exists a negative curve on $X_k(a, b, c)$ for any field k.

Lemma 3.4 *Let a, b and c be pairwise coprime positive integers.*

1. *Let K/k be a field extension. Then, for any integers n and r,*

$$[\mathfrak{p}_k(a, b, c)^{(r)}]_n \otimes_k K = [\mathfrak{p}_K(a, b, c)^{(r)}]_n.$$

2. *For any integers n, r and any prime number p,*

$$\dim_{\mathbb{F}_p}[\mathfrak{p}_{\mathbb{F}_p}(a,b,c)^{(r)}]_n \geq \dim_{\mathbb{Q}}[\mathfrak{p}_{\mathbb{Q}}(a,b,c)^{(r)}]_n$$

holds, where $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{F}_p$ is the prime field of characteristic p. Here, $\dim_{\mathbb{F}_p}$ (resp. $\dim_{\mathbb{Q}}$) denotes the dimension as an $\mathbb{F}_p$-vector space (resp. $\mathbb{Q}$-vector space).

Proof. Since $S \to S \otimes_k K$ is flat, it is easy to prove the assertion (1).

We shall prove the assertion (2). Let $\mathbb{Z}$ be the ring of rational integers. Set $S_{\mathbb{Z}} = \mathbb{Z}[x,y,z]$. Let $\mathfrak{p}_{\mathbb{Z}}$ be the kernel of the ring homomorphism

$$\phi_{\mathbb{Z}} : S_{\mathbb{Z}} \longrightarrow \mathbb{Z}[t]$$

given by $\phi_{\mathbb{Z}}(x) = t^a$, $\phi_{\mathbb{Z}}(y) = t^b$ and $\phi_{\mathbb{Z}}(z) = t^c$. Since the cokernel of $\phi_{\mathbb{Z}}$ is $\mathbb{Z}$-free module, we know

$$\mathfrak{p}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k = \mathrm{Ker}(\phi_{\mathbb{Z}}) \otimes_{\mathbb{Z}} k = \mathrm{Ker}(\phi_k) = \mathfrak{p}_k$$

for any field k.

Consider the following exact sequence of $\mathbb{Z}$-free modules:

$$0 \longrightarrow {\mathfrak{p}_{\mathbb{Z}}}^{(r)} \longrightarrow S_{\mathbb{Z}} \longrightarrow S_{\mathbb{Z}}/{\mathfrak{p}_{\mathbb{Z}}}^{(r)} \longrightarrow 0$$

For any field k, the following sequence is exact:

$$0 \longrightarrow {\mathfrak{p}_{\mathbb{Z}}}^{(r)} \otimes_{\mathbb{Z}} k \longrightarrow S \longrightarrow S_{\mathbb{Z}}/{\mathfrak{p}_{\mathbb{Z}}}^{(r)} \otimes_{\mathbb{Z}} k \longrightarrow 0$$

Since $\mathfrak{p}_{\mathbb{Z}} S_{\mathbb{Z}}[x^{-1}]$ is generated by a regular sequence, we know

$${\mathfrak{p}_{\mathbb{Z}}}^{(r)} S_{\mathbb{Z}}[x^{-1}] = {\mathfrak{p}_{\mathbb{Z}}}^{r} S_{\mathbb{Z}}[x^{-1}]$$

for any $r \geq 0$. Therefore, for any $f \in {\mathfrak{p}_{\mathbb{Z}}}^{(r)}$, there is a positive integer u such that

$$x^u f \in {\mathfrak{p}_{\mathbb{Z}}}^{r}.$$

Let p be a prime number. Consider the natural surjective ring homomorphism

$$\eta : S_{\mathbb{Z}} \longrightarrow S_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

Suppose $f \in {\mathfrak{p}_{\mathbb{Z}}}^{(r)}$. Since $x^u f \in {\mathfrak{p}_{\mathbb{Z}}}^{r}$ for some positive integer u, we obtain

$$x^u \eta(f) \in \eta({\mathfrak{p}_{\mathbb{Z}}}^{r}) = {\mathfrak{p}_{\mathbb{F}_p}}^{r}.$$

Hence we know

$${\mathfrak{p}_{\mathbb{Z}}}^{(r)} \otimes_{\mathbb{Z}} \mathbb{F}_p = \eta({\mathfrak{p}_{\mathbb{Z}}}^{(r)}) \subset {\mathfrak{p}_{\mathbb{F}_p}}^{(r)}.$$

We obtain

$$\mathrm{rank}_{\mathbb{Z}}[{\mathfrak{p}_{\mathbb{Z}}}^{(r)}]_n = \dim_{\mathbb{F}_p}[{\mathfrak{p}_{\mathbb{Z}}}^{(r)}]_n \otimes_{\mathbb{Z}} \mathbb{F}_p \leq \dim_{\mathbb{F}_p}[{\mathfrak{p}_{\mathbb{F}_p}}^{(r)}]_n$$

for any $r \geq 0$ and $n \geq 0$. Here, $\mathrm{rank}_{\mathbb{Z}}$ denotes the rank as a $\mathbb{Z}$-module.

