Numerical equivalence defined on Chow groups of Noetherian local rings

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Abstract

In the present paper, we define a notion of numerical equivalence on Chow groups or Grothendieck groups of Noetherian local rings, which is an analogue of that on smooth projective varieties. Under a mild condition, it is proved that the Chow group modulo numerical equivalence is a finite dimensional $\mathbb{Q}$-vector space, as in the case of smooth projective varieties. Numerical equivalence on local rings is deeply related to that on smooth projective varieties. For example, if Grothendieck's standard conjectures are true, then a vanishing of Chow group (of local rings) modulo numerical equivalence can be proven.

Using the theory of numerical equivalence, the notion of numerically Roberts rings is defined. It is proved that a Cohen-Macaulay local ring of positive characteristic is a numerically Roberts ring if and only if the Hilbert-Kunz multiplicity of a maximal primary ideal of finite projective dimension is always equal to its colength. Numerically Roberts rings satisfy the vanishing property of intersection multiplicities.

We shall prove another special case of the vanishing of intersection multiplicities using a vanishing of localized Chern characters.

1 Introduction

For a smooth projective variety X over a field, the Chow ring $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$ with rational coefficients is defined. However, Chow rings are usually large and difficult to understand. In order to study Chow rings, we define numerical equivalence and sometimes consider the Chow ring modulo numerical equivalence, denoted by $\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}$. $\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space and a graded Gorenstein ring. Compared with the ordinary Chow ring $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$, $\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}$ is easier to study.

In the present paper, we define a notion of numerical equivalence on Chow groups or Grothendieck groups of Noetherian local rings (Definition 2.2), which is an analogue of that on smooth projective varieties. We denote by $\overline{\mathrm{A}_*(R)_{\mathbb{Q}}} = \oplus_i \overline{\mathrm{A}_i(R)_{\mathbb{Q}}}$ (resp. $\overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$) the Chow group (resp. Grothendieck group) of a Noetherian local ring R modulo numerical equivalence. In many cases, the Chow group

$\mathrm{A}_*(R)_{\mathbb{Q}}$ and the Grothendieck group $\mathrm{G}_0(R)_{\mathbb{Q}}$ are infinite dimensional $\mathbb{Q}$-vector spaces. The main theorem of the present paper is as follows:

Theorem 3.1 *Let $(R, \mathfrak{m})$ be a Noetherian excellent local ring that satisfies one of the following two conditions; (1) R contains $\mathbb{Q}$, (2) R is essentially of finite type over a field, $\mathbb{Z}$ or a complete discrete valuation ring.*

Then, we have $\dim \overline{\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}} = \dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} = \dim \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} < \infty$.

Numerical equivalence on Noetherian local rings is deeply related to that on smooth projective varieties. (This will be discussed in Section 7.) In fact, let $A = \oplus_{n \geq 0} A_n$ be a standard graded ring over a field $k = A_0$, and let $R = A_{A_+}$, where $A_+ = \oplus_{n>0} A_n$. Assume that $X = \mathrm{Proj}(A)$ is smooth over k. Let h be the very ample divisor under the embedding. It is known (cf. [10]) that there is an isomorphism

$$\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}/h \cdot \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \simeq \mathrm{A}_*(R)_{\mathbb{Q}}.$$

As will be shown in Section 7, this isomorphism induces the natural surjection

$$\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}/h \cdot \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} \longrightarrow \overline{\mathrm{A}_*(R)_{\mathbb{Q}}},$$

that is not always an isomorphism. It is an isomorphism if and only if the natural map

$$\text{(1.1)} \quad \mathrm{Ker}\left[\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \xrightarrow{h} \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}\right] \to \mathrm{Ker}\left[\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} \xrightarrow{h} \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}\right]$$

is surjective, as will be shown in Section 7. (The map (1.1) is studied in depth by Roberts and Srinivas [29].) In particular, if the natural map $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}$ is an isomorphism, so is $\mathrm{A}_*(R)_{\mathbb{Q}} \longrightarrow \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$, as shown in Theorem 7.7. We shall see in Remark 7.12 that, if Grothendieck's standard conjectures are true, then

$$\overline{\mathrm{A}_j(R)_{\mathbb{Q}}} = 0 \text{ for } j \leq \dim R/2\ .$$

This is equivalent to the condition that $\chi_{\mathbb{F}.}(M)$ is equal to 0 for any $\mathbb{F}. \in C^{\mathfrak{m}}(R)$ and any finitely generated R-module M with $\dim M \leq \dim R/2$. We refer the reader to Section 2 for the definition of $C^{\mathfrak{m}}(R)$ and $\chi_{\mathbb{F}.}(M)$.

In Section 4, we shall study the invariant $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ of a local ring R.

Let R be a homomorphic image of a regular local ring S. Then, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\tau_{R/S} : \mathrm{G}_0(R)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(R)_{\mathbb{Q}}$$

by the singular Riemann-Roch theorem with base regular ring S (18.2 and 20.1 in [2]). Recall that the map $\tau_{R/S}$ is determined by both R and S. (The author does not know an explicit example such that $\tau_{R/S}$ actually depends on the choice of S.) By Proposition 2.4, we have an induced homomorphism $\overline{\tau_{R/S}}$ that makes the following diagram commutative:

$$
\begin{array}{ccccc}
\mathrm{G}_0(R)_{\mathbb{Q}} & \stackrel{\tau_{R/S}}{\longrightarrow} & \mathrm{A}_*(R)_{\mathbb{Q}} & & \\
\downarrow & & \downarrow & & \\
\overline{\mathrm{G}_0(R)_{\mathbb{Q}}} & \stackrel{\overline{\tau_{R/S}}}{\longrightarrow} & \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} & = & \oplus_{i=0}^{\dim R} \overline{\mathrm{A}_i(R)_{\mathbb{Q}}}
\end{array}
$$

We shall prove that the map $\overline{\tau_{R/S}}$ is independent of the choice of S in Section 5.

Using the map $\overline{\tau_{R/S}}$, the notion of numerically Roberts rings is defined in Section 6. Numerically Roberts rings are characterized in terms of Dutta multiplicity or Hilbert-Kunz multiplicity as follows:

Theorem 6.4 *Let $(R, \mathfrak{m})$ be a homomorphic image of a regular local ring.*

(1) Then, R is a numerically Roberts ring if and only if the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ coincides with the alternating sum of length of homology

$$\chi(\mathbb{F}.) = \sum_i (-1)^i \ell_R(H_i(\mathbb{F}.))$$

for any $\mathbb{F}. \in C^{\mathfrak{m}}(R)$.

(2) Assume that R is a Cohen-Macaulay ring of characteristic p, where p is a prime number. Then, R is a numerically Roberts ring if and only if the Hilbert-Kunz multiplicity $e_{HK}(J)$ of J coincides with the colength $\ell_R(R/J)$ for any $\mathfrak{m}$-primary ideal J of finite projective dimension.

Roberts rings defined in [12] are numerically Roberts rings. As in Remark 6.5, numerically Roberts rings satisfy the vanishing property of intersection multiplicities. As in [12], the category of Roberts rings contains rings of dimension at most 1, complete intersections, quotient singularities, Galois extensions of regular local rings, affine cones of abelian varieties, and many others. As in Example 6.6, the category of numerically Roberts rings contains integral domains of dimension at most 2, Gorenstein rings of dimension 3, and many others.

A vanishing of Chow groups modulo numerical equivalence will be discussed in Section 8. We shall prove the following theorem:

Theorem 8.1 *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local domain that is a homomorphic image of a regular local ring. Assume that there exists a regular alteration $\pi : Z \to \mathrm{Spec}(R)$. Then, we have $\overline{\mathrm{A}_t(R)_{\mathbb{Q}}} = 0$ for $t < d - \dim \pi^{-1}(\mathfrak{m})$.*

By this theorem, we have

$$\dim \pi^{-1}(\mathfrak{m}) \geq d - \min\{t \mid \overline{\mathrm{A}_t(R)_{\mathbb{Q}}} \neq 0\}$$

for any resolution of singularities $\pi : Z \to \mathrm{Spec}(R)$.

In Theorem 9.1, we shall prove a special case of the vanishing property of intersection multiplicities as follows:

Theorem 9.1 *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local domain. Assume that R is a homomorphic image of an excellent regular local ring S and there exists a regular alteration $\pi : Z \to \mathrm{Spec}(R)$. Let Y be a closed subset of $\mathrm{Spec}(R)$ such that*

$$\pi|_{Z \setminus \pi^{-1}(Y)} : Z \setminus \pi^{-1}(Y) \to \mathrm{Spec}(R) \setminus Y$$

is finite. If $\dim \pi^{-1}(Y) \leq d/2$, then R satisfies the vanishing property, that is,

$$\sum_i (-1)^i \ell_R(\mathrm{Tor}_i^R(M, N)) = 0$$

for finitely generated R-modules M and N such that $\mathrm{pd}_R M < \infty$, $\mathrm{pd}_R N < \infty$, $\ell_R(M \otimes_R N) < \infty$ and $\dim M + \dim N < d$, where pd_R denotes the projective dimension as an R-module.

In Theorem 9.4, we give an another proof of the vanishing theorem of the first localized Chern characters due to Roberts [28].

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2 Numerical equivalence

Throughout the present paper, we assume that all local rings are homomorphic images of regular local rings. We denote by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ the ring of integers, the field of rational numbers and the field of complex numbers, respectively.

Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring. Let $C^{\mathfrak{m}}(R)$ be the category of bounded complexes of finitely generated R-free modules and R-linear maps such that its support is in $\{\mathfrak{m}\}$, i.e., each homology is of finite length as an R-module. A morphism in $C^{\mathfrak{m}}(R)$ is a chain homomorphism of R-linear maps. We sometimes call $C^{\mathfrak{m}}(R)$ the category of bounded R-free complexes with support in $\{\mathfrak{m}\}$.

