The singular Riemann-Roch theorem and Hilbert-Kunz functions

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Abstract

In the paper, via the singular Riemann-Roch theorem, it is proved that the class of the e-th Frobenius power ${}^eA$ can be described using the class of the canonical module ω_A for a normal local ring A of positive characteristic. As a corollary, we prove that the coefficient $\beta(I, M)$ of the second term of the Hilbert-Kunz function $\ell_A(M/I^{[p^e]}M)$ of e vanishes if A is a $\mathbb{Q}$-Gorenstein ring and M is a finitely generated A-module of finite projective dimension.

For a normal algebraic variety X over a perfect field of positive characteristic, it is proved that the first Chern class of the e-th Frobenius power $F^e_*\mathcal{O}_X$ can be described using the canonical divisor K_X.

1 Introduction

Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring of characteristic p, where p is a prime integer. Here, $\mathfrak{m}$ is the unique maximal primary ideal of A. For an $\mathfrak{m}$-primary ideal I and a positive integer e, we set

$$I^{[p^e]} = (a^{p^e} \mid a \in I)A.$$

It is easy to see that $I^{[p^e]}$ is an $\mathfrak{m}$-primary ideal of A. For a finitely generated A-module M, the function $\ell_A(M/I^{[p^e]}M)$ of e is called the *Hilbert-Kunz function* of M with respect to I, where $\ell_A(\ )$ stands for the length of the given A-module. It is known that

$$\lim_{e\to\infty} \frac{\ell_A(M/I^{[p^e]}M)}{p^{de}}$$

exists [9], and this limit is called the *Hilbert-Kunz multiplicity*, which is denoted by $e_{HK}(I, M)$. Several properties of $e_{HK}(I, M)$ have been studied by many authors (Monsky, Watanabe, Yoshida, Huneke, Enescu, etc.).

Recently Huneke, McDermott and Monsky (Theorem 1, Corollary 1.10 and Theorem 1.11 in [5]) proved the following exciting theorem:

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Theorem 1.1 (Huneke, McDermott and Monsky) *Let $(A, \mathfrak{m})$ be a d-dimensional excellent normal local ring of characteristic p, where p is a prime integer. Assume that the residue class field of A is perfect.*

Let I be an $\mathfrak{m}$-primary ideal of A and M be a finitely generated A-module.

1. *There exists a real number $\beta(I, M)$ that satisfies the following equation:*[1]

$$\ell_A(M/I^{[p^e]}M) = e_{HK}(I, M) \cdot p^{de} + \beta(I, M) \cdot p^{(d-1)e} + O(p^{(d-2)e})$$

2. *Assume that A is F-finite.*[2] *Then, there exists a $\mathbb{Q}$-homomorphism τ_I : $\mathrm{Cl}(A)_{\mathbb{Q}} \longrightarrow \mathbb{R}$ that satisfies*

$$\beta(I, M) = \tau_I \left(\mathrm{cl}(M) - \frac{\mathrm{rank}_A M}{p^d - p^{d-1}} \mathrm{cl}({}^1 A) \right)$$

for any finitely generated torsion-free A-module M. In particular, we have

$$\beta(I, A) = -\frac{1}{p^d - p^{d-1}} \tau_I \left(\mathrm{cl}({}^1 A) \right).$$

We denote by $\mathbb{Q}$ (resp. $\mathbb{R}$) the field of rational numbers (resp. real numbers). For an abelian group N, $N_{\mathbb{Q}}$ stands for $N \otimes_{\mathbb{Z}} \mathbb{Q}$.

The map $\mathrm{cl} : \mathrm{G}_0(A) \longrightarrow \mathrm{Cl}(A)$ is defined by Bourbaki [1] and sometimes called the *determinant map* (see Remark 2.1 below).

It is natural to ask the following questions:

Question 1.2

1. When does $\mathrm{cl}({}^1 A)$ vanish?
2. How do the $\mathrm{cl}({}^e A)$'s behave?

Using the singular Riemann-Roch formula, we obtain the following theorem:

Theorem 1.3 *Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian normal local ring of characteristic p, where p is a prime integer. Assume the following three conditions; (1) A is a homomorphic image of a regular local ring, (2) the residue class field of A is perfect, and (3) A is F-finite.*

Then, for each integer $e > 0$, we have

$$\mathrm{cl}({}^e A) = \frac{p^{de} - p^{(d-1)e}}{2} \mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$, where ω_A is the canonical module of A.