On the other hand, it is easy to see that

$$\mathfrak{p}_{\mathbb{Z}}{}^{(r)} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathfrak{p}_{\mathbb{Q}}{}^{(r)}$$

for any $r \geq 0$. Therefore, we have

$$\mathrm{rank}_{\mathbb{Z}}[\mathfrak{p}_{\mathbb{Z}}{}^{(r)}]_n = \dim_{\mathbb{Q}}[\mathfrak{p}_{\mathbb{Q}}{}^{(r)}]_n$$

for any $r \geq 0$ and $n \geq 0$.

Hence, we obtain

$$\dim_{\mathbb{Q}}[\mathfrak{p}_{\mathbb{Q}}{}^{(r)}]_n \leq \dim_{\mathbb{F}_p}[\mathfrak{p}_{\mathbb{F}_p}{}^{(r)}]_n$$

for any $r \geq 0$, $n \geq 0$, and any prime number p. **q.e.d.**

Remark 3.5 Let a, b, c be pairwise coprime positive integers. Assume that there exists a negative curve on $X_{k_0}(a,b,c)$ for a field k_0 of characteristic zero.

By Lemma 3.4, we know that there exists a negative curve on $X_k(a,b,c)$ for any field k. Therefore, if k is a field of characteristic positive, then the symbolic Rees ring $R_s(\mathfrak{p}_k)$ is finitely generated over k by a result of Cutkosky [3]. However, if k is a field of characteristic zero, then $R_s(\mathfrak{p}_k)$ is not necessary Noetherian. In fact, assume that k is of characteristic zero and $(a,b,c) = (7n-3, (5n-2)n, 8n-3)$ with $n \not\equiv 0$ (3) and $n \geq 4$ as in Goto-Nishida-Watanabe [4]. Then there exists a negative curve, but $R_s(\mathfrak{p}_k)$ is not Noetherian.

Definition 3.6 Let a, b, c be pairwise coprime positive integers. Let k be a field.

We define the following three conditions:

(C1) There exists a negative curve on $X_k(a,b,c)$, i.e., $[\mathfrak{p}_k(a,b,c)^{(r)}]_n \neq 0$ for some positive integers n, r satisfying $n/r < \sqrt{abc}$.

(C2) There exist positive integers n, r satisfying $n/r < \sqrt{abc}$ and $\dim_k S_n > r(r+1)/2$.

(C3) There exist positive integers q, r satisfying $abcq/r < \sqrt{abc}$ and $\dim_k S_{abcq} > r(r+1)/2$.

Here, $\dim_k$ denotes the dimension as a k-vector space.

By the following lemma, we know the implications

$$(C3) \Longrightarrow (C2) \Longrightarrow (C1)$$

since $\dim_k[\mathfrak{p}^{(r)}]_n = \dim_k S_n - \dim_k[S/\mathfrak{p}^{(r)}]_n$.

Lemma 3.7 *Let a, b, c be pairwise coprime positive integers. Let r and n be non-negative integers. Then,*

$$\dim_k[S/\mathfrak{p}^{(r)}]_n \leq r(r+1)/2$$

holds true for any field k.

Proof. Since x, y, z are non-zero divisors on $S/\mathfrak{p}^{(r)}$, we have only to prove that

$$\dim_k [S/\mathfrak{p}^{(r)}]_{abcq} = r(r+1)/2$$

for $q \gg 0$.

The left-hand side is the multiplicity of the abc-th Veronese subring

$$[S/\mathfrak{p}^{(r)}]^{(abc)} = \oplus_{q \geq 0} [S/\mathfrak{p}^{(r)}]_{abcq}.$$

Therefore, for $q \gg 0$, we have

$$\begin{aligned}
\dim_k [S/\mathfrak{p}^{(r)}]_{abcq} &= \ell([S/\mathfrak{p}^{(r)} + (x^{bc})]^{(abc)}) \\
&= e((x^{bc}), [S/\mathfrak{p}^{(r)}]^{(abc)}) \\
&= \frac{1}{abc} e((x^{bc}), S/\mathfrak{p}^{(r)}) \\
&= \frac{1}{a} e((x), S/\mathfrak{p}^{(r)}) \\
&= \frac{1}{a} e((x), S/\mathfrak{p}) \ell_{S_\mathfrak{p}}(S_\mathfrak{p}/\mathfrak{p}^r S_\mathfrak{p}) \\
&= \frac{r(r+1)}{2}
\end{aligned}$$

q.e.d.

Remark 3.8 It is easy to see that $[\mathfrak{p}_k(a,b,c)]_n \neq 0$ if and only if $\dim_k S_n \geq 2$. Therefore, if we restrict ourselves to $r = 1$, then (C1) and (C2) are equivalent.

However, even if $[\mathfrak{p}_k(a,b,c)^{(2)}]_n \neq 0$, $\dim_k S_n$ is not necessary bigger than 3. In fact, since $\mathfrak{p}_k(5,6,7)$ contains $y^2 - zx$, we know $[\mathfrak{p}_k(5,6,7)^2]_{24} \neq 0$. In this case, $\dim_k S_{24}$ is equal to three.