We define the *Grothendieck group* of complexes with support in $\{\mathfrak{m}\}$ as

$$\mathrm{K}_0^{\mathfrak{m}}(R) = \bigoplus_{\mathbb{F}. \in C^{\mathfrak{m}}(R)} \mathbb{Z} \cdot [\mathbb{F}.] \Big/ P,$$

where $[\mathbb{F}.]$ is a free basis corresponding to the isomorphism class containing a complex

$$\mathbb{F}. \ : \ \cdots \to F_n \to F_{n-1} \to \cdots$$

contained in $C^{\mathfrak{m}}(R)$, and P is the subgroup generated by both

$$\{[\mathbb{G}.] - [\mathbb{F}.] - [\mathbb{H}.] \mid 0 \to \mathbb{F}. \to \mathbb{G}. \to \mathbb{H}. \to 0 \text{ is exact in } C^{\mathfrak{m}}(R)\}$$

and

$$\{[\mathbb{F}.] - [\mathbb{G}.] \mid \mathbb{F}. \to \mathbb{G}. \text{ is a quasi-isomorphism in } C^{\mathfrak{m}}(R)\}.$$

We define the *Grothendieck group* of finitely generated R-modules as

$$\mathrm{G}_0(R) = \bigoplus_{M \in \mathcal{M}(R)} \mathbb{Z} \cdot [M] \Big/ Q,$$

where $\mathcal{M}(R)$ is the category of finitely generated R-modules and R-linear maps, $[M]$ is a free basis corresponding to the isomorphism class containing M, and Q is the subgroup generated by

$$\{[M] - [L] - [N] \mid \ 0 \to L \to M \to N \to 0 \text{ is exact in } \mathcal{M}(R)\}.$$

Let $\mathrm{A}_*(R) = \oplus_{i=0}^{d} \mathrm{A}_i(R)$ be the *Chow group* of the affine scheme $\mathrm{Spec}(R)$, i.e.,

$$\mathrm{A}_i(R) = \bigoplus \mathbb{Z} \cdot [\mathrm{Spec}(R/\mathfrak{p})] \Big/ \mathrm{Rat}_i(R),$$

where the sum as above is taken over all prime ideals $\mathfrak{p}$ of $\dim R/\mathfrak{p} = i$, $[\mathrm{Spec}(R/\mathfrak{p})]$ is a free basis corresponding to a prime ideal $\mathfrak{p}$, and $\mathrm{Rat}_i(R)$ is the subgroup generated by rational equivalence (cf. Chapter 1 of [2]).

For an additive group A, we denote by $A_{\mathbb{Q}}$ the tensor product $A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let Z be a Noetherian scheme and let X be a closed subset of Z. We denote by $C^X(Z)$ the category of bounded complexes of vector bundles on Z with support in X. Let $\mathrm{K}_0^X(Z)$ be the Grothendieck group of $C^X(Z)$. Let $\mathrm{G}_0(Z)$ be the Grothendieck group of coherent sheaves on Z. Let $\mathrm{A}_*(Z)$ be the Chow group of Z. We refer the reader to Fulton [2], Gillet-Soulé [3] and Srinivas [31] for definitions and basic properties on $\mathrm{K}_0^X(Z)$, $\mathrm{G}_0(Z)$ and $\mathrm{A}_*(Z)$. For $\mathbb{G}. \in C^X(Z)$, the *localized Chern characters* (Chapter 18 of [2])

$$\mathrm{ch}(\mathbb{G}.) = \sum_{i \geq 0} \mathrm{ch}_i(\mathbb{G}.) : \mathrm{A}_*(Z)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(X)_{\mathbb{Q}}$$

are defined as operators on the Chow group, and for $\eta \in \mathrm{A}_j(Z)_{\mathbb{Q}}$, we have

$$\mathrm{ch}_i(\mathbb{G}.)(\eta) \in \mathrm{A}_{j-i}(X)_{\mathbb{Q}}$$

for each i.

For $\mathbb{F}. \in C^{\mathfrak{m}}(R)$ and $M \in \mathcal{M}(R)$, we set

$$\chi_{\mathbb{F}.}(M) = \sum_i (-1)^i \ell_R(H_i(\mathbb{F}. \otimes_A M)),$$

where $\ell_R(\ )$ denotes the *length* as an R-module. We obtain well-defined maps

$$\begin{aligned} e &: \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} \otimes \mathrm{G}_0(R)_{\mathbb{Q}} \longrightarrow \mathbb{Q} \\ v &: \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} \otimes \mathrm{A}_*(R)_{\mathbb{Q}} \longrightarrow \mathbb{Q} \end{aligned} \tag{2.1}$$

that satisfy $e([\mathbb{F}.]\otimes[M]) = \chi_{\mathbb{F}.}(M)$ and $v([\mathbb{F}.]\otimes[\mathrm{Spec}(R/\mathfrak{p})]) = \mathrm{ch}(\mathbb{F}.)([\mathrm{Spec}(R/\mathfrak{p})])$, where

$$\mathrm{ch}(\mathbb{F}.) : \mathrm{A}_*(R)_{\mathbb{Q}} \to \mathrm{A}_*(R/\mathfrak{m})_{\mathbb{Q}} = \mathbb{Q}\cdot[\mathrm{Spec}(R/\mathfrak{m})] = \mathbb{Q}$$

is the localized Chern character of $\mathbb{F}. \in C^{\mathfrak{m}}(R)$ (cf. Proposition 18.1 (b) and Example 18.1.4 in Fulton [2]).

We define cycles that are numerically equivalent to 0 and groups modulo numerical equivalence as follows.

Definition 2.2 We define subgroups consisting of elements numerically equivalent to 0 as

$$\begin{array}{rcl} \mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} & = & \{\alpha \in \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} \mid e(\alpha\otimes\beta) = 0 \text{ for any } \beta \in \mathrm{G}_0(R)_{\mathbb{Q}}\}, \\ \mathrm{N}\,\mathrm{G}_0(R)_{\mathbb{Q}} & = & \{\beta \in \mathrm{G}_0(R)_{\mathbb{Q}} \mid e(\alpha\otimes\beta) = 0 \text{ for any } \alpha \in \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}\}, \\ \mathrm{N}\,\mathrm{A}_i(R)_{\mathbb{Q}} & = & \{\gamma \in \mathrm{A}_i(R)_{\mathbb{Q}} \mid v(\alpha\otimes\gamma) = 0 \text{ for any } \alpha \in \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}\} \end{array}$$

for $i = 0, \ldots, d$.

We define groups modulo numerical equivalence as

$$\begin{array}{rcl} \overline{\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}} & = & \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}/\,\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}, \\ \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} & = & \mathrm{G}_0(R)_{\mathbb{Q}}/\,\mathrm{N}\,\mathrm{G}_0(R)_{\mathbb{Q}}, \\ \overline{\mathrm{A}_i(R)_{\mathbb{Q}}} & = & \mathrm{A}_i(R)_{\mathbb{Q}}/\,\mathrm{N}\,\mathrm{A}_i(R)_{\mathbb{Q}} \end{array}$$

for $i = 0, \ldots, d$.

By definition, e induces a map

$$\overline{e} : \overline{\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}} \otimes \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \longrightarrow \mathbb{Q}$$

that satisfies $\overline{e}(\overline{\alpha}\otimes\overline{\beta}) = e(\alpha\otimes\beta)$ where $\overline{\alpha}$ and $\overline{\beta}$ denote the images of α and β, respectively.

Let R be a homomorphic image of a regular local ring S. By the singular Riemann-Roch theorem with base regular ring S (18.2 and 20.1 in [2]), we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\tau_{R/S} : \mathrm{G}_0(R)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(R)_{\mathbb{Q}}.$$

Recall that the map $\tau_{R/S}$ as above is defined using not only R but also S.[1]

[1] Let Z be a regular scheme. For a scheme X of finite type over Z, we have an isomorphism of $\mathbb{Q}$-vector spaces $\tau_{X/Z} : \mathrm{G}_0(X)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(X)_{\mathbb{Q}}$ by the singular Riemann-Roch theorem with regular base scheme Z (20.1 in [2]).

If R is a homomorphic image of a regular local ring S, we denote $\tau_{\mathrm{Spec}(R)/\mathrm{Spec}(S)}$ simply by $\tau_{R/S}$.

The map $\tau_{X/Z}$ usually depends on the choice of Z, as will be shown in the following example.

Note that

$$\mathrm{N}\,\mathrm{K}_0^{\mathtt{m}}(R)_{\mathbb{Q}} = \{\alpha \in \mathrm{K}_0^{\mathtt{m}}(R)_{\mathbb{Q}} \mid v(\alpha \otimes \gamma) = 0 \text{ for any } \gamma \in \mathrm{A}_*(R)_{\mathbb{Q}}\}$$

because the diagram

$$\begin{array}{ccc} \mathrm{G}_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_{R/S}} & \mathrm{A}_*(R)_{\mathbb{Q}} \\ {\scriptstyle \chi_{\mathbb{F}.}}\downarrow & & {\scriptstyle \mathrm{ch}(\mathbb{F}.)}\downarrow \\ \mathbb{Q} & = & \mathbb{Q} \end{array} \tag{2.3}$$

is commutative for each $\mathbb{F}. \in C^{\mathtt{m}}(R)$ by the local Riemann-Roch theorem (Example 18.3.12 in [2]).

In order to construct an isomorphism $\overline{\tau_{R/S}}$ between $\overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ and $\oplus_i \overline{\mathrm{A}_i(R)_{\mathbb{Q}}}$, we need the following proposition.

Proposition 2.4 *With notation as above, we have*

$$\tau_{R/S}(\mathrm{N}\,\mathrm{G}_0(R)_{\mathbb{Q}}) = \bigoplus_{i=0}^{d} \mathrm{N}\,\mathrm{A}_i(R)_{\mathbb{Q}}.$$

Proof. Since diagram (2.3) is commutative for each $\mathbb{F}. \in C^{\mathtt{m}}(R)$, we have

$$\tau_{R/S}(\mathrm{N}\,\mathrm{G}_0(R)_{\mathbb{Q}}) = \{\gamma \in \mathrm{A}_*(R)_{\mathbb{Q}} \mid \mathrm{ch}(\mathbb{F}.)(\gamma) = 0 \text{ for any } \mathbb{F}. \in C^{\mathtt{m}}(R)\}.$$

By definition, it is easy to see that $\tau_{R/S}(\mathrm{N}\,\mathrm{G}_0(R)_{\mathbb{Q}}) \supseteq \oplus_{i=0}^{d} \mathrm{N}\,\mathrm{A}_i(R)_{\mathbb{Q}}$.

We shall prove the opposite containment. Let β be an element of $\mathrm{N}\,\mathrm{G}_0(R)_{\mathbb{Q}}$. Set $\tau_{R/S}(\beta) = \gamma_d + \gamma_{d-1} + \cdots + \gamma_0$, where $\gamma_i \in \mathrm{A}_i(R)_{\mathbb{Q}}$ for each i. We want to show $\mathrm{ch}(\mathbb{F}.)(\gamma_i) = 0$ for any $\mathbb{F}. \in C^{\mathtt{m}}(R)$ and any i.