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[1] Let $f(e)$ and $g(e)$ be functions of e. We denote $f(e) = O(g(e))$ if there exists a real number K that satisfies $|f(e)| < Kg(e)$ for all $e \gg 0$.

[2] We say that A is F-finite if the Frobenius map $F : A \to A = {}^1 A$ is module-finite. We sometimes denote the e-th iteration of F by $F^e : A \to A = {}^e A$.

The following corollary is an immediate consequence of Theorem 1.1 and Theorem 1.3. A Noetherian normal local domain A is called $\mathbb{Q}$-*Gorenstein* if $\mathrm{cl}(\omega_A)$ is a torsion element in $\mathrm{Cl}(A)$.

Corollary 1.4 *Under the same assumptions as in Theorem 1.3, if A is a $\mathbb{Q}$-Gorenstein ring, then $\beta(I, A) = 0$ for any maximal primary ideal I.*

Remark 1.5 If A is a $\mathbb{Q}$-Gorenstein ring, then we have $\beta(I, M) = \tau_I(\mathrm{cl}(M))$ by Theorem 1.1 (2).

Furthermore, assume that M is a finitely generated A-module of finite projective dimension. Then, we have $\mathrm{cl}(M) = \mathrm{rank}_A M \cdot \mathrm{cl}(A) = 0$. Therefore, in this case, $\beta(I, M)$ is equal to 0.

The following is an analogue of Theorem 1.3 for normal algebraic varieties.

Theorem 1.6 *Let k be a perfect field of characteristic p, where p is a prime integer. Let X be a normal algebraic variety over k of dimension d. Let $F : X \to X$ be the absolute Frobenius map.*[3]

Then, we have

$$c_1(F_*^e \mathcal{O}_X) = \frac{p^{de} - p^{(d-1)e}}{2} K_X$$

in $\mathrm{A}_{d-1}(X)_{\mathbb{Q}}$, where $c_1(\ )$ is the first Chern class[4] *and K_X is the canonical divisor of X.*

Here, $A_{d-1}(X)$ is the Chow group of X consisting of cycles of dimension $d-1$. We refer the reader to [2] for Chow groups. If A (in Theorem 1.3) is a local ring at a closed point of a normal algebraic variety over a perfect field of positive characteristic, then Theorem 1.3 follows from Theorem 1.6.

We give a proof of Theorem 1.3 and Theorem 1.6 in the next section.

2 A proof of Theorem 1.3 and Theorem 1.6

Before proving Theorem 1.3, we recall basic properties on the determinant map.

Remark 2.1 The map cl in Theorem 1.3 is called the *determinant map* which is defined by Bourbaki [1]. Here, recall basic properties on cl which are used later.

Let R be a Noetherian normal domain. The group of isomorphism classes of reflexive R-modules of rank 1 is called the *divisor class group* of R, and denoted by $\mathrm{Cl}(R)$. Let $\mathrm{G}_0(R)$ be the Grothendieck group of finitely generated R-modules.

[3] Remark that, under the assumption, F is a finite morphism.

[4] Set $U = X \setminus \mathrm{Sing}(X)$. Since $\mathrm{codim}_X \mathrm{Sing}(X) \geq 2$, the restriction $\mathrm{A}_{d-1}(X) \to \mathrm{A}_{d-1}(U)$ is an isomorphism. Here, remark that $(F_*^e \mathcal{O}_X)|_U = (F|_U)_*^e \mathcal{O}_U$ is a locally free sheaf on U. Thus, $c_1(F_*^e \mathcal{O}_X)$ is defined as the first Chern class $c_1((F_*^e \mathcal{O}_X)|_U) \in \mathrm{A}_{d-1}(U) = \mathrm{A}_{d-1}(X)$.

For an R-module M, we denote by $[M]$ the element in $\mathrm{G}_0(R)$ corresponding to the isomorphism class which M belongs to. Then, there exists the unique map

$$\mathrm{cl} : \mathrm{G}_0(R) \longrightarrow \mathrm{Cl}(R)$$

that satisfies the following two conditions:

(1) If M is a reflexive module of rank 1, then $\mathrm{cl}([M])$ is just the isomorphism class which M belongs to.

(2) Let M be a finitely generated R-module. If the height of the annihilator of M is greater than 1, then $\mathrm{cl}([M]) = 0$.

For an R-module M, we denote $\mathrm{cl}([M])$ simply by $\mathrm{cl}(M)$ as usual.