Here assume that (C1) is satisfied for $r = 2$. Furthermore, we assume that the characteristic of k is zero. Then, there exists $f \neq 0$ in $[\mathfrak{p}_k(a,b,c)^{(2)}]_n$ such that $n < 2\sqrt{abc}$ for some $n > 0$. Let $f = f_1 \cdots f_s$ be the irreducible decomposition. Then, at least one of f_i's satisfies the condition (C1). If it satisfies (C1) with $r = 1$, then (C2) is satisfied as above. Suppose that the irreducible component satisfies (C2) with $r = 2$. For the simplicity of notation, we assume that f itself is irreducible. We want to show $\dim_k S_n \geq 4$. Assume the contrary. By Lemma 3.4 (1), we may assume that f is a polynomial with rational coefficients. Set

$$f = k_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} - k_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + k_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3}.$$

Furthermore, we may assume that k_1, k_2, k_3 are non-negative integers such that $GCD(k_1, k_2, k_3) = 1$. Since

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \in \mathfrak{p}_k(a,b,c)$$

as in Remark 5.1, we have

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \begin{pmatrix} k_1 \\ -k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, we have

$$(x^{\alpha_1}y^{\beta_1}z^{\gamma_1})^{k_1}(x^{\alpha_3}y^{\beta_3}z^{\gamma_3})^{k_3} = (x^{\alpha_2}y^{\beta_2}z^{\gamma_2})^{k_2}.$$

Since f is irreducible, $x^{\alpha_1}y^{\beta_1}z^{\gamma_1}$ and $x^{\alpha_3}y^{\beta_3}z^{\gamma_3}$ have no common divisor. Note that $k_2 = k_1 + k_3$ since $f \in \mathfrak{p}_k(a,b,c)$. Since k_1 and k_3 are relatively prime, there exist monomials N_1 and N_3 such that $x^{\alpha_1}y^{\beta_1}z^{\gamma_1} = N_1^{k_1+k_3}$, $x^{\alpha_3}y^{\beta_3}z^{\gamma_3} = N_3^{k_1+k_3}$ and $x^{\alpha_2}y^{\beta_2}z^{\gamma_2} = N_1^{k_1}N_3^{k_3}$. Then

$$f = k_1N_1^{k_1+k_3} - (k_1+k_3)N_1^{k_1}N_3^{k_3} + k_3N_3^{k_1+k_3}.$$

Then, f is divisible by $N_1 - N_3$. Since f is irreducible, f is equal to $N_1 - N_3$. It contradicts to

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \in \mathfrak{p}_k(a,b,c).$$

Consequently, if (C1) is satisfied with $r \le 2$ for a field k of characteristic zero, then (C2) is satisfied.

We shall discuss the difference between (C1) and (C2) in Section 6.1.

Remark 3.9 Let a, b and c be pairwise coprime positive integers. Assume that $\mathfrak{p}_k(a,b,c)$ is a complete intersection, i.e., generated by two elements.

Permuting a, b and c, we may assume that

$$\mathfrak{p}_k(a,b,c) = (x^b - y^a,\ z - x^\alpha y^\beta)$$

for some $\alpha, \beta \ge 0$ satisfying $\alpha a + \beta b = c$. If $ab < c$, then

$$\deg(x^b - y^a) = ab < \sqrt{abc}.$$

If $ab > c$, then

$$\deg(z - x^\alpha y^\beta) = c < \sqrt{abc}.$$

If $ab = c$, then (a,b,c) must be equal to $(1,1,1)$. Ultimately, there exists a negative curve if $(a,b,c) \ne (1,1,1)$.

4 The case where $(a+b+c)^2 > abc$

In the rest of this paper, we set $\xi = abc$ and $\eta = a+b+c$ for pairwise coprime positive integers a, b and c.

For $v = 0, 1, \ldots, \xi - 1$, we set

$$S^{(\xi,v)} = \oplus_{q \geq 0} S_{\xi q + v}.$$

This is a module over $S^{(\xi)} = \oplus_{q \geq 0} S_{\xi q}$.

Lemma 4.1

$$\dim_k [S^{(\xi,v)}]_q = \dim_k S_{\xi q + v} = \frac{1}{2}\left\{\xi q^2 + (\eta + 2v)q + 2\dim_k S_v\right\}$$

holds for any $q \geq 0$.

The following simple proof is due to Professor Kei-ichi Watanabe. We appreciate him very much.

Proof. We set $a_n = \dim_k S_n$ for each integer n. Set

$$f(t) = \sum_{n \in \mathbb{Z}} a_n t^n.$$

Here we put $a_n = 0$ for $n < 0$. Then, the equality

$$f(t) = \frac{1}{(1-t^a)(1-t^b)(1-t^c)}$$

holds.

Set $b_n = a_n - a_{n-\xi}$. Then, b_n is equal to the coefficient of t^n in $(1-t^\xi)f(t)$ for each n. Furthermore, $b_n - b_{n-1}$ is equal to the coefficient of t^n in $(1-t)(1-t^\xi)f(t)$ for each n.