For a positive integer n, we denote by $\psi^n : \mathrm{K}_0^{\mathtt{m}}(R)_{\mathbb{Q}} \to \mathrm{K}_0^{\mathtt{m}}(R)_{\mathbb{Q}}$ the n-th Adams operation defined by Gillet and Soulé [3]. For n, i and $\mathbb{F}.$, we have $\mathrm{ch}_i(\psi^n(\mathbb{F}.)) = n^i \mathrm{ch}_i(\mathbb{F}.)$ by Theorem 3.1 in [14]. Then, we have

$$\begin{aligned} 0 &= \chi_{\psi^n(\mathbb{F}.)}(\beta) \\ &= \mathrm{ch}(\psi^n(\mathbb{F}.))(\tau_{R/S}(\beta)) \\ &= \sum_{i=0}^{d} \mathrm{ch}_i(\psi^n(\mathbb{F}.))(\gamma_i) \\ &= \sum_{i=0}^{d} n^i \cdot \mathrm{ch}_i(\mathbb{F}.)(\gamma_i) \\ &= \sum_{i=0}^{d} n^i \cdot \mathrm{ch}(\mathbb{F}.)(\gamma_i) \end{aligned}$$

Assume that X is a smooth algebraic variety over a field k. Set $Z = \mathrm{Spec}(k)$. Then, by 18.2 in [2], we have $\tau_{X/Z}([\mathcal{O}_X]) = \mathrm{td}(\Omega_{X/k}^{\vee}) = 1 - c_1(K_X)/2 + \cdots$, that is, the Todd class (Example 3.2.4 in [2]) of the tangent sheaf of X. On the other hand, we have $\tau_{X/X}([\mathcal{O}_X]) = 1$ by definition. Therefore, if X is not $\mathbb{Q}$-Gorenstein, then $\tau_{X/Z}([\mathcal{O}_X]) \neq \tau_{X/X}([\mathcal{O}_X])$.

However, in the case of $X = \mathrm{Spec}(R)$ such that R is a Noetherian local ring, the author does not know any example such that $\tau_{X/Z}$ actually depends on the choice of a regular base scheme Z (cf. Section 4 in [12]).

for any positive integer n. Therefore, we have $\mathrm{ch}(\mathbb{F}.)(\gamma_i) = 0$ for each i. **q.e.d.**

We denote by $\overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$ the direct sum of $\overline{\mathrm{A}_i(R)_{\mathbb{Q}}}$'s. By the previous proposition, we have an isomorphism $\overline{\tau_{R/S}}$ that makes the following diagram commutative:

$$
\begin{array}{ccccc}
\mathrm{G}_0(R)_{\mathbb{Q}} & \stackrel{\tau_{R/S}}{\longrightarrow} & \mathrm{A}_*(R)_{\mathbb{Q}} & & \\
\downarrow & & \downarrow & & \\
\overline{\mathrm{G}_0(R)_{\mathbb{Q}}} & \stackrel{\overline{\tau_{R/S}}}{\longrightarrow} & \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} & = & \oplus_{i=0}^{d} \overline{\mathrm{A}_i(R)_{\mathbb{Q}}}
\end{array}
$$

where the vertical maps are the natural projections.

The map $\tau_{R/S}$ is constructed using not only R but also S. However, it will be proved in Section 5 that the map $\overline{\tau_{R/S}}$ is independent of the choice of S.

Remark 2.5 For any local domain $(R, \mathfrak{m})$ of $\dim R \leq 2$, we shall show that $\overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ is spanned by $\overline{[R]}$ as a $\mathbb{Q}$-vector space by Proposition 3.7. If R is an affine cone of a smooth curve over $\mathbb{C}$ of positive genus, then $\dim \mathrm{G}_0(R)_{\mathbb{Q}} = \infty$ as in Example 6.7. Therefore, $\mathrm{N}\,\mathrm{G}_0(R)_{\mathbb{Q}} \neq 0$ in this case.

Let $\mathbb{K}.$ be the Koszul complex with respect to a system of parameters $\underline{a}$ of a local ring R. Then, it is well known that $\chi_{\mathbb{K}.}([R])$ is the multiplicity of R with respect to the ideal $(\underline{a})$. Therefore, $\mathbb{K}.$ is not contained in $\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}$, and $\overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ never coincides with 0. In particular, if R is a regular local ring, then we have $\mathrm{G}_0(R)_{\mathbb{Q}} = \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} = \mathbb{Q}$.

For a finitely generated R-module M with $\dim M < \dim R$, we have

$$
\chi_{\mathbb{K}.}([M]) = 0. \tag{2.6}
$$

However, for an arbitrary $\mathbb{F}. \in C^{\mathfrak{m}}(R)$, the equality (2.6) does not always hold true (cf. Dutta-Hochster-MacLaughlin [1], Levin [16], Miller-Singh [20] and Roberts-Srinivas [29]). By Example 7.9, we know the following:

Let m and n be positive integers such that $n \geq m \geq 2$. Suppose

$$
R = k[x_{ij} \mid i = 1, \ldots, m;\ j = 1, \ldots, n]_{(x_{ij} | i,\ j)} / I_2(x_{ij}),
$$

where $I_2(x_{ij})$ is the ideal generated by all the 2×2 minors of the $m \times n$ matrix (x_{ij}). It is well known that the dimension of R is $m + n - 1$. Then, for $s = n, n+1, \ldots, m+n-1$, there is a complex $\mathbb{H}\{s\}. \in C^{\mathfrak{m}}(R)$ that satisfies the following two properties:

1. For any finitely generated R-module M with $\dim M < s$, $\chi_{\mathbb{H}\{s\}.}([M]) = 0$.

2. There exists a finitely generated R-module N_s of dimension s such that $\chi_{\mathbb{H}\{s\}.}([N_s]) \neq 0$.

3 Proof of the main theorem

The aim of this section is to prove the following theorem:

Theorem 3.1 *Let $(R, \mathfrak{m})$ be a Noetherian excellent local ring that satisfies one of the following two conditions; (1) R contains $\mathbb{Q}$, (2) R is essentially of finite type over a field, $\mathbb{Z}$ or a complete discrete valuation ring.*

Then, we have $\dim \overline{\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}} = \dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} = \dim \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} < \infty$.

Since $\overline{\tau_{R/S}}$ is an isomorphism, we have $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} = \dim \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$. Furthermore, since the pairing $\overline{e} : \overline{\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}} \otimes \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \longrightarrow \mathbb{Q}$ is perfect, we have $\dim \overline{\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}} = \dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ if $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} < \infty$. Therefore, it is sufficient to prove $\dim \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} < \infty$.

Lemma 3.2 *Let $(A, \mathfrak{p}) \xrightarrow{f} (B, \mathfrak{q})$ be a finite morphism of Noetherian local rings, that is, B is a finitely generated A-module. We denote by $f_* : \mathrm{A}_*(B)_{\mathbb{Q}} \to \mathrm{A}_*(A)_{\mathbb{Q}}$ the induced map by the proper morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. *Then, there is a map $\overline{f_*} : \overline{\mathrm{A}_*(B)_{\mathbb{Q}}} \to \overline{\mathrm{A}_*(A)_{\mathbb{Q}}}$ that makes the following diagram commutative:*

$$\begin{array}{ccc} \mathrm{A}_*(B)_{\mathbb{Q}} & \xrightarrow{f_*} & \mathrm{A}_*(A)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \overline{\mathrm{A}_*(B)_{\mathbb{Q}}} & \xrightarrow{\overline{f_*}} & \overline{\mathrm{A}_*(A)_{\mathbb{Q}}} \end{array}$$

where the vertical maps are the natural projections.

Proof. It is sufficient to show $f_*(\alpha) \in \mathrm{N}\,\mathrm{A}_i(A)_{\mathbb{Q}}$ for each i and for each $\alpha \in \mathrm{N}\,\mathrm{A}_i(B)_{\mathbb{Q}}$.

For $\mathbb{F}. \in C^{\mathfrak{p}}(A)$, the complex $\mathbb{F}. \otimes_A B$ is contained in $C^{\mathfrak{q}}(B)$ since the closed fibre of the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ coincides with $\{\mathfrak{q}\}$. By Definition 17.1 (C_1) and Theorem 18.1 in [2], we have

$$\begin{array}{cl} & \mathrm{ch}(\mathbb{F}.)(f_*(\alpha)) \\ = & [B/\mathfrak{q} : A/\mathfrak{p}] \cdot \mathrm{ch}(\mathbb{F}. \otimes B)(\alpha) \\ = & 0 \end{array}$$

because $\alpha \in \mathrm{N}\,\mathrm{A}_i(B)_{\mathbb{Q}}$. Thus, we obtain $f_*(\alpha) \in \mathrm{N}\,\mathrm{A}_i(A)_{\mathbb{Q}}$. **q.e.d.**

Let $(R, \mathfrak{m})$ be a local ring that satisfies the assumptions in Theorem 3.1. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ be the set of minimal prime ideals of R. Let $R_i = R/\mathfrak{p}_i$, and let $f_i : R \to R_i$ denote the projection for $i = 1, \ldots, r$. By the previous proposition, we have the following commutative diagram:

$$\begin{array}{ccc} \oplus_i \mathrm{A}_*(R_i)_{\mathbb{Q}} & \xrightarrow{\sum_i f_{i*}} & \mathrm{A}_*(R)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \oplus_i \overline{\mathrm{A}_*(R_i)_{\mathbb{Q}}} & \xrightarrow{\sum_i \overline{f_{i*}}} & \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} \end{array}$$

where all maps are surjections. Therefore, we have only to show that $\dim \overline{\mathrm{A}_*(R_i)_{\mathbb{Q}}} < \infty$ for each i.

Hence, we may assume that the given local ring $(R, \mathfrak{m})$ is an integral domain.

Then, by Hironaka [5] or de Jong [7], there exists a projective regular alteration $\pi : Z \to \mathrm{Spec}(R)$, that is, a projective generically finite morphism such that Z is a regular scheme. Furthermore, we may assume that $\pi^{-1}(\mathfrak{m})_{\mathrm{red}}$ is a simple normal crossing divisor. Let $\pi^{-1}(\mathfrak{m})_{\mathrm{red}} = E_1 \cup \cdots \cup E_t$ be the irreducible decomposition. Then, each E_l is a regular projective variety over $R/\mathfrak{m}$ with $\mathrm{codim}_Z E_l = 1$ for each $l = 1, \ldots, t$.

We denote by $\mathrm{CH}^{\cdot}_{\mathrm{num}}(E_l)_{\mathbb{Q}}$ the Chow group of E_l (with rational coefficients) modulo numerical equivalence (Chapter 19 in [2]). Then, $\dim \mathrm{CH}^{\cdot}_{\mathrm{num}}(E_l)_{\mathbb{Q}}$ is finite as in Example 19.1.4 in [2]. (In Example 19.1.4, it is assumed that the base field is algebraically closed. However, it is easy to remove this assumption.) Then, Theorem 3.1 follows from the following claim:

Claim 3.3 *With notation as above,* $\overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$ *is a subquotient of* $\oplus_{l=1}^{t} \mathrm{CH}^{\cdot}_{\mathrm{num}}(E_l)_{\mathbb{Q}}$.

The claim is proven as follows.