Let $(A, \mathfrak{m})$ be a Noetherian local ring that satisfies the assumption in Theorem 1.3. It is enough to prove Theorem 1.3 for complete local rings. Therefore, in the rest of this section, we assume that A is a d-dimensional local normal domain which is a homomorphic image of a formal power series ring S over a perfect field k of positive characteristic unless otherwise specified. By the singular Riemann-Roch theorem (cf., Chapter 18 and 20 in [2]), we have an isomorphism

$$\tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S)} : \mathrm{G}_0(A)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(A)_{\mathbb{Q}}$$

of $\mathbb{Q}$-vector spaces. Here, remark that the Riemann-Roch map $\tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S)}$ is determined not only by $\mathrm{Spec}(A)$ but also by the regular base scheme $\mathrm{Spec}(S)$ as in 20.1 in [2]. Let

$$p_i : \mathrm{A}_*(A)_{\mathbb{Q}} \longrightarrow \mathrm{A}_i(A)_{\mathbb{Q}}$$

be the projection for $i = 0, 1, \ldots, d$. We set

$$\tau_i = p_i \circ \tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S)} : \mathrm{G}_0(A)_{\mathbb{Q}} \longrightarrow \mathrm{A}_i(A)_{\mathbb{Q}}.$$

For a prime ideal $\mathfrak{p}$ of A, $[\mathrm{Spec}(A/\mathfrak{p})]$ stands for the element in $\mathrm{A}_*(A)$ corresponding to the closed subscheme $\mathrm{Spec}(A/\mathfrak{p})$ of $\mathrm{Spec}(A)$.

Lemma 2.2 *Let A be a d-dimensional normal local ring which is a homomorphic image of a regular local ring.*

(i) There exists a natural isomorphism $\mathrm{A}_{d-1}(A) = \mathrm{Cl}(A)$ by identifying $[\mathrm{Spec}(A/\mathfrak{p})]$ with $\mathrm{cl}(\mathfrak{p})$ for any prime ideal $\mathfrak{p}$ of height 1. Then, for any prime ideal $\mathfrak{q} \neq 0$, $\tau_{d-1}([A/\mathfrak{q}])$ is equal to $-\mathrm{cl}(A/\mathfrak{q})$.

(ii) We have the equality

$$\tau_{d-1}([A]) = \frac{1}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{A}_{d-1}(A)_{\mathbb{Q}} = \mathrm{Cl}(A)_{\mathbb{Q}}$.

(iii) Furthermore, assume that A is a homomorphic image of a formal power series ring S over a perfect field of positive characteristic. Then, for each $e > 0$ and $i = 0, 1, \dots, d$, the equality

$$\tau_i([{}^eA]) = p^{ie}\tau_i([A])$$

is satisfied.

Proof. First we prove (i). It is well-known that there exists an isomorphism $\mathrm{A}_{d-1}(A) \to \mathrm{Cl}(A)$ by $[\mathrm{Spec}(A/\mathfrak{p})] \mapsto \mathrm{cl}(\mathfrak{p})$ (cf., Bourbaki [1]). Suppose that $\mathfrak{p}$ is a prime ideal of height 1. By the exact sequence

$$0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0,$$

we have

$$\mathrm{cl}(\mathfrak{p}) = \mathrm{cl}(A) - \mathrm{cl}(A/\mathfrak{p}) = -\mathrm{cl}(A/\mathfrak{p}).$$

On the other hand, by the top-term property (Theorem 18.3 (5) in [2]), we have $\tau_{d-1}([A/\mathfrak{p}]) = [\mathrm{Spec}(A/\mathfrak{p})]$. Thus, we obtain

$$\tau_{d-1}([A/\mathfrak{p}]) = [\mathrm{Spec}(A/\mathfrak{p})] = \mathrm{cl}(\mathfrak{p}) = -\mathrm{cl}(A/\mathfrak{p}).$$

Let $\mathfrak{q}$ be a prime ideal of height at least 2. By the top-term property, we have $\tau_{d-1}([A/\mathfrak{q}]) = 0$. In this case, we also have $\mathrm{cl}(A/\mathfrak{q}) = 0$ by Remark 2.1 (2). The proof of (i) is completed.

We refer the reader to Lemma 3.5 of [6] for a proof of (ii).