On the other hand, we have the equality

$$(1-t)(1-t^\xi)f(t) = g(t) \times \frac{1}{1-t} = g(t) \times (1 + t + t^2 + \cdots), \tag{4}$$

where

$$g(t) = \frac{1 + t + \cdots + t^{\xi-1}}{(1 + t + \cdots + t^{a-1})(1 + t + \cdots + t^{b-1})(1 + t + \cdots + t^{c-1})}.$$

Since a, b and c are pairwise coprime, $g(t)$ is a polynomial of degree $\xi - \eta + 2$. Therefore, the coefficient of t^n in $(1-t)(1-t^\xi)f(t)$ is equal to $g(1)$ for $n \geq \xi - \eta + 2$ by the equation (4). It is easy to see $g(1) = 1$.

Since $b_n - b_{n-1} = 1$ for $n \geq \xi + 1$,

$$b_n = b_\xi + (n - \xi)$$

holds for any $n \geq \xi$. Then,

$$\begin{aligned}
a_{\xi q+v} - a_v &= \sum_{i=1}^{q}(a_{\xi i+v} - a_{\xi(i-1)+v}) \\
&= \sum_{i=1}^{q} b_{\xi i+v} \\
&= \sum_{i=1}^{q} (b_\xi + \xi(i-1) + v) \\
&= b_\xi q + \xi\frac{(q-1)q}{2} + vq \\
&= \frac{\xi}{2}q^2 + \left(b_\xi - \frac{\xi}{2} + v\right) q.
\end{aligned}$$

Recall that b_ξ is the coefficient of t^ξ in

$$(1-t^\xi)f(t) = \frac{g(t)}{(1-t)^2} = g(t) \times (1 + 2t + \cdots + (n+1)t^n + \cdots) \,. \tag{5}$$

Setting

$$g(t) = c_0 + c_1 t + \cdots + c_{\xi-\eta+2} t^{\xi-\eta+2},$$

it is easy to see

$$c_i = c_{\xi-\eta+2-i} \tag{6}$$

for each i. Therefore, by the equations (5) and (6), we have

$$b_\xi = c_0(\xi+1) + c_1\xi + \cdots + c_{\xi-\eta+2}(\eta-1) = (c_0 + c_1 + \cdots + c_{\xi-\eta+2}) \times \frac{\xi+\eta}{2}.$$

Since $g(1) = 1$, we have $b_\xi = \frac{\xi+\eta}{2}$. Thus,

$$a_{\xi q+v} = \frac{\xi}{2}q^2 + \left(\frac{\xi+\eta}{2} - \frac{\xi}{2} + v\right) q + a_v.$$

q.e.d.

Before proving Theorem 4.3, we need the following lemma:

Lemma 4.2 Assume that a, b and c are pairwise coprime positive integers such that $(a,b,c) \neq (1,1,1)$. Then, $\eta - \sqrt{\xi} \neq 0, 1, 2$.

Proof. We may assume that all of a, b and c are squares of integers. It is sufficient to show

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma \neq 0, 1, 2$$

for pairwise coprime positive integers α, β, γ such that $(\alpha, \beta, \gamma) \neq (1, 1, 1)$.

Assume the contrary. Suppose that $(\alpha_0, \beta_0, \gamma_0)$ is a counterexample such that $\alpha_0 + \beta_0 + \gamma_0$ is minimum. We may assume $1 \le \alpha_0 \le \beta_0 \le \gamma_0$.

Set

$$f(x) = x^2 - \alpha_0\beta_0 x + \alpha_0^2 + \beta_0^2.$$

First suppose $\alpha_0\beta_0 \le \gamma_0$. Then,

$$f(\gamma_0) \ge f(\alpha_0\beta_0) = \alpha_0^2 + \beta_0^2 \ge 2.$$

Since $f(\gamma_0) = 0, 1$, or 2, we have

$$\gamma_0 = \alpha_0\beta_0 \ \text{ and } \ \alpha_0^2 + \beta_0^2 = 2.$$

Then, we obtain the equality $\alpha_0 = \beta_0 = \gamma_0 = 1$ immediately. It is a contradiction.

Next, suppose $\frac{\alpha_0\beta_0}{2} < \gamma_0 < \alpha_0\beta_0$. Then, $0 < \alpha_0\beta_0 - \gamma_0 < \gamma_0$ and

$$f(\alpha_0\beta_0 - \gamma_0) = f(\gamma_0) = 0, 1, \text{ or } 2.$$

It is easy to see that α_0, β_0, $\alpha_0\beta_0 - \gamma_0$ are pairwise coprime positive integers. By the minimality of $\alpha_0 + \beta_0 + \gamma_0$, we have $\alpha_0 = \beta_0 = \alpha_0\beta_0 - \gamma_0 = 1$. Then, γ_0 must be zero. It is a contradiction.