Since $\pi_* : \mathrm{A}_*(Z)_{\mathbb{Q}} \to \mathrm{A}_*(R)_{\mathbb{Q}}$ is surjective, we have a map $s : \mathrm{A}_*(R)_{\mathbb{Q}} \to \mathrm{A}_*(Z)_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces such that $\pi_* \cdot s = 1$.

For each $l = 1, \ldots, t$, we denote by j_l the inclusion $E_l \to Z$. We denote by φ the composite map

$$\mathrm{A}_*(R)_{\mathbb{Q}} \xrightarrow{s} \mathrm{A}_*(Z)_{\mathbb{Q}} \xrightarrow{\sum j_l^*} \oplus_l \mathrm{A}_*(E_l)_{\mathbb{Q}} \longrightarrow \oplus_l \mathrm{CH}^{\cdot}_{\mathrm{num}}(E_l)_{\mathbb{Q}},$$

where the last map is the natural projection and j_l^* is the map taking the intersection with the effective Cartier divisor E_l (see Chapter 2 in [2]).

In order to prove the claim, it is sufficient to show that the kernel of φ is contained in $\oplus_i \mathrm{N}\,\mathrm{A}_i(R)_{\mathbb{Q}}$.

Assume that γ is an element of $\mathrm{A}_*(R)_{\mathbb{Q}}$ such that $\varphi(\gamma) = 0$. We shall prove $\mathrm{ch}(\mathbb{F}.)(\gamma) = 0$ for any $\mathbb{F}. \in C^{\mathfrak{m}}(R)$. Note that, since $\varphi(\gamma) = 0$,

(3.4) $\quad j_l^* \cdot s(\gamma)$ is equal to 0 in $\mathrm{CH}^{\cdot}_{\mathrm{num}}(E_l)_{\mathbb{Q}}$ for each l.

Set $E = \pi^{-1}(\mathfrak{m})$. Since the diagram

$$\begin{array}{ccc} E & \longrightarrow & Z \\ {\scriptstyle \pi'} \downarrow & & {\scriptstyle \pi} \downarrow \\ \mathrm{Spec}(R/\mathfrak{m}) & \to & \mathrm{Spec}(R) \end{array}$$

is a fibre square, we have

$$\begin{aligned} \mathrm{ch}(\mathbb{F}.)(\gamma) &= \mathrm{ch}(\mathbb{F}.)(\pi_* \cdot s(\gamma)) \\ &= \pi'_* \mathrm{ch}(\pi^* \mathbb{F}.)(s(\gamma)). \end{aligned}$$

Note that $\pi^*\mathbb{F}.$ is a complex in $C^E(Z)$, that is, $\pi^*\mathbb{F}.$ is a bounded complex of vector bundles on Z with support in E. Consider the following commutative diagram:

$$\begin{array}{ccc} \oplus_l \mathrm{K}_0^{E_l}(Z)_{\mathbb{Q}} & \xrightarrow{\sum i_{l*}} & \mathrm{K}_0^E(Z)_{\mathbb{Q}} \\ \oplus\chi \downarrow & & \chi\downarrow \\ \oplus_l \mathrm{G}_0(E_l)_{\mathbb{Q}} & \xrightarrow{\sum i_{l*}} & \mathrm{G}_0(E)_{\mathbb{Q}} \end{array}$$

where $\chi : \mathrm{K}_0^Y(Z)_{\mathbb{Q}} \to \mathrm{G}_0(Y)_{\mathbb{Q}}$ is a map defined by $\chi(\mathbb{G}.) = \sum_i (-1)^i [H_i(\mathbb{G}.)]$ for each closed subset Y of Z and for each $\mathbb{G}. \in C^Y(Z)$, $i_{l*} : \mathrm{K}_0^{E_l}(Z)_{\mathbb{Q}} \to \mathrm{K}_0^E(Z)_{\mathbb{Q}}$ is a map given by $i_{l*}([\mathbb{H}.]) = [\mathbb{H}.]$ for $\mathbb{H}. \in C^{E_l}(Z)$, and $i_{l*} : \mathrm{G}_0(E_l)_{\mathbb{Q}} \to \mathrm{G}_0(E)_{\mathbb{Q}}$ is a map given by $i_{l*}([\mathcal{F}]) = [\mathcal{F}]$ for each coherent $\mathcal{O}_{E_l}$-module $\mathcal{F}$. Since Z is a regular scheme, the vertical maps are isomorphisms by Lemma 1.9 in [3]. The bottom map is surjective, because $E_{\mathrm{red}} = E_1 \cup \cdots \cup E_t$. Therefore, the top map is also surjective and there exist $\delta_l \in \mathrm{K}_0^{E_l}(Z)_{\mathbb{Q}}$ for $l = 1, \ldots, t$ such that

$$[\pi^*\mathbb{F}.] = \sum_l i_{l*}(\delta_l) \ \text{ in } \mathrm{K}_0^E(Z)_{\mathbb{Q}}.$$

For each l, $\pi_l : E_l \to \mathrm{Spec}(R/\mathfrak{m})$ denotes the structure morphism. Then, we have

$$\pi'_*\mathrm{ch}(\pi^*\mathbb{F}.)(s(\gamma)) = \sum_l \pi'_*\mathrm{ch}(i_{l*}(\delta_l))(s(\gamma)) = \sum_l \pi'_* i_{l*}\mathrm{ch}(\delta_l)(s(\gamma)) = \sum_l \pi_{l*}\mathrm{ch}(\delta_l)(s(\gamma))$$

since $\mathrm{ch}(i_{l*}(\delta_l)) = i_{l*}\mathrm{ch}(\delta_l)$.

Here, we shall prove $\pi_{l*}\mathrm{ch}(\delta_l)(s(\gamma)) = 0$ for each l. We denote by g_l the composite map

$$\mathrm{K}_0^{E_l}(Z)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{G}_0(E_l)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{K}_0^{E_l}(E_l)_{\mathbb{Q}}.$$

Set $\epsilon_l = g_l(\delta_l)$ for $l = 1, \ldots, t$. Then, by Corollary 18.1.2 in [2], we have

$$\pi_{l*}\mathrm{ch}(\delta_l)(s(\gamma)) = \pi_{l*}\left(\mathrm{ch}(\epsilon_l) \cdot \mathrm{td}(j_l^*\mathcal{O}(E_l))^{-1} \cdot j_l^*(s(\gamma))\right) = 0$$

since $j_l^* \cdot s(\gamma)$ is equal to 0 in $\mathrm{CH}^{\cdot}_{\mathrm{num}}(E_l)_{\mathbb{Q}}$ as (3.4).

We have obtained $\mathrm{ch}(\mathbb{F}.)(\gamma) = 0$.

We have completed the proof of Theorem 3.1.

Remark 3.5 Let $(R, \mathfrak{m})$ be a d-dimensional local ring that satisfies the assumptions as in Theorem 3.1. Set

$$\begin{aligned} \mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R) &= \{\alpha \in \mathrm{K}_0^{\mathfrak{m}}(R) \mid e(\alpha \otimes \beta) = 0 \text{ for any } \beta \in \mathrm{G}_0(R)\}, \\ \mathrm{N}\,\mathrm{G}_0(R) &= \{\beta \in \mathrm{G}_0(R) \mid e(\alpha \otimes \beta) = 0 \text{ for any } \alpha \in \mathrm{K}_0^{\mathfrak{m}}(R)\}, \\ \mathrm{N}\,\mathrm{A}_*(R) &= \{\gamma \in \mathrm{A}_*(R) \mid v(\alpha \otimes \gamma) = 0 \text{ for any } \alpha \in \mathrm{K}_0^{\mathfrak{m}}(R)\}, \end{aligned}$$

where e and v are the maps as in (2.1). It is easy to see that $(\mathrm{K}_0^{\mathfrak{m}}(R)/\,\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R))_{\mathbb{Q}}$, $(\mathrm{G}_0(R)/\,\mathrm{N}\,\mathrm{G}_0(R))_{\mathbb{Q}}$ and $(\mathrm{A}_*(R)/\,\mathrm{N}\,\mathrm{A}_*(R))_{\mathbb{Q}}$ coincide with $\overline{\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}}$, $\overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ and

$\overline{\mathrm{A}_i(R)_{\mathbb{Q}}}$, respectively. Therefore, by Theorem 3.1, $\mathrm{K}_0^{\mathfrak{m}}(R)/\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R)$, $\mathrm{G}_0(R)/\mathrm{N}\,\mathrm{G}_0(R)$ and $\mathrm{A}_*(R)/\mathrm{N}\,\mathrm{A}_*(R)$ are torsion-free abelian groups of finite rank. Note that the pairing

$$\overline{e} : (\mathrm{K}_0^{\mathfrak{m}}(R)/\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R))_{\mathbb{Q}} \otimes_{\mathbb{Q}} (\mathrm{G}_0(R)/\mathrm{N}\,\mathrm{G}_0(R))_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

is perfect and it satisfies that $\overline{e}(a \otimes b) \in \mathbb{Z}$ for any $a \in \mathrm{K}_0^{\mathfrak{m}}(R)/\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R)$ and any $b \in \mathrm{G}_0(R)/\mathrm{N}\,\mathrm{G}_0(R)$.

Suppose that L_1 and L_2 are torsion-free abelian groups of finite rank with a perfect pairing $p : L_{1\mathbb{Q}} \otimes L_{2\mathbb{Q}} \to \mathbb{Q}$ such that $p(a \otimes b) \in \mathbb{Z}$ for any $a \in L_1$ and any $b \in L_2$. Then, L_1 and L_2 can easily be proven to be finitely generated free abelian groups.

Based on this fact, we know that $\mathrm{K}_0^{\mathfrak{m}}(R)/\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R)$ and $\mathrm{G}_0(R)/\mathrm{N}\,\mathrm{G}_0(R)$ are finitely generated free abelian groups. By 18.1 (14) in Fulton [2], we have $\mathrm{ch}(\mathbb{F}.)(\gamma) \in \mathbb{Z}/d!$ for any $\mathbb{F}. \in C^{\mathfrak{m}}(R)$ and $\gamma \in \mathrm{A}_*(R)$. Thus, we can prove similarly that $\mathrm{A}_*(R)/\mathrm{N}\,\mathrm{A}_*(R)$ is also a finitely generated free abelian group.

Remark 3.6 Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring. By definition, for $\mathbb{F}. \in C^{\mathfrak{m}}(R)$ and for a non-negative integer n, the following two conditions are equivalent:

1. $\mathrm{ch}_i(\mathbb{F}.) : \mathrm{A}_i(R)_{\mathbb{Q}} \to \mathbb{Q}$ is equal to 0 for $i = 0, \ldots, n$,

2. $\chi_{\mathbb{F}.}(M)$ is equal to 0 for any finitely generated R-module M with $\dim M \leq n$.