Now we start to prove (iii). Consider the following commutative diagrams:

$$\begin{array}{ccc} {}^eS & \longrightarrow & {}^eA \\ {\scriptstyle F^e}\uparrow & & \uparrow{\scriptstyle F^e} \\ S & \longrightarrow & A \end{array} \qquad \begin{array}{ccccc} S^{p^e} & \longrightarrow & S & \longrightarrow & A \\ \| & & & & \| \\ S & \longrightarrow & A & \xrightarrow{F^e} & {}^eA \end{array}$$

The lefthand diagram above and the covariance with the proper map $F^e : \mathrm{Spec}({}^eA) \to \mathrm{Spec}(A)$ (Theorem 18.3 (1) in [2]) imply that the bottom half of the following diagram commutes. The righthand diagram above implies that the top half of the following diagram commutes.

$$\begin{array}{ccc} \mathrm{G}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S^{p^e})}} & \mathrm{A}_*(A)_{\mathbb{Q}} \\ \| & & \| \\ \mathrm{G}_0({}^eA)_{\mathbb{Q}} & \xrightarrow{\tau_{\mathrm{Spec}({}^eA)/\mathrm{Spec}(S)}} & \mathrm{A}_*({}^eA)_{\mathbb{Q}} \\ {\scriptstyle F^e_*}\downarrow & & {\scriptstyle F^e_*}\downarrow \\ \mathrm{G}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S)}} & \mathrm{A}_*(A)_{\mathbb{Q}} \end{array} \tag{2.3}$$

Here, $S^{p^e} = \{x^{p^e} \mid x \in S\} \subset S$. Remark that S^{p^e} is a regular local ring and S is a finite module over S^{p^e}. Therefore, $\tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S^{p^e})}$ and $\tau_{\mathrm{Spec}(S)/\mathrm{Spec}(S^{p^e})}$ can be defined (cf. Chapter 18 and 20 in Fulton [2]).

Here, we shall prove

$$\tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S^{p^e})} = \tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S)} \tag{2.4}$$

for any $e > 0$.

Since S is a regular local ring, we have $\mathrm{G}_0(S)_{\mathbb{Q}} = \mathbb{Q}[S]$ and $\mathrm{A}_*(S)_{\mathbb{Q}} = \mathbb{Q}[\mathrm{Spec}(S)]$ since $\mathrm{G}_0(S)_{\mathbb{Q}}$ is isomorphic to $\mathrm{A}_*(S)_{\mathbb{Q}}$ by the singular Riemann-Roch theorem. By the top term property (Theorem 18.3 (5) in [2]), we have

$$\tau_{\mathrm{Spec}(S)/\mathrm{Spec}(S^{p^e})}([S]) = [\mathrm{Spec}(S)]. \tag{2.5}$$

Let M be a finitely generated A-module and $\mathbb{F}.$ be a finite S-free resolution of M. By definition of $\tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S)}$ (18.3 in [2]), we have

$$\tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S)}([M]) = \mathrm{ch}^{\mathrm{Spec}(S)}_{\mathrm{Spec}(A)}(\mathbb{F}.) \cap [\mathrm{Spec}(S)] \tag{2.6}$$

in $\mathrm{A}_*(A)_{\mathbb{Q}}$, where

$$\mathrm{ch}^{\mathrm{Spec}(S)}_{\mathrm{Spec}(A)}(\mathbb{F}.) : \mathrm{A}_*(S)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(A)_{\mathbb{Q}}$$

is the localized Chern character of the complex $\mathbb{F}.$ (18.1 in [2]). Therefore, we have

$$\begin{aligned}
\tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S^{p^e})}([M]) &= \tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S^{p^e})}\left(\sum_i (-1)^i [H_i(\mathbb{F}.)]\right) \\
&= \mathrm{ch}^{\mathrm{Spec}(S)}_{\mathrm{Spec}(A)}(\mathbb{F}.) \cap \tau_{\mathrm{Spec}(S)/\mathrm{Spec}(S^{p^e})}([S]) \\
&= \mathrm{ch}^{\mathrm{Spec}(S)}_{\mathrm{Spec}(A)}(\mathbb{F}.) \cap [\mathrm{Spec}(S)] \\
&= \tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S)}([M]),
\end{aligned}$$

where the second equality follows from the local Riemann-Roch formula (Example 18.3.12 in [2]), the third from (2.5) and the fourth from (2.6). Thus, (2.4) has been proved.