Finally, suppose $0 < \gamma_0 \le \frac{\alpha_0\beta_0}{2}$. Since $\beta_0 \le \gamma_0 \le \frac{\alpha_0\beta_0}{2}$, we have $\alpha_0 \ge 2$. If $\alpha_0 = 2$, then $2 \le \beta_0 = \gamma_0$. It contradicts to $(\beta_0, \gamma_0) = 1$. Assume $\alpha_0 \ge 3$. Since $\beta_0 < \gamma_0$,

$$f(\gamma_0) < f(\beta_0) = (2 - \alpha_0)\beta_0^2 + \alpha_0^2 \le 0.$$

It is a contradiction. **q.e.d.**

Theorem 4.3 *Let a, b and c be pairwise coprime integers such that $(a, b, c) \neq (1, 1, 1)$.*

Then, we have the following:

1. *Assume that $\sqrt{abc} \notin \mathbb{Z}$. Then, (C3) holds if and only if $(a + b + c)^2 > abc$.*
2. *Assume that $\sqrt{abc} \in \mathbb{Z}$. Then, (C3) holds if and only if $(a + b + c)^2 > 9abc$.*
3. *If $(a + b + c)^2 > abc$, then, (C2) holds. In particular, a negative curve exists in this case.*

Proof. Remember that, by Lemma 4.1, we obtain

$$\dim_k S_{\xi q} = \frac{1}{2}(\xi q^2 + \eta q + 2)$$

for any $q \ge 0$.

First we shall prove the assertion (1). Assume that (C3) is satisfied. Then,

$$\begin{cases} \sqrt{\xi} > \frac{\xi q}{r} \\ \frac{\xi q^2+\eta q+2}{2} > \frac{r(r+1)}{2} \end{cases}$$

is satisfied for some positive integers r and q. The second inequality is equivalent to $\xi q^2 + \eta q \geq r(r+1)$ since both integers are even. Since

$$\xi q^2 + \eta q \geq r^2 + r > \xi q^2 + \sqrt{\xi} q,$$

we have $\eta > \sqrt{\xi}$ immediately.

Assume $\eta > \sqrt{\xi}$ and $\sqrt{\xi} \notin \mathbb{Z}$. Let ϵ be a real number satisfying $0 < \epsilon < 1$ and

$$2\epsilon\sqrt{\xi} < \frac{\eta - \sqrt{\xi}}{2}. \tag{7}$$

Since $\sqrt{\xi} \notin \mathbb{Q}$, there exist positive integers r and q such that

$$\epsilon > r - \sqrt{\xi} q > 0.$$

Then,

$$\frac{r}{q} < \sqrt{\xi} + \frac{\epsilon}{q} \leq \sqrt{\xi} + \epsilon < \sqrt{\xi} + \frac{\eta - \sqrt{\xi}}{2} = \frac{\eta + \sqrt{\xi}}{2}.$$

Since $\sqrt{\xi} q + \epsilon > r$, we have

$$\xi q^2 + 2\epsilon\sqrt{\xi} q + \epsilon^2 > r^2.$$

Therefore

$$r^2 + r < \xi q^2 + 2\epsilon\sqrt{\xi} q + \epsilon^2 + \frac{\eta + \sqrt{\xi}}{2} q < \xi q^2 + \eta q + \epsilon^2 < \xi q^2 + \eta q + 2$$

by the equation (7).

Next we shall prove the assertion (2). Suppose $\sqrt{\xi} \in \mathbb{Z}$. Since $r > \sqrt{\xi} q$, we may assume that $r = \sqrt{\xi} q + 1$. Then,

$$(\xi q^2 + \eta q + 2) - (r^2 + r) = (\eta - 3\sqrt{\xi}) q.$$

Therefore, the assertion (2) immediately follows from this.

Now, we shall prove the assertion (3). Assume $\eta > \sqrt{\xi}$. Since $(a, b, c) \neq (1, 1, 1)$, we know $\xi > 1$. If $\sqrt{\xi} \notin \mathbb{Z}$, then the assertion immediately follows from the assertion (1). Therefore, we may assume $\sqrt{\xi} \in \mathbb{Z}$.

Let n, q and v be integers such that

$$n = \xi q + v, \quad v = \sqrt{\xi} - 1.$$

We set

$$r = \sqrt{\xi}q + 1.$$

Then,

$$\sqrt{\xi}r = \xi q + \sqrt{\xi} > n.$$

Furthermore, by Lemma 4.1,

$$\begin{aligned} 2\dim_k S_n - (r^2 + r) &= \left(\xi q^2 + (\eta + 2v)q + 2\dim_k S_v\right) - \left(\xi q^2 + 3\sqrt{\xi}q + 2\right) \\ &= (\eta - \sqrt{\xi} - 2)q + (2\dim_k S_v - 2). \end{aligned}$$

Since $\eta - \sqrt{\xi}$ is a non-negative integer, we know $\eta - \sqrt{\xi} \geq 3$ by Lemma 4.2. Consequently, we have $2\dim_k S_n - (r^2 + r) > 0$ for $q \gg 0$. **q.e.d.**

Remark 4.4 If $(a+b+c)^2 > abc$, then $R_s(\mathfrak{p})$ is Noetherian by a result of Cutkosky [3].

If $(a+b+c)^2 > abc$ and $\sqrt{abc} \notin \mathbb{Q}$, then the existence of a negative curve follows from Nakai's criterion for ampleness, Kleimann's theorem and the cone theorem (e.g. Theorem 1.2.23 and Theorem 1.4.23 in [11], Theorem 4-2-1 in [8]).

The condition $(a+b+c)^2 > abc$ is equivalent to $(-K_X)^2 > 0$. If $-K_X$ is ample, then the finite generation of the total coordinate ring follows from Proposition 2.9 and Corollary 2.16 in Hu-Keel [6].