A Koszul complex of a system of parameters satisfies the conditions as above with $n = d - 1$.

On the other hand, for $s = n, n+1, \ldots, m+n-1$, the complex $\mathbb{H}\{s\}.$ in Remark 2.5 satisfies following two conditions: (1) $\mathrm{ch}_i(\mathbb{H}\{s\}.) : \mathrm{A}_i(R)_{\mathbb{Q}} \to \mathbb{Q}$ is equal to 0 for $i < s$, and (2) $\mathrm{ch}_s(\mathbb{H}\{s\}.) \neq 0$.

Proposition 3.7 *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring.*

(1) If $d > 0$, then we have $\mathrm{A}_0(R)_{\mathbb{Q}} = \overline{\mathrm{A}_0(R)_{\mathbb{Q}}} = 0$.

(2) Assume that $\dim R/\mathfrak{p}$ is at least 2 for each minimal prime ideal $\mathfrak{p}$ of R. Then, $\overline{\mathrm{A}_1(R)_{\mathbb{Q}}} = 0$.

(3) The natural map $\mathrm{A}_d(R)_{\mathbb{Q}} \to \overline{\mathrm{A}_d(R)_{\mathbb{Q}}}$ is an isomorphism.

Proof. By definition, if $d > 0$, then we have $\mathrm{A}_0(R)_{\mathbb{Q}} = 0$. Since $\overline{\mathrm{A}_0(R)_{\mathbb{Q}}}$ is a homomorphic image of $\mathrm{A}_0(R)_{\mathbb{Q}}$, we have $\overline{\mathrm{A}_0(R)_{\mathbb{Q}}} = 0$ in this case.

Roberts [28] proved $\mathrm{ch}_1(\mathbb{F}.) = 0$ for any $\mathbb{F}. \in C^{\mathfrak{m}}(R)$ under the assumption in (2) as above. Therefore, we have $\overline{\mathrm{A}_1(R)_{\mathbb{Q}}} = 0$ in this case. (We shall give an another proof of the vanishing theorem of the first localized Chern characters due to Roberts in Theorem 9.4.)

We shall prove (3). Set $\operatorname{Assh} R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$, that is the set of minimal prime ideals of coheight d. Then, $\mathrm{A}_d(R)_{\mathbb{Q}}$ is the $\mathbb{Q}$-vector space with basis $\{[\operatorname{Spec}(R/\mathfrak{p}_j)] \mid j = 1, \ldots, t\}$. Suppose that $\sum_j n_j [\operatorname{Spec}(R/\mathfrak{p}_j)] \in \mathrm{N}\,\mathrm{A}_d(R)_{\mathbb{Q}}$. We want to show that $n_1 = \cdots = n_t = 0$.

For each $j = 1, \ldots, t$, take

$$x_j \in \left(\bigcap_{\substack{\mathfrak{q} \in \operatorname{Min}(R) \\ \mathfrak{q} \neq \mathfrak{p}_j}} \mathfrak{q} \right) \setminus \mathfrak{p}_j .$$

Then, $\sum_k x_k \notin \mathfrak{p}_j$ for $j = 1, \ldots, t$. Therefore, we can choose $y_2, \ldots, y_d \in \mathfrak{m}$ such that $\sum_k x_k, y_2, \ldots, y_d$ is a system of parameters for R. Then, it is easy to check that $\sum_k x_k^{s_k}, y_2, \ldots, y_d$ is also a system of parameters for R for any positive integers $s_1, \ldots, s_t$. Let $\mathbb{K}(\sum_k x_k^{s_k}, y_2, \ldots, y_d).$ be the Koszul complex with respect to $\sum_k x_k^{s_k}, y_2, \ldots, y_d$.

Since R is a homomorphic image of a regular local ring S, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\tau_{R/S} : \mathrm{G}_0(R)_{\mathbb{Q}} \longrightarrow \mathrm{A}_*(R)_{\mathbb{Q}}$$

by the singular Riemann-Roch theorem [2]. By the top term property (Theorem 18.3 (5) in [2]), we have

$$\tau_{R/S}([R/\mathfrak{p}_j]) = [\operatorname{Spec}(R/\mathfrak{p}_j)] + \gamma_{j,d-1} + \cdots + \gamma_{j0},$$

where $\gamma_{ji} \in \mathrm{A}_i(R)_{\mathbb{Q}}$. Then, by the local Riemann-Roch formula (Example 18.3.12 in [2]), we have

$$\begin{aligned} \chi_{\mathbb{K}(\sum_k x_k^{s_k}, y_2, \ldots, y_d).}([R/\mathfrak{p}_j]) &= \operatorname{ch}_d(\mathbb{K}(\sum_k x_k^{s_k}, y_2, \ldots, y_d).)([\operatorname{Spec}(R/\mathfrak{p}_j)]) \\ &\quad + \sum_{i=0}^{d-1} \operatorname{ch}_i(\mathbb{K}(\sum_k x_k^{s_k}, y_2, \ldots, y_d).)(\gamma_{ji}). \end{aligned}$$

On the other hand, we have

$$\operatorname{ch}_i(\mathbb{K}(\sum_k x_k^{s_k}, y_2, \ldots, y_d).)(\gamma_{ji}) = 0$$

for $i = 0, \ldots, d-1$ by Remark 3.6. Therefore, we have

$$\chi_{\mathbb{K}(\sum_k x_k^{s_k}, y_2, \ldots, y_d).}(\sum_j n_j [R/\mathfrak{p}_j]) = \operatorname{ch}_d(\mathbb{K}(\sum_k x_k^{s_k}, y_2, \ldots, y_d).)(\sum_j n_j [\operatorname{Spec}(R/\mathfrak{p}_j)]) = 0$$

since $\sum_j n_j [\operatorname{Spec}(R/\mathfrak{p}_j)] \in \mathrm{N}\,\mathrm{A}_d(R)_{\mathbb{Q}}$.

By Lech's Lemma and a theorem of Auslander-Buchsbaum (cf. p110, p111 in Matsumura [19]), we have

$$\begin{aligned} 0 &= \chi_{\mathbb{K}(\sum_k x_k^{s_k}, y_2, \ldots, y_d).}(\sum_j n_j [R/\mathfrak{p}_j]) \\ &= \sum_j n_j \chi_{\mathbb{K}(\sum_k x_k^{s_k}, y_2, \ldots, y_d).}([R/\mathfrak{p}_j]) \\ &= \sum_j n_j \chi_{\mathbb{K}(x_j^{s_j}, y_2, \ldots, y_d).}([R/\mathfrak{p}_j]) \\ &= \sum_j n_j e((x_j^{s_j}, y_2, \ldots, y_d), R/\mathfrak{p}_j) \\ &= \sum_j n_j s_j \cdot e((x_j, y_2, \ldots, y_d), R/\mathfrak{p}_j) \end{aligned}$$

for any $s_1, \ldots, s_t > 0$, where $e(\ ,\ )$ denotes the multiplicity. Since $e((x_j, y_2, \ldots, y_d), R/\mathfrak{p}_j) > 0$ for each j, we have $n_1 = \cdots = n_t = 0$. **q.e.d.**

4 Dimension of $\overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ as a $\mathbb{Q}$-vector space

By Theorem 3.1, we know that the dimension of $\overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ as a $\mathbb{Q}$-vector space is finite for a local ring R that satisfies a mild condition. It is expected that $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} < \infty$ holds for an arbitrary local ring R.

In this section, we study the dimension of $\overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ as a $\mathbb{Q}$-vector space.

As was shown in Remark 2.5, $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ is always positive. More precisely, by Theorem 3.1 and Proposition 3.7 (3), $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ is at least the number of prime ideals contained in $\mathrm{Assh}\, R$. If R is a regular local ring, then $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ is equal to one since $\mathrm{G}_0(R)_{\mathbb{Q}} = \mathbb{Q}$.

For a local domain R of dimension ≤ 2, $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ is equal to 1 by Proposition 3.7.

First of all, we give some examples.

Example 4.1 Let m and n be positive integers such that $n \geq m \geq 2$. Suppose

$$R = k[x_{ij} \mid i = 1, \ldots, m;\ j = 1, \ldots, n]_{(x_{ij} \mid i,\ j)} \big/ I_2(x_{ij})$$

as in Remark 2.5. In Example 7.9, we will show that $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ is equal to m.

Let X be the blowing-up of the projective space $\mathbb{P}^n_{\mathbb{C}}$ at r distinct points, where we suppose that $n \geq 2$. Let R be the local ring (at the homogeneous maximal ideal) of an affine cone of X. Then, by Example 7.8, $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ is equal to $r+1$.

Hence, $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}}$ has no upper bound even if $\dim R = 3$.

In the remainder of this section, we compare $\dim \overline{\mathrm{G}_0(A)_{\mathbb{Q}}}$ with $\dim \overline{\mathrm{G}_0(B)_{\mathbb{Q}}}$ for a homomorphism $A \to B$ of Noetherian local rings.

Lemma 4.2 *Let $f : (A, \mathfrak{p}) \to (B, \mathfrak{q})$ be a flat local homomorphism of Noetherian local rings such that $B/\mathfrak{q}$ is a finite algebraic extension of $A/\mathfrak{p}$.*

(1) The map $f^* : \mathrm{G}_0(A)_{\mathbb{Q}} \to \mathrm{G}_0(B)_{\mathbb{Q}}$ induces a map $\overline{f^*} : \overline{\mathrm{G}_0(A)_{\mathbb{Q}}} \to \overline{\mathrm{G}_0(B)_{\mathbb{Q}}}$ that makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{G}_0(A)_{\mathbb{Q}} & \xrightarrow{f^*} & \mathrm{G}_0(B)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \overline{\mathrm{G}_0(A)_{\mathbb{Q}}} & \xrightarrow{\overline{f^*}} & \overline{\mathrm{G}_0(B)_{\mathbb{Q}}} \end{array}$$

where the vertical maps are the projections.

(2) Assume $\sqrt{\mathfrak{p}B} = \mathfrak{q}$. Then, the map $\overline{f^*} : \overline{\mathrm{G}_0(A)_{\mathbb{Q}}} \to \overline{\mathrm{G}_0(B)_{\mathbb{Q}}}$ is injective. In particular, $\dim \overline{\mathrm{G}_0(A)_{\mathbb{Q}}} \leq \dim \overline{\mathrm{G}_0(B)_{\mathbb{Q}}}$ in this case.

Proof. Recall that $f^* : \mathrm{G}_0(A) \to \mathrm{G}_0(B)$ is a map defined by $f^*([M]) = [M \otimes_A B]$ for each finitely generated A-module M.