We denote the composite maps of the vertical arrows of the diagram (2.3) by $F^e_* : \mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(A)_{\mathbb{Q}}$ and $F^e_* : \mathrm{A}_*(A)_{\mathbb{Q}} \to \mathrm{A}_*(A)_{\mathbb{Q}}$, respectively. Then, by the definition of F_* (cf., Chapter 1 in [2]), it is easy to see that the restriction $F^e_*|_{\mathrm{A}_i(A)_{\mathbb{Q}}}$ is just the multiplication by p^{ie} for $i = 0, 1, \ldots, d$ and any $e > 0$. By (2.3) and (2.4), we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{G}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S)}} & \mathrm{A}_*(A)_{\mathbb{Q}} \\
{\scriptstyle F^e_*}\downarrow & & {\scriptstyle F^e_*}\downarrow \\
\mathrm{G}_0(A)_{\mathbb{Q}} & \xrightarrow{\tau_{\mathrm{Spec}(A)/\mathrm{Spec}(S)}} & \mathrm{A}_*(A)_{\mathbb{Q}}
\end{array} \tag{2.7}$$

Thus, for an A-module M, we obtain

$$\tau_i F^e_*([M]) = p^{ie}\tau_i([M]) \tag{2.8}$$

in $\mathrm{A}_i(A)_{\mathbb{Q}}$ for each i and e. Since $F^e_*([A]) = [{}^eA]$, (iii) has been proved. **q.e.d.**

Before proving Theorem 1.3, we prove the following lemma:

Lemma 2.9 *Let $(A, \mathfrak{m})$ be a d-dimensional normal local ring that is a homomorphic image of a regular local ring. Then, for a finitely generated A-module M, we have*

$$\tau_{d-1}([M]) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A M}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$.

Proof. Set $r = \mathrm{rank}_A M$. Then we have an exact sequence

$$0 \to A^r \to M \to T \to 0,$$

where T is a torsion module. By this exact sequence, we obtain

$$\mathrm{cl}(M) = r \cdot \mathrm{cl}(A) + \mathrm{cl}(T) = \mathrm{cl}(T).$$

On the other hand, by (ii) in Lemma 2.2, we obtain

$$\tau_{d-1}([M]) = r \cdot \tau_{d-1}([A]) + \tau_{d-1}([T]) = \frac{r}{2}\mathrm{cl}(\omega_A) + \tau_{d-1}([T]).$$

Therefore, it is enough to show $\tau_{d-1}([T]) = -\mathrm{cl}(T)$ for any torsion module T.

We may assume that $T = A/\mathfrak{q}$, where $\mathfrak{q} \neq 0$ is a prime ideal of A. Then, by (i) in Lemma 2.2, we have

$$\tau_{d-1}([A/\mathfrak{q}]) = -\mathrm{cl}(A/\mathfrak{q})$$

as required. **q.e.d.**

Now we start to prove Theorem 1.3.

By (iii) and (ii) in Lemma 2.2, we obtain

$$\tau_{d-1}([{}^eA]) = p^{(d-1)e}\tau_{d-1}([A]) = \frac{p^{(d-1)e}}{2}\mathrm{cl}(\omega_A).$$

By Lemma 2.9, we have

$$\tau_{d-1}([{}^eA]) = -\mathrm{cl}({}^eA) + \frac{\mathrm{rank}_A {}^eA}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$. Since $\mathrm{rank}_A {}^eA = p^{de}$, we obtain

$$\mathrm{cl}({}^eA) = \frac{p^{de} - p^{(d-1)e}}{2}\mathrm{cl}(\omega_A)$$

in $\mathrm{Cl}(A)_{\mathbb{Q}}$. **q.e.d.**

Remark 2.10 By Lemma 2.9 and Theorem 1.3, we have

$$\tau_{d-1}([M]) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A M}{2}\mathrm{cl}(\omega_A) = -\mathrm{cl}(M) + \frac{\mathrm{rank}_A M}{p^d - p^{d-1}}\mathrm{cl}({}^1A).$$

Therefore, by Theorem 1.1, we have

$$\beta(I, M) = -\tau_I(\tau_{d-1}([M])) \quad \text{and} \quad \beta(I, A) = -\frac{1}{2}\tau_I(\mathrm{cl}(\omega_A))$$

for any torsion-free A-module M.