If $(a,b,c) = (5,6,7)$, then the negative curve C is the proper transform of the curve defined by $y^2 - zx$. Therefore, C is linearly equivalent to $12A - E$. Since $(a+b+c)^2 > abc$, $(-K_X)^2 > 0$. Since

$$-K_X.C = (18A - E).(12A - E) = 0.028\cdots > 0,$$

$-K_X$ is ample by Nakai's criterion.

If $(a,b,c) = (7,8,9)$, then the negative curve C is the proper transform of the curve defined by $y^2 - zx$. Therefore, C is linearly equivalent to $16A - E$. Since $(a+b+c)^2 > abc$, $(-K_X)^2 > 0$. Since

$$-K_X.C = (24A - E).(16A - E) = -0.23\cdots < 0,$$

$-K_X$ is not ample by Nakai's criterion.

5 Degree of a negative curve

Remark 5.1 Let k be a field of characteristic zero, and R be a polynomial ring over k with variables $x_1, x_2, \ldots, x_m$. Suppose that P is a prime ideal of R. By [12], we have

$$P^{(r)} = \left\{ h \in R \;\middle|\; 0 \leq \alpha_1 + \cdots + \alpha_m < r \Longrightarrow \frac{\partial^{\alpha_1 + \cdots + \alpha_m} h}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} \in P \right\}.$$

In particular, if $f \in P^{(r)}$, then

$$\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m} \in P^{(r-1)}.$$

Proposition 5.2 *Let a, b and c be pairwise coprime integers, and k be a field of characteristic zero. Suppose that a negative curve exists, i.e., there exist positive integers n and r satisfying $[\mathfrak{p}_k(a,b,c)^{(r)}]_n \neq 0$ and $n/r < \sqrt{abc}$.*

Set n_0 and r_0 to be

$$\begin{array}{rcl} n_0 & = & \min\{n \in \mathbb{N} \mid \exists r > 0 \text{ such that } n/r < \sqrt{\xi} \text{ and } [\mathfrak{p}^{(r)}]_n \neq 0\} \\ r_0 & = & \lfloor \frac{n_0}{\sqrt{\xi}} \rfloor + 1, \end{array}$$

where $\lfloor \frac{n_0}{\sqrt{\xi}} \rfloor$ is the maximum integer which is less than or equal to $\frac{n_0}{\sqrt{\xi}}$.

Then, the negative curve C is linearly equivalent to $n_0A - r_0E$.

Proof. Suppose that the negative curve C is linearly equivalent to $n_1A - r_1E$. Since $n_1/r_1 < \sqrt{\xi}$ and $[\mathfrak{p}^{(r_1)}]_{n_1} \neq 0$, we have $n_1 \geq n_0$. Since $H^0(X, \mathcal{O}(n_0A - r_0E)) \neq 0$ with $n_0/r_0 < \sqrt{abc}$, $n_0A - r_0E - C$ is linearly equivalent to an effective divisor. Therefore, $n_0 \geq n_1$. Hence, $n_0 = n_1$.

Since $n_0/r_1 < \sqrt{\xi}$, $r_0 \leq r_1$ holds. Now, suppose $r_0 < r_1$. Let f be the defining equation of $\pi(C)$, where $\pi : X \to \mathbb{P}$ is the blow-up at $V_+(\mathfrak{p})$. Then, we have

$$[\mathfrak{p}^{(r_1-1)}]_{n_0} = [\mathfrak{p}^{(r_1)}]_{n_0} = k\ f.$$

If n is an integer less than n_0, then $[\mathfrak{p}^{(r_1-1)}]_n = 0$ because

$$\frac{n}{r_1 - 1} < \frac{n_0}{r_1 - 1} \leq \frac{n_0}{r_0} < \sqrt{\xi}.$$

By Remark 5.1, we have

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \in \mathfrak{p}^{(r_1-1)}.$$

Since their degrees are strictly less than n_0, we know

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0.$$

On the other hand, the equality

$$ax\frac{\partial f}{\partial x} + by\frac{\partial f}{\partial y} + cz\frac{\partial f}{\partial z} = n_0 f$$

holds. Remember that k is a field of characteristic zero. It is a contradiction. **q.e.d.**

Remark 5.3 Let a, b and c be pairwise coprime integers, and k be a field of characteristic zero. Assume that the negative curve C exists, and C is linearly equivalent to $n_0A - r_0E$.

Then, by Proposition 5.2, we obtain

$$\begin{aligned} n_0 &= \min\{n \in \mathbb{N} \mid [\mathfrak{p}^{(\lfloor \frac{n}{\sqrt{\xi}} \rfloor + 1)}]_n \neq 0\} \\ r_0 &= \lfloor \frac{n_0}{\sqrt{\xi}} \rfloor + 1. \end{aligned}$$

Theorem 5.4 *Let a, b and c be pairwise coprime positive integers such that $\sqrt{\xi} > \eta$. Assume that (C2) is satisfied, i.e., there exist positive integers n_1 and r_1 such that $n_1/r_1 < \sqrt{\xi}$ and $\dim_k S_{n_1} > r_1(r_1+1)/2$. Suppose $n_1 = \xi q_1 + v_1$, where q_1 and v_1 are integers such that $0 \leq v_1 < \xi$.*

Then, $q_1 < \frac{2\dim_k S_{v_1}}{\sqrt{\xi}-\eta}$ holds.