First, we prove (1). It is sufficient to show that, for any $c \in \mathrm{N}\,\mathrm{G}_0(A)_{\mathbb{Q}}$, $f^*(c)$ is contained in $\mathrm{N}\,\mathrm{G}_0(B)_{\mathbb{Q}}$.

It is sufficient to show that, for any $\mathbb{F}. \in C^{\mathfrak{q}}(B)$, $\chi_{\mathbb{F}.}(f^*(c)) = 0$.

First we shall prove that there exist $\mathbb{G}. \in C^{\mathfrak{p}}(A)$ and a chain map $\varphi : \mathbb{G}. \to \mathbb{F}.$ of A-linear maps such that φ is a quasi-isomorphism. Recall that each homology of $\mathbb{F}. \in C^{\mathfrak{q}}(B)$ is of finite length as an A-module since $B/\mathfrak{q}$ is a finite algebraic extension of $A/\mathfrak{p}$. By killing homology modules of $\mathbb{F}.$, there exists an A-free complex $\mathbb{G}.$ and a chain map $\phi : \mathbb{G}. \to \mathbb{F}.$ of A-linear maps such that

1. each G_i is a finitely generated free A-module,
2. $\mathbb{G}.$ is bounded below, i.e., $G_i = 0$ for $i << 0$, and
3. $\phi : \mathbb{G}. \to \mathbb{F}.$ is a quasi-isomorphism.

Furthermore, we may assume that $\mathbb{G}.$ is a minimal complex, that is, all boundaries of $\mathbb{G}. \otimes_A A/\mathfrak{p}$ are 0. Note that each homology of $\mathbb{G}.$ is of finite length as an A-module. We want to show that $\mathbb{G}.$ is bounded. Since $\phi : \mathbb{G}. \to \mathbb{F}.$ is a quasi-isomorphism of complexes of flat A-modules which are bounded below, $\phi \otimes 1 : \mathbb{G}. \otimes_A A/\mathfrak{p} \to \mathbb{F}. \otimes_A A/\mathfrak{p}$ is also a quasi-isomorphism. Therefore, we have

$$(A/\mathfrak{p})^{\mathrm{rank}_A G_i} \simeq H_i(\mathbb{G}. \otimes_A A/\mathfrak{p}) \simeq H_i(\mathbb{F}. \otimes_A A/\mathfrak{p})$$

for any i. Since $\mathbb{F}.$ is bounded, so is $\mathbb{G}.$.

Since $\phi : \mathbb{G}. \to \mathbb{F}.$ is a quasi-isomorphism of bounded complexes of flat A-modules, $\phi \otimes 1 : \mathbb{G}. \otimes_A M \to \mathbb{F}. \otimes_A M$ is a quasi-isomorphism for any A-module M.

Since $\mathbb{F}.\otimes_A M = \mathbb{F}.\otimes_B (B\otimes_A M)$, $H_i(\mathbb{G}.\otimes_A M)$ is isomorphic to $H_i(\mathbb{F}.\otimes_B (B\otimes_A M))$ as an A-module. Therefore, we have

$$\chi_{\mathbb{F}.}(f^*(c)) = \frac{1}{[B/\mathfrak{q} : A/\mathfrak{p}]}\chi_{\mathbb{G}.}(c) = 0$$

since $c \in \mathrm{N}\,\mathrm{G}_0(A)_{\mathbb{Q}}$. We have completed the proof of (1) in Lemma 4.2.

Next, we shall prove (2). Assume that $\beta \in \mathrm{G}_0(A)_{\mathbb{Q}}$ satisfies $f^*(\beta) \in \mathrm{N}\,\mathrm{G}_0(B)_{\mathbb{Q}}$. We want to show that $\beta \in \mathrm{N}\,\mathrm{G}_0(A)_{\mathbb{Q}}$. Let $\mathbb{G}.$ be a complex contained in $C^{\mathfrak{p}}(A)$. Since $\sqrt{\mathfrak{p}B} = \mathfrak{q}$, $\mathbb{G}. \otimes_A B$ is contained in $C^{\mathfrak{q}}(B)$. Then, we have

$$\chi_{\mathbb{G}.}(\beta) = \frac{1}{\ell_B(B/\mathfrak{p}B)}\chi_{\mathbb{G}.\otimes_A B}(f^*(\beta)) = 0.$$

Therefore, we have $\beta \in \mathrm{N}\,\mathrm{G}_0(A)_{\mathbb{Q}}$. **q.e.d.**

Theorem 4.3 *Let $f : (A, \mathfrak{p}) \to (B, \mathfrak{q})$ be a local homomorphism of Noetherian local rings.*

(1) If f is finite and injective, then $\dim \overline{\mathrm{G}_0(A)_{\mathbb{Q}}} \leq \dim \overline{\mathrm{G}_0(B)_{\mathbb{Q}}}$.

(2) Assume that A is an excellent normal local ring. Suppose that the Frobenius map $A \to A$ is finite if A is of positive characteristic. Let K be a finite normal (algebraic) extension of $Q(A)$, where $Q(\ )$ is the field of fractions. Assume that B is a local ring at a maximal ideal of the integral closure of A in K. Then, we have $\dim \overline{\mathrm{G}_0(A)_{\mathbb{Q}}} \leq \dim \overline{\mathrm{G}_0(B)_{\mathbb{Q}}}$.

Proof. Assume that f is finite and injective. Then, by the lying-over theorem, $f_* : \mathrm{A}_*(B)_{\mathbb{Q}} \to \mathrm{A}_*(A)_{\mathbb{Q}}$ is surjective. By Lemma 3.2, we obtain a surjective map $\overline{f_*} : \overline{\mathrm{A}_*(B)_{\mathbb{Q}}} \to \overline{\mathrm{A}_*(A)_{\mathbb{Q}}}$. Therefore, by Theorem 3.1, we have $\dim \overline{\mathrm{G}_0(A)_{\mathbb{Q}}} \leq \dim \overline{\mathrm{G}_0(B)_{\mathbb{Q}}}$.

Next, we shall prove (2). Let L be the intermediate field such that $Q(B)/L$ is a Galois extension and $L/Q(A)$ is a purely inseparable extension. Let C (resp. D) be the integral closure of A in L (resp. $Q(B)$). Since A is excellent, both C and D are finite A-modules.

Suppose that $L \neq Q(A)$. Then, the characteristic of A is a prime number p and, by the assumption, the Frobenius map $A \to A$ is finite. Therefore, for a sufficiently large e,

$$A \subset C \subset A^{1/p^e} \simeq A.$$

Since the maps as above are finite, we have

$$\dim \overline{\mathrm{G}_0(A)_{\mathbb{Q}}} \leq \dim \overline{\mathrm{G}_0(C)_{\mathbb{Q}}} \leq \dim \overline{\mathrm{G}_0(A^{1/p^e})_{\mathbb{Q}}} = \dim \overline{\mathrm{G}_0(A)_{\mathbb{Q}}}$$

by (1). Therefore, $\dim \overline{\mathrm{G}_0(A)_{\mathbb{Q}}}$ coincides with $\dim \overline{\mathrm{G}_0(C)_{\mathbb{Q}}}$. Replacing C with A, we may assume that $Q(B)$ is a Galois extension of $Q(A)$.

Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the set of maximal ideals of D. Suppose $B = D_{\mathfrak{m}_1}$. Let G be the Galois group of the Galois extension $Q(B)/Q(A)$. Let H be the splitting group of $\mathfrak{m}_1$, that is

$$H = \{\sigma \in G \mid \sigma(\mathfrak{m}_1) = \mathfrak{m}_1\}.$$

Set $K = Q(B)^H$. Let E be the integral closure of A in K. Set $\mathfrak{n} = \mathfrak{m}_1 \cap E$. Since the induced map $E_{\mathfrak{n}} \to D_{\mathfrak{m}_1}$ is finite injective (e.g., (41.2) (1) in Nagata [22]), we have

$$\dim \overline{\mathrm{G}_0(E_{\mathfrak{n}})_{\mathbb{Q}}} \leq \dim \overline{\mathrm{G}_0(D_{\mathfrak{m}_1})_{\mathbb{Q}}}.$$

By (41.2) (2), (3) in [22], we obtain $\mathfrak{p}E_{\mathfrak{n}} = \mathfrak{n}E_{\mathfrak{n}}$ and $A/\mathfrak{p} = E_{\mathfrak{n}}/\mathfrak{n}E_{\mathfrak{n}}$. Then, by (43.1) in [22], $E_{\mathfrak{n}}$ is flat over A. Then, by Lemma 4.2, we have

$$\dim \overline{\mathrm{G}_0(A)_{\mathbb{Q}}} \leq \dim \overline{\mathrm{G}_0(E_{\mathfrak{n}})_{\mathbb{Q}}}.$$

The assertion (2) follows immediately from the above inequalities. **q.e.d.**

5 $\overline{\tau_{R/S}}$ is independent of S

This section is devoted to proving that the map $\overline{\tau_{R/S}} : \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$ is independent of the choice of S.

The essential point is to prove that the induced map $\overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{G}_0(\hat{R})_{\mathbb{Q}}}$ by completion $R \to \hat{R}$ is injective as follows.

Theorem 5.1 *(1) Let R be a Noetherian local ring. Then, the induced map $\overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{G}_0(\hat{R})_{\mathbb{Q}}}$ by the completion $R \to \hat{R}$ is injective.*

(2) Let R be a homomorphic image of a regular local ring S. Then, the induced map $\overline{\tau_{R/S}} : \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$ is independent of the choice of S.

Proof. The first assertion follows immediately from Lemma 4.2.

Let $f : R \to \hat{R}$ be the completion. By Lemma 4.1 in [12], we have a map $f^* : \mathrm{A}_*(R)_{\mathbb{Q}} \to \mathrm{A}_*(\hat{R})_{\mathbb{Q}}$ that makes the following diagram commutative:

(5.2)
$$\begin{array}{ccc} \mathrm{G}_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_{R/S}} & \mathrm{A}_*(R)_{\mathbb{Q}} \\ f^* \downarrow & & \downarrow f^* \\ \mathrm{G}_0(\hat{R})_{\mathbb{Q}} & \xrightarrow{\tau_{\hat{R}/\hat{S}}} & \mathrm{A}_*(\hat{R})_{\mathbb{Q}} \end{array}$$

Note that the maps $f^* : \mathrm{G}_0(R)_{\mathbb{Q}} \to \mathrm{G}_0(\hat{R})_{\mathbb{Q}}$ and $f^* : \mathrm{A}_*(R)_{\mathbb{Q}} \to \mathrm{A}_*(\hat{R})_{\mathbb{Q}}$ are independent of the choice of S. Furthermore, the map $\tau_{\hat{R}/\hat{S}}$ as above does not depend on the choice of $\hat{S}$ since $\hat{R}$ is complete (cf. Section 4 in [12]).