In the rest of this section, we shall give an outline of a proof of Theorem 1.6. Let X be an algebraic variety that satisfies the assumptions in Theorem 1.6. Removing singularities of X, we may assume that X is a smooth algebraic variety over a perfect field k of characteristic $p > 0$.[5] Applying much the same method as in the proof of the commutativity of the diagram (2.7), one can prove that the diagram

$$\begin{array}{ccc} \mathrm{G}_0(X)_{\mathbb{Q}} & \stackrel{\tau_{X/\operatorname{Spec}(k)}}{\longrightarrow} & \mathrm{A}_*(X)_{\mathbb{Q}} \\ {\scriptstyle F_*^e}\downarrow & & {\scriptstyle F_*^e}\downarrow \\ \mathrm{G}_0(X)_{\mathbb{Q}} & \stackrel{\tau_{X/\operatorname{Spec}(k)}}{\longrightarrow} & \mathrm{A}_*(X)_{\mathbb{Q}} \end{array}$$

is also commutative.

Set

$$\tau_{X/\operatorname{Spec}(k)}([\mathcal{O}_X]) = t_d + t_{d-1} + \cdots + t_0$$

where $t_i \in \mathrm{A}_i(X)_{\mathbb{Q}}$ for $i = 0, 1, \cdots, d$. By the commutative diagram above, we have

$$\tau_{X/\operatorname{Spec}(k)}([F_*^e\mathcal{O}_X]) = p^{de}t_d + p^{(d-1)e}t_{d-1} + \cdots + p^0 t_0. \tag{2.11}$$

On the other hand, by Theorem 18.3 (2) in [2], we have

$$\begin{aligned} \tau_{X/\operatorname{Spec}(k)}([F_*^e\mathcal{O}_X]) &= \mathrm{ch}(F_*^e\mathcal{O}_X) \cap \tau_{X/\operatorname{Spec}(k)}([\mathcal{O}_X]) \\ &= \left(p^{de} + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \cdots\right) \cap (t_d + t_{d-1} + t_{d-2} + \cdots) \qquad (2.12) \end{aligned}$$

where c_i stands for $c_i(F_*^e\mathcal{O}_X)$ for $i = 1, 2, \ldots$ (cf. Example 3.2.3 in [2]). Here, remark that $F_*^e\mathcal{O}_X$ is a locally free sheaf on X since X is a non-singular variety.

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[5] Set $U = X \setminus \mathrm{Sing}(X)$. Since $\mathrm{codim}_X \mathrm{Sing}(X) \geq 2$, the restriction $\mathrm{A}_{d-1}(X) \to \mathrm{A}_{d-1}(U)$ is an isomorphism. On the other hand, we have $(F_*^e\mathcal{O}_X)|_U = (F|_U)_*^e\mathcal{O}_U$ and $K_X|_U = K_U$. Therefore, we have only to show $c_1((F|_U)_*^e\mathcal{O}_U) = \frac{p^{de} - p^{(d-1)e}}{2}K_U$.

In the case of Theorem 1.3, the proof does not become easier even if we remove singularities of $\mathrm{Spec}(A)$. The reason is that $\mathrm{Spec}(A) \setminus \mathrm{Sing}(A)$ is not smooth over the base regular scheme $\mathrm{Spec}(S)$.

Comparing (2.11) with (2.12), we have the following equalities:

$$\begin{aligned} p^{de}t_d &= p^{de}t_d \\ p^{(d-1)e}t_{d-1} &= c_1 t_d + p^{de}t_{d-1} \qquad (2.13) \\ p^{(d-2)e}t_{d-2} &= \frac{1}{2}(c_1^2 - 2c_2)t_d + c_1 t_{d-1} + p^{de}t_{d-2} \qquad (2.14) \\ &\vdots \end{aligned}$$

Note that $t_d = [X]$, and it is sometimes denoted by 1 since $[X]$ is the unit element of the Chow ring of X. By the definition of t_i (Example 3.2.4 and Chapter 18 in [2]), we have

$$\begin{aligned} t_{d-1} &= \mathrm{td}_1(\Omega_X^\vee) = \frac{1}{2}c_1(\Omega_X^\vee) = -\frac{1}{2}c_1(\Omega_X) = -\frac{1}{2}c_1(\omega_X) = -\frac{1}{2}K_X \qquad (2.15) \\ t_{d-2} &= \mathrm{td}_2(\Omega_X^\vee) = \frac{1}{12}\left(c_1(\Omega_X^\vee)^2 + c_2(\Omega_X^\vee)\right) = \frac{1}{12}\left(K_X^2 + c_2(\Omega_X^\vee)\right). \qquad (2.16) \end{aligned}$$

Substituting (2.15) and (2.16) for (2.13) and (2.14), we have

$$\begin{aligned} c_1(F_*^e\mathcal{O}_X) &= \frac{p^{de} - p^{(d-1)e}}{2}K_X \\ c_2(F_*^e\mathcal{O}_X) &= \frac{3p^{2de} - 6p^{(2d-1)e} + 3p^{2(d-1)e} - 4p^{de} + 6p^{(d-1)e} - 2p^{(d-2)e}}{24}K_X^2 \\ &\quad + \frac{p^{de} - p^{(d-2)e}}{12}c_2(\Omega_X^\vee) \\ &\vdots \end{aligned}$$

We have completed the proof of Theorem 1.6.