In particular,

$$n_1 = \xi q_1 + v_1 < \frac{2\xi \max\{\dim_k S_t \mid 0 \leq t < \xi\}}{\sqrt{\xi} - \eta} + \xi.$$

Proof. We have

$$r_1 > \frac{n_1}{\sqrt{\xi}} = \sqrt{\xi} q_1 + \frac{v_1}{\sqrt{\xi}}.$$

Therefore,

$$2\dim_k S_{n_1} > r_1^2 + r_1 > \xi q_1^2 + 2v_1 q_1 + \frac{v_1^2}{\xi} + \sqrt{\xi} q_1 + \frac{v_1}{\sqrt{\xi}}.$$

By Lemma 4.1, we have

$$(\sqrt{\xi} - \eta) q_1 < 2\dim_k S_{v_1} - \frac{v_1^2}{\xi} - \frac{v_1}{\sqrt{\xi}} \leq 2\dim_k S_{v_1}.$$

q.e.d.

Remember that, if $\sqrt{\xi} < \eta$, then (C2) is always satisfied by Theorem 4.3 (3).

6 Calculation by computer

In this section, we assume that the characteristic of k is zero.

6.1 Examples that do not satisfy (C2)

Suppose that (C2) is satisfied, i.e., there exist positive integers n_1 and r_1 such that $n_1/r_1 < \sqrt{\xi}$ and $\dim_k S_{n_1} > r_1(r_1+1)/2$. Put $n_1 = \xi q_1 + v_1$, where q_1 and v_1 are integers such that $0 \leq v_1 < \xi$. If $\sqrt{\xi} > \eta$, then $q_1 < \frac{2\dim_k S_{v_1}}{\sqrt{\xi}-\eta}$ holds by Theorem 5.4.

By the following programming on MATHEMATICA, we can check whether (C2) is satisfied or not in the case where $\sqrt{\xi} > \eta$.

```
c2[a_, b_, c_] :=
Do[
  If[(a + b + c)^2 > a b c , Print["-K: self-int positive"]; Goto[fin]];
  s = Series[((1 - t^a)(1 - t^b)(1 - t^c))^(-1), {t, 0, a b c}];
  Do[   h = SeriesCoefficient[s, k];
        m = IntegerPart[2 h/(Sqrt[a b c] - a - b - c)];
        Do[  r = IntegerPart[(a b c q + k)(Sqrt[a b c]^(-1))] + 1;
             If[2 h + q(a + b + c) + a b c q^2 + 2q k > r (r + 1),
               Print[StringForm["n=‘‘, r=‘‘", a b c q + k, r]];
               Goto[fin]],
          {q, 0, m}],
    {k, 0, a b c - 1}];
  Print["c2 is not satisfied"];
  Label[fin];
  Print["finished"]]
```

Calculations by a computer show that (C2) is not satisfied in some cases, for example, $(a, b, c) = (5, 33, 49), (7, 11, 20), (9, 10, 13), \cdots$.

The examples due to Goto-Nishida-Watanabe [4] have negative curves with $r = 1$. Therefore, by Remark 3.8, they satisfy the condition (C2).

In the case where $(a, b, c) = (5, 33, 49), (7, 11, 20), (9, 10, 13), \cdots$, the authors do not know whether $R_s(\mathfrak{p}_k)$ is Noetherian or not.

Remark 6.1 Set

$$\begin{aligned} A &= \{(a, b, c) \mid 0 < a \le b \le c \le 50, a, b, c \text{ are pairwise coprime}\} \\ B &= \{(a, b, c) \in A \mid a + b + c > \sqrt{abc}\} \\ C &= \{(a, b, c) \in A \mid (a, b, c) \text{ does not satisfy (C2)}\}. \end{aligned}$$

$\sharp A = 6156$, $\sharp B = 1950$, $\sharp C = 457$. By Theorem 4.3, we know $B \cap C = \emptyset$.