By Lemma 4.2, it is easy to see that there is an induced map $\overline{f^*} : \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} \to \overline{\mathrm{A}_*(\hat{R})_{\mathbb{Q}}}$ that is independent of the choice of S, and makes the following diagrams commutative:

$$
\begin{array}{ccc}
\mathrm{A}_*(R)_{\mathbb{Q}} & \xrightarrow{f^*} & \mathrm{A}_*(\hat{R})_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\overline{\mathrm{A}_*(R)_{\mathbb{Q}}} & \xrightarrow{\overline{f^*}} & \overline{\mathrm{A}_*(\hat{R})_{\mathbb{Q}}}
\end{array}
\qquad
\begin{array}{ccc}
\overline{\mathrm{G}_0(R)_{\mathbb{Q}}} & \xrightarrow{\overline{\tau_{R/S}}} & \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} \\
\overline{f^*}\downarrow & & \downarrow\overline{f^*} \\
\overline{\mathrm{G}_0(\hat{R})_{\mathbb{Q}}} & \xrightarrow{\overline{\tau_{\hat{R}/\hat{S}}}} & \overline{\mathrm{A}_*(\hat{R})_{\mathbb{Q}}}
\end{array}
\tag{5.3}
$$

Since $\tau_{\hat{R}/\hat{S}}$ is independent of the choice of S, so is $\overline{\tau_{\hat{R}/\hat{S}}}$. Since both $\overline{\tau_{R/S}}$ and $\overline{\tau_{\hat{R}/\hat{S}}}$ are isomorphisms (see Section 2) and $\overline{f^*} : \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{G}_0(\hat{R})_{\mathbb{Q}}}$ is injective by (1), it follows that the map $\overline{\tau_{R/S}} : \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$ is independent of the choice of S. **q.e.d.**

We hereafter denote the map $\overline{\tau_{R/S}}$ simply by $\overline{\tau_R}$.

Note that, if the maps f^* in the diagram (5.2) are injective, then $\tau_{R/S}$ itself is independent of the choice of S. The author does not know any example such that f^* is not injective. We refer the reader to [8] for some sufficient conditions of the injectivity of the map $f^* : \mathrm{G}_0(R)_{\mathbb{Q}} \to \mathrm{G}_0(\hat{R})_{\mathbb{Q}}$.

We proved that $\overline{f^*} : \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{G}_0(\hat{R})_{\mathbb{Q}}}$ is injective in Theorem 5.1 (1). The remainder of this section will be devoted to investigating some sufficient conditions of the surjectivity of the map $\overline{f^*} : \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{G}_0(\hat{R})_{\mathbb{Q}}}$. Note that $\overline{f^*} : \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{G}_0(\hat{R})_{\mathbb{Q}}}$ is surjective if and only if so is $\overline{f^*} : \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} \to \overline{\mathrm{A}_*(\hat{R})_{\mathbb{Q}}}$.

Here, we give an easy example where $\overline{f^*}$ is not surjective. Set $R = \mathbb{C}[x, y]_{(x,y)}/(xy - x^3 - y^3)$. Since $xy - x^3 - y^3$ is an irreducible polynomial, we have

$$\mathbb{Q} = \mathrm{A}_1(R)_{\mathbb{Q}} = \overline{\mathrm{A}_1(R)_{\mathbb{Q}}}$$

by Proposition 3.7 (3). However, since the ring $\hat{R} = \mathbb{C}[[x, y]]/(xy - x^3 - y^3)$ has two minimal prime ideals, we have

$$\mathbb{Q} \oplus \mathbb{Q} = \mathrm{A}_1(\hat{R})_{\mathbb{Q}} = \overline{\mathrm{A}_1(\hat{R})_{\mathbb{Q}}}$$

by Proposition 3.7 (3).

Remark 5.4 Let $(R, \mathfrak{m})$ be a Noetherian local ring that satisfies the assumptions in Theorem 3.1.

Then, $\overline{f^*} : \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{G}_0(\hat{R})_{\mathbb{Q}}}$ is an isomorphism if and only if the equality $\dim \overline{\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}} = \dim \overline{\mathrm{K}_0^{\mathfrak{m}\hat{R}}(\hat{R})_{\mathbb{Q}}}$ holds.

On the other hand, the natural map $f^* : \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} \to \mathrm{K}_0^{\mathfrak{m}\hat{R}}(\hat{R})_{\mathbb{Q}}$ is an isomorphism by Theorem 7.1 in Thomason-Trobaugh [33]. By the surjectivity of f^*, it is easy to see that $f^*(\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}) \supseteq \mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}\hat{R}}(\hat{R})_{\mathbb{Q}}$.

Therefore, we know that $\overline{f^*} : \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{G}_0(\hat{R})_{\mathbb{Q}}}$ is an isomorphism if and only if $f^*(\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}) \subseteq \mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}\hat{R}}(\hat{R})_{\mathbb{Q}}$.

Proposition 5.5 *Assume that R is a local ring that satisfies the assumptions in Theorem 3.1. Furthermore, assume that R is henselian or that R is the local ring (at the homogeneous maximal ideal) of an affine cone of a smooth projective variety over a field. Then, $\overline{f^*} : \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{G}_0(\hat{R})_{\mathbb{Q}}}$ is an isomorphism.*

Proof. First assume that R is henselian. Let $\mathbb{F}.$ be a complex contained in $C^{\mathfrak{m}}(R)$. Let M be a finitely generated $\hat{R}$-module. Then, using the approximation theorem due to Popescu-Ogoma ([24], [25]), there exists a finitely generated R-module N such that $\chi_{\mathbb{F}.}([N]) = \chi_{\mathbb{F}.\otimes_R \hat{R}}([M])$. Here, assume that $\mathbb{F}. \in \mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}$. Then, we have $\chi_{\mathbb{F}.}([N]) = 0$. Therefore, we have $\mathbb{F}. \otimes_R \hat{R} \in \mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}\hat{R}}(\hat{R})_{\mathbb{Q}}$.

Next, assume that R is the local ring at the homogeneous maximal ideal of an affine cone of a smooth projective variety over a field, i.e., let $A = \oplus_{n\geq 0} A_n = k[A_1]$ be a standard Noetherian graded ring over a field k such that $X = \mathrm{Proj}(A)$ is smooth over k, and set $R = A_{A_+}$, $\mathfrak{m} = A_+R$, where $A_+ = \oplus_{n>0} A_n$. Let $\pi : Z \to \mathrm{Spec}(R)$ be the blowing-up with center $\mathfrak{m}$. Then, π gives a resolution of singularities of $\mathrm{Spec}(R)$. Since $R \to \hat{R}$ is a regular homomorphism, $\pi \times 1 : Z \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\hat{R}) \to \mathrm{Spec}(\hat{R})$ is also a resolution of singularities. It is easy to see that X is isomorphic to $\pi^{-1}(\mathfrak{m})$. We have the following two fibre squares:

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ \mathrm{Spec}(R/\mathfrak{m}) & \to & \mathrm{Spec}(R) \end{array} \qquad \begin{array}{ccc} X & \to & Z \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\hat{R}) \\ \downarrow & & {\scriptstyle \pi\times 1}\downarrow \\ \mathrm{Spec}(\hat{R}/\mathfrak{m}\hat{R}) & \to & \mathrm{Spec}(\hat{R}) \end{array}$$

By the argument in Roberts-Srinivas [29] (see (7.4) below), for $\alpha \in \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}$, α is contained in $\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}$ if and only if α is contained in the kernel of the composite map

$$\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} \xrightarrow{\pi^*} \mathrm{K}_0^X(Z)_{\mathbb{Q}} \xrightarrow{\chi} \mathrm{G}_0(X)_{\mathbb{Q}} \xrightarrow{\tau_{X/Z}} \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}},$$

where χ is a map defined in the proof of Claim 3.3, and $\tau_{X/Z}$ is an isomorphism given by the singular Riemann-Roch theorem (Chapter 18 in [2]) with base regular scheme Z.

In the same way, one can prove that $f^*(\alpha)$ is contained in $\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}\hat{R}}(\hat{R})_{\mathbb{Q}}$ if and only if $f^*(\alpha)$ is contained in the kernel of the composite map

$$\mathrm{K}_0^{\mathfrak{m}\hat{R}}(\hat{R})_{\mathbb{Q}} \xrightarrow{(\pi\times 1)^*} \mathrm{K}_0^X(Z\times\mathrm{Spec}(\hat{R}))_{\mathbb{Q}} \xrightarrow{\chi} \mathrm{G}_0(X)_{\mathbb{Q}} \xrightarrow{\tau_{X/(Z\times\mathrm{Spec}(\hat{R}))}} \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}.$$

By Corollary 18.1.2 in [2] and the definition of the maps, we have

$$\tau_{X/Z} = \mathrm{td}(N)^{-1} \cdot \tau_{X/X} = \tau_{X/(Z\times\mathrm{Spec}(\hat{R}))},$$

where N is the normal bundle of the embedding $X \to Z$. Note that the diagram

$$\begin{tikzcd} \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} \arrow[r, "\pi^*"] \arrow[d, "f^*"'] & \mathrm{K}_0^X(Z)_{\mathbb{Q}} \arrow[d, "(1\times f)^*"'] \arrow[dr] & \\ \mathrm{K}_0^{\mathfrak{m}\hat{R}}(\hat{R})_{\mathbb{Q}} \arrow[r, "(\pi\times 1)^*"] & \mathrm{K}_0^X(Z \times \mathrm{Spec}(\hat{R}))_{\mathbb{Q}} \arrow[r] & \mathrm{G}_0(X)_{\mathbb{Q}} \end{tikzcd}$$

is commutative. Thus, if α is contained in $\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}$, then $f^*(\alpha)$ is in $\mathrm{N}\,\mathrm{K}_0^{\mathfrak{m}\hat{R}}(\hat{R})_{\mathbb{Q}}$.
q.e.d.

6 Numerically Roberts rings

Since $(R, \mathfrak{m})$ is a homomorphic image of a regular local ring S, we have an isomorphism

$$\overline{\tau_R} : \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} \to \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} = \bigoplus_{i=0}^{\dim R} \overline{\mathrm{A}_i(R)_{\mathbb{Q}}}$$

that is independent of the choice of a base regular local ring S, as was shown in Section 5. Using the map as above, we shall define a notion of *numerically Roberts* rings and study these rings in this section.

Definition 6.1 We say that R is a *numerically Roberts* ring if $\overline{\tau_R}(\overline{[R]}) \in \overline{\mathrm{A}_{\dim R}(R)_{\mathbb{Q}}}$.