3 Some examples

Example 3.1 1. This example is due to Han-Monsky [3]. Set

$$A = \mathbb{F}_5[[x_1, \ldots, x_4]]/(x_1^4 + \cdots + x_4^4)$$

and $\mathfrak{m} = (x_1, \ldots, x_4)A$. Then, we have

$$\ell_A(A/\mathfrak{m}^{[5^e]}) = \frac{168}{61}5^{3e} - \frac{107}{61}3^e.$$

Therefore, in this case, $e_{HK}(\mathfrak{m}, A) = \frac{168}{61}$ and $\beta(\mathfrak{m}, A) = 0$. We thus know that there is no hope to extend Theorem 1.1 under the same assumptions to get a third term in the Hilbert-Kunz function of the form $\gamma \cdot p^{(d-2)e} + O(p^{(d-3)e})$ in place of $O(p^{(d-2)e})$.

2. Set

$$A = k[[x_{ij} \mid i = 1, \dots, m;\ j = 1, \dots, n]]/I_2(x_{ij}), \tag{3.2}$$

where k is a perfect field of characteristic $p > 0$, and $I_2(x_{ij})$ is the ideal generated by all the 2 by 2 minors of the generic m by n matrix (x_{ij}).

Suppose $m = 2$ and $n = 3$. Then, K.-i. Watanabe [10] proved

$$\ell_A(A/\mathfrak{m}^{[p^e]}) = (13p^{4e} - 2p^{3e} - p^{2e} - 2p^e)/8.$$

Therefore, we have $e_{HK}(\mathfrak{m}, A) = \frac{13}{8}$ and $\beta(\mathfrak{m}, A) = -\frac{1}{4} \neq 0$.

3. Let $(A, \mathfrak{m})$ be a homomorphic image of a regular local ring S. Then, by the singular Riemann-Roch theorem (Chapter 18 and 20 in Fulton [2]), we have an isomorphism

$$\tau_{\mathrm{Spec}(A)/\,\mathrm{Spec}(S)} : \mathrm{G}_0(A)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(A)_{\mathbb{Q}}$$

of $\mathbb{Q}$-vector spaces. For a finitely generated A-module M, put

$$\tau_{\mathrm{Spec}(A)/\,\mathrm{Spec}(S)}([M]) = \tau_d([M]) + \tau_{d-1}([M]) + \cdots + \tau_0([M]),$$

where $\tau_i([M]) \in \mathrm{A}_i(A)_{\mathbb{Q}}$ for $i = 0, 1, \dots, d$. Let $\mathbb{F}.$ be a bounded finite A-free complex such that each homology module has finite length. Then, by the local Riemann-Roch formula (Example 18.3.12 in [2]), we have

$$\sum_j (-1)^j \ell_A(H_j(\mathbb{F}. \otimes_A M)) = \sum_i \mathrm{ch}(\mathbb{F}.) \cap \tau_i([M]).$$

Furthermore, assume that A is a Cohen-Macaulay ring of characteristic p, where p is a prime number, and the residue class field of A is perfect. Let I be a maximal primary ideal of finite projective dimension. Let $\mathbb{F}_I.$ be a finite A-free resolution of A/I. Then, if the depth of M is equal to d, we have

$$\begin{aligned} \ell_A(M/I^{[p^e]}M) &= \ell_A(F^e_*(M)/IF^e_*(M)) \\ &= \sum_j (-1)^j \ell_A(H_j(F^e_*(M) \otimes \mathbb{F}_I.)) \\ &= \sum_i \mathrm{ch}(\mathbb{F}_I.) \cap \tau_i F^e_*([M]) \\ &= \sum_i \mathrm{ch}(\mathbb{F}_I.) \cap p^{ie}\tau_i([M]) \\ &= \sum_i \left(\mathrm{ch}(\mathbb{F}_I.) \cap \tau_i([M])\right) p^{ie} \end{aligned}$$

by the equality (2.8). Therefore, in this case, we have $e_{HK}(I, M) = \mathrm{ch}(\mathbb{F}_I.) \cap \tau_d([M])$ and $\beta(I, M) = \mathrm{ch}(\mathbb{F}_I.) \cap \tau_{d-1}([M])$.