6.2 Does a negative curve exist?

By the following simple computer programming on MATHEMATICA, it is possible to know whether a negative curve exists or not.

```
n[a1_, b1_, c1_, r1_, d1_] := (V = 0;
  Do[
   mono = {};
   Do[ e1 = d1 - i*a1;
     Do[ h1 = e1 - j*b1; k1 = Floor[h1/c1];
       If[ h1 / c1 == k1,
         mono = Join[mono, {x^i y^j z^(k1)}]],
```

```
          {j, 0, Floor[e1/b1]}
        ], {i, 0, Floor[d1/a1]}
      ];
    w = Length[mono];
    If[w > N[r1*(r1 + 1)/2],
      V = 1; W = w; J = d1; R = r1; H = N[r1*(r1 + 1)/2],
      If[ w > 0,
        f[x_, y_, z_] := mono;
        mat = {};
        Do[
          Do[
            mat = Join[mat, { D[f[x, y, z], {x, j}, {y, i - j}] }], {j, 0, i}
            ], {i, 0, r1 - 1}
          ];
        mat = mat /. x -> 1 /. y -> 1 /. z -> 1;
        q = MatrixRank[mat];
        If[ q < w, V = 1; W = w; J = d1; R = r1; H = N[r1*(r1 + 1)/2] ]
      ]
    ]
  ]);

t[a_, b_, c_, rmade_] := (
  Do[
    W = 0;
    p = Ceiling[r*Sqrt[a*b*c]] - 1;
    Do[
      n[a, b, c, r, p - u];
      If[V == 1,
        J1 = J; Break[]], {u, 0, a - 1}
      ];
    If[
      V == 1,
      Do[
        n[a, b, c, r, J1 - a*u];
        If[V == 0,
          J1 = J; Break[]], {u, 1, b*c}
        ];

      Do[
        n[a, b, c, r, J1 - b*u];
        If[V == 0,
          J1 = J; Break[]], {u, 1, a*c}
        ];
```

```
      Do[
        n[a, b, c, r, J1 - c*u];
        If[V == 0,
          J1 = J; Break[]], {u, 1, c*a}
        ]
      ];
    If[W > 0, Break[]];

    Print["r th symbolic power does not contain a negative curve if r <= ",
      r], {r, 1, rmade}
    ];
  If[W == 0,
    Print["finished"],
    Print["There exists a negative curve. Degree = ", J, ",  r = ", R,
      ",  Dimension of homog. comp. = ", W, ",  # of equations = ", H]
    ]
  )
```

By the command t[a,b,c,r], we can check whether $\mathfrak{p}(a,b,c)^{(m)}$ contains an equation of a negative curve for $m = 1, 2, \cdots, r$.

$\mathfrak{p}(9,10,13)^{(m)}$ does not contain an equation of a negative curve if $m \leq 24$. Remember that $(9,10,13)$ does not satisfy (C2). Our computer gave up computation of $\mathfrak{p}(9,10,13)^{(25)}$ for scarcity of memories. We don't know whether ther exists a negative curve in the case $(9,10,13)$.

On the other hand, there are examples that (C2) is not satisfied but there exists a negative curve.

- Suppose $(a,b,c) = (5,33,49)$. Then (C2) is not satisfied, but $[\mathfrak{p}(5,33,49)^{(18)}]_{1617}$ contatins a negative curve.
- Suppose $(a,b,c) = (8,15,43)$. Then (C2) is not satisfied, but $[\mathfrak{p}(8,15,43)^{(9)}]_{645}$ contains a negative curve.

References

[1] M. Artin, *Some numerical criteria for contractability of curves on algebraic surfaces*, Amer. J. Math. **84** (1962), 485–496.

[2] R. Cowsik, *Symbolic powers and the number of defining equations*, Algebra and its Applications, Lect. Notes in Pure and Appl. Math. **91** (1985), 13–14.

[3] S. D. Cutkosky, *Symbolic algebras of monomial primes*, J. reine angew. Math. **416** (1991), 71–89.

[4] S. Goto, K, Nishida and K.-i. Watanabe, *Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik's question*, Proc. Amer. Math. Soc. **120** (1994), 383–392.

[5] J. Herzog, *Generators and relations of Abelian semigroups and semigroup rings*, Manuscripta Math. **3** (1970), 175–193.

[6] Y. Hu and S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348.

[7] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987), 293–318.

[8] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, in "Algebraic Geometry, Sendai," Adv. Stud. Pure Math. vol. 10 (1987), 283–360.

[9] K. Kurano, *On finite generation of Rees rings defined by filtrations of ideals*, J. Math. Kyoto Univ. **34** (1994), 73–86.

[10] K. Kurano, *Positive characteristic finite generation of symbolic Rees algebras and Roberts' counterexamples to the fourteenth problem of Hilbert*, Tokyo J. Math. **16** (1993), 473–496.

[11] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics , 48. Springer-Verlag, Berlin, 2004.

[12] F. Lazzeri, O. Stănăsilă and T. Tognoli, *Some remarks on q-flat C^∞-functions*, Bollettino U. M. I., (4) **9** (1974), 402–415.

[13] M. Morimoto and S. Goto, *Non-Cohen-Macaulay symbolic blow-ups for space monomial curves*, Proc. Amer. Math. Soc. **116** (1992), 305–311.

[14] M. Nagata, *On the fourteenth problem of Hilbert*, Proc. Internat. Congress Math. (1958), Cambridge Univ. Press, 1960.

[15] P. Roberts, *A prime ideal in a polynomial ring whose symbolic blow-up is not Noetherian*, Proc. Amer. Math. Soc. **94** (1985), 589–592.

[16] P. Roberts, *An infinitely generated symbolic blow-ups in a power series ring and a new counterexample to Hilbert's fourteenth problem*, J. Algebra **132** (1990), 461–473.

Kazuhiko Kurano
Department of Mathematics
Faculty of Science and Technology
Meiji University
Higashimita 1-1-1, Tama-ku
Kawasaki 214-8571, Japan
kurano@math.meiji.ac.jp
http://www.math.meiji.ac.jp/~kurano

Naoyuki Matsuoka
Department of Mathematics
Faculty of Science and Technology
Meiji University
Higashimita 1-1-1, Tama-ku
Kawasaki 214-8571, Japan
matsuoka@math.meiji.ac.jp