One can prove that there exists a maximal primary ideal I of finite projective dimension such that $\mathrm{ch}(\mathbb{F}_{I}.) \cap \tau_i([M]) \neq 0$ if and only if $\tau_i([M])$ is not numerically equivalent to 0 (cf., Theorem 6.4 in [8]). We refer the reader to [8] for the theory of numerical equivalence.

Suppose that A is the ring in (3.2) as above. In this case, $\tau_{d-1}([A])$ is not numerically equivalent to 0 if and only if $m \neq n$ (cf., Section 3 in [7] and Example 7.9 in [8]). Therefore, if $m \neq n$, then there exists a maximal primary ideal I of finite projective dimension such that $\beta(I, A) \neq 0$.

On the other hand, if $m = n$, then A is a Gorenstein ring. By Corollary 1.4, $\beta(I, A) = 0$ for any maximal primary ideal I of A.

Example 3.3 1. Set

$$A = k[[x_1, x_2, x_3, y_1, y_2, y_3]] \Big/ I_2 \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix},$$

$\mathfrak{p} = (x_1, x_2, x_3)A$ and $\mathfrak{q} = (x_1, y_1)A$. Here, assume that k is a perfect field of characteristic 2. Then, applying Hirano's formula[6] (Theorem 2 in [4]), one can prove that

$$^1A \simeq A^{\oplus 10} \oplus \mathfrak{p} \oplus \mathfrak{q}^{\oplus 5}.$$

Here, remark that $\mathrm{rank}_A {}^1A = p^{\dim A} = 2^4 = 16$.

Then, we have

$$\mathrm{cl}(^1A) = 10 \cdot \mathrm{cl}(A) + \mathrm{cl}(\mathfrak{p}) + 5 \cdot \mathrm{cl}(\mathfrak{q}) = 4 \cdot \mathrm{cl}(\mathfrak{q})$$

since $\mathrm{cl}(A) = 0$ and $\mathrm{cl}(\mathfrak{p}) + \mathrm{cl}(\mathfrak{q}) = 0$.

On the other hand, it is well known that $\omega_A \simeq \mathfrak{q}$. By Theorem 1.3, we have

$$\mathrm{cl}(^1A) = \frac{2^4 - 2^3}{2} \mathrm{cl}(\omega_A) = 4 \cdot \mathrm{cl}(\mathfrak{q}).$$

2. Let k be a perfect field of characteristic p, where p is a prime integer. Put $X = \mathbb{P}^1_k$. Let $F : X \to X$ be the absolute Frobenius map. Then, we have $F_*\mathcal{O}_X \simeq \mathcal{O}_X \oplus \mathcal{O}_X(-1)^{\oplus(p-1)}$, and

$$c_1(F_*\mathcal{O}_X) = c_1(\wedge^p F_*\mathcal{O}_X) = c_1(\mathcal{O}_X(1-p)) = 1 - p.$$

[6] Hirano proved the following: Let X be a d-dimensional toric variety over a perfect field of characteristic $p > 0$ defined by a fan in $N = \mathbb{Z}^d$. Let $F : X \to X$ be the absolute Frobenius map. Then, for any positive integer e, we have

$$F^e_*\mathcal{O}_X = \bigoplus_{0 \leq s_1, \ldots, s_d \leq p^e} \mathcal{O}_X \left(\frac{1}{p^e} \mathrm{div}_X(u_1^{s_1} \cdots u_d^{s_d}) \right) u_1^{s_1/p^e} \cdots u_d^{s_d/p^e},$$

where $\{u_1, \ldots, u_d\}$ is the dual basis of $N = \mathbb{Z}^d$.

Here, remark that the natural map

$$\deg : \mathrm{Cl}(X) \to \mathbb{Z}$$

is an isomorphism in this case.

On the other hand, it is well known that $\omega_X \simeq \mathcal{O}_X(-2)$. Therefore, we have $K_X = -2$. By Theorem 1.6, we have

$$c_1(F_*\mathcal{O}_X) = \frac{p-1}{2}K_X = 1-p.$$

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