Set $d = \dim R$ and

$$\tau_{R/S}([R]) = \tau_d + \cdots + \tau_0, \tag{6.2}$$

where $\tau_i \in \mathrm{A}_i(R)_{\mathbb{Q}}$ for $i = 0, \cdots, d$. By the top term property (Theorem 18.3 (5) in [2]), we have

$$\tau_d = \sum_{\mathfrak{p} \in \mathrm{Assh}(R)} \ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})[\mathrm{Spec}(R/\mathfrak{p})].$$

As in Definition 2.1 in [12], R is said to be a *Roberts* ring if $\tau_{d-1} = \cdots = \tau_0 = 0$ with some base regular local ring S. Since the diagram

$$\begin{array}{ccccc}
\mathrm{G}_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_{R/S}} & \mathrm{A}_*(R)_{\mathbb{Q}} & = & \oplus_{i=0}^d \mathrm{A}_i(R)_{\mathbb{Q}} \\
\downarrow & & \downarrow & & \\
\overline{\mathrm{G}_0(R)_{\mathbb{Q}}} & \xrightarrow{\overline{\tau_R}} & \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} & = & \oplus_{i=0}^d \overline{\mathrm{A}_i(R)_{\mathbb{Q}}}
\end{array}$$

is commutative, we have

$$\overline{\tau_R}(\overline{[R]}) = \overline{\tau_d} + \cdots + \overline{\tau_0}. \tag{6.3}$$

Therefore, R is a numerically Roberts ring if and only if $\tau_i \in \mathrm{N}\,\mathrm{A}_i(R)_{\mathbb{Q}}$ for $i = 0, 1, \ldots, d-1$. In particular, if R is a Roberts ring, then it is a numerically Roberts ring. (The converse is not true. See Example 6.7.)

Assume that the natural map $\mathrm{A}_*(R)_{\mathbb{Q}} \to \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$ is an isomorphism. Then, R is a numerically Roberts ring if and only if R is a Roberts ring.

For a complex $\mathbb{F}. \in C^{\mathfrak{m}}(R)$, the rational number

$$\mathrm{ch}(\mathbb{F}.)(\tau_d) = \mathrm{ch}_d(\mathbb{F}.)(\tau_d)$$

is called the *Dutta multiplicity* of the complex $\mathbb{F}.$ and is denoted by $\chi_\infty(\mathbb{F}.)$.

The following proposition characterizes numerically Roberts rings. We refer the reader to [21] for Hilbert-Kunz multiplicity.

Theorem 6.4 *Let $(R, \mathfrak{m})$ be a homomorphic image of a regular local ring.*

(1) Then, R is a numerically Roberts ring if and only if the Dutta multiplicity $\chi_\infty(\mathbb{F}.)$ coincides with the alternating sum of length of homology

$$\chi(\mathbb{F}.) = \sum_i (-1)^i \ell_R(H_i(\mathbb{F}.))$$

for any $\mathbb{F}. \in C^{\mathfrak{m}}(R)$.

(2) Assume that R is a Cohen-Macaulay ring of characteristic p, where p is a prime number. Then, R is a numerically Roberts ring if and only if the Hilbert-Kunz multiplicity $e_{HK}(J)$ of J coincides with the colength $\ell_R(R/J)$ for any $\mathfrak{m}$-primary ideal J of finite projective dimension.

Proof. With notation as in (6.2), R is a numerically Roberts ring if and only if $(\tau_{R/S})^{-1}(\tau_{d-1} + \cdots + \tau_0) \in \mathrm{N}\,\mathrm{G}_0(R)_{\mathbb{Q}}$ (cf. Proposition 2.4). Note that

$$\begin{aligned} \chi_\infty(\mathbb{F}.) &= \chi_{\mathbb{F}.}\left((\tau_{R/S})^{-1}(\tau_d)\right) \\ &= \chi_{\mathbb{F}.}\left([R] - (\tau_{R/S})^{-1}(\tau_{d-1} + \cdots + \tau_0)\right) \\ &= \chi(\mathbb{F}.) - \chi_{\mathbb{F}.}\left((\tau_{R/S})^{-1}(\tau_{d-1} + \cdots + \tau_0)\right). \end{aligned}$$

Therefore, R is a numerically Roberts ring if and only if $\chi_\infty(\mathbb{F}.)$ coincides with $\chi(\mathbb{F}.)$ for any $\mathbb{F}. \in C^{\mathfrak{m}}(R)$.

Assume that R is a Cohen-Macaulay ring of characteristic p. Let $F(R)$ be the category of R-modules of finite length and of finite projective dimension. Since R is a Cohen-Macaulay local ring, $\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}$ is generated by free resolutions of modules in $F(R)$ (cf. Proposition 2 in [29]). For $M \in F(R)$, $\mathbb{F}_M. \in C^{\mathfrak{m}}(R)$ denotes the minimal free resolution of M. By (1), R is a numerically Roberts ring if and only if $\chi_\infty(\mathbb{F}_M.)$ coincides with $\chi(\mathbb{F}_M.) = \ell_R(M)$ for any $M \in F(R)$. Suppose that $M \in F(R)$. Using the method of Lemma 9.10 in [31], $\mathfrak{m}$-primary ideals J, I_1, $\ldots$, I_t of finite projective dimension can be found such that I_1, $\ldots$, I_t are parameter ideals and

$$[M] = [R/J] - \sum_{i=1}^{t} [R/I_i]$$

in $\mathrm{K}_0(F(R))$. Then, we have

$$\chi([\mathbb{F}_M.]) = \ell_R(M) = \ell_R(R/J) - \sum_{i=1}^{t} \ell_R(R/I_i)$$

and

$$[\mathbb{F}_M.] = [\mathbb{F}_{R/J}.] - \sum_{i=1}^{t} [\mathbb{F}_{R/I_i}.]$$

in $\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}$. Therefore, we have

$$\begin{aligned}\chi_\infty(\mathbb{F}_M.) &= \chi_\infty(\mathbb{F}_{R/J}.) - \sum_{i=1}^{t} \chi_\infty(\mathbb{F}_{R/I_i}.) \\ &= e_{HK}(J) - \sum_{i=1}^{t} \ell_R(R/I_i),\end{aligned}$$

by Remark 2.7 in [9] and Theorem 1.2 (1) in [11]. Thus, R is a numerically Roberts ring if and only if $e_{HK}(J)$ coincides with $\ell_R(R/J)$ for any $\mathfrak{m}$-primary ideal J of finite projective dimension. **q.e.d.**

Remark 6.5 Let R be a numerically Roberts ring. Then, R satisfies the vanishing property of intersection multiplicities, that is,

$$\sum_i (-1)^i \ell_R(\mathrm{Tor}_i^R(M, N)) = 0$$

for finitely generated R-modules M and N such that $\mathrm{pd}_R M < \infty$, $\mathrm{pd}_R N < \infty$, $\ell_R(M \otimes_R N) < \infty$ and $\dim M + \dim N < \dim R$.

The proof is the same as that due to Roberts [27].

Example 6.6 Let R be a homomorphic image of a regular local ring and let $d = \dim R$.

Suppose $d = 0$. Since $\overline{\mathrm{A}_*(R)_{\mathbb{Q}}} = \overline{\mathrm{A}_0(R)_{\mathbb{Q}}}$, R is a numerically Roberts ring.

Suppose $d = 1$. Since $\overline{\mathrm{A}_0(R)_{\mathbb{Q}}} = 0$ by Proposition 3.7 (1), R is a numerically Roberts ring.

Assume that R is equi-dimensional with $d = 2$. Since $\overline{\mathrm{A}_0(R)_{\mathbb{Q}}} = \overline{\mathrm{A}_1(R)_{\mathbb{Q}}} = 0$ by Proposition 3.7, R is a numerically Roberts ring.

Assume that R is a Gorenstein ring with $d = 3$. Set

$$\overline{\tau_R}(\overline{[R]}) = \overline{\tau_3} + \overline{\tau_2} + \overline{\tau_1} + \overline{\tau_0}$$

as (6.3). Since $\overline{\mathrm{A}_0(R)_{\mathbb{Q}}} = \overline{\mathrm{A}_1(R)_{\mathbb{Q}}} = 0$ by Proposition 3.7, we have $\overline{\tau_0} = \overline{\tau_1} = 0$. Furthermore, since R is a Gorenstein ring, we have $\tau_2 = 0$ by Proposition 2.8 in [10]. Therefore, R is a numerically Roberts ring.

Using an example due to Dutta-Hochster-MacLaughlin [1], we can construct a three-dimensional Cohen-Macaulay normal ring R and $\mathbb{F}. \in C^{\mathfrak{m}}(R)$ such that $\chi_\infty(\mathbb{F}.) \neq \chi(\mathbb{F}.)$. By Proposition 6.4 (1), R is not a numerically Roberts ring.

A five-dimensional Gorenstein ring constructed by Miller-Singh [20] is not a numerically Roberts ring.

Next, we give an example of a numerically Roberts ring that is not a Roberts ring.

Example 6.7 We give an example of a two-dimensional Noetherian local domain that is not a Roberts ring. (Recall that two-dimensional Noetherian local domains are numerically Roberts rings by Example 6.6.)

Let X be a smooth projective curve over $\mathbb{C}$ with $g(X) \geq 2$. Since $\mathrm{Pic}^0(X)$ is an abelian variety of dimension $g(X)$, $\mathrm{Pic}(X)_{\mathbb{Q}}$ is an infinite dimensional $\mathbb{Q}$-vector space. Furthermore, we have $\deg K_X = 2g(X) - 2 > 0$.

Take a divisor $H \in \mathrm{Div}(X)$ such that H and K_X are linearly independent in $\mathrm{Pic}(X)_{\mathbb{Q}}$. We assume $\deg H >> 0$. Then, H is an very ample divisor, and we may assume that the graded ring

$$A = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nH))$$

is generated by elements of degree 1 over $\mathbb{C} = H^0(X, \mathcal{O}_X)$.

Set $R = A_{A_+}$. Then, R is a two-dimensional Noetherian local domain and we have an exact sequence

$$\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \xrightarrow{H} \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \xrightarrow{\xi} \mathrm{A}_*(R)_{\mathbb{Q}} \longrightarrow 0,$$

where ξ is defined by $\xi([\mathrm{Proj}(A/P)]) = [\mathrm{Spec}(R/PR)]$ for each homogeneous prime ideal P of A not equal to A_+. The map $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \xrightarrow{H} \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$ denotes the multiplication by H. Then, by Theorem 1.3 in [10], we have

$$\xi(\mathrm{td}(\Omega_X^\vee)) = \tau_R([R]) = \tau_2 + \tau_1 + \tau_0.$$

Furthermore, by definition of Todd classes of vector bundles, we have

$$\mathrm{td}(\Omega_X^\vee) = 1 + \frac{1}{2}c_1(\Omega_X^\vee) = 1 - \frac{1}{2}c_1(\omega_X) = 1 - \frac{1}{2}K_X.$$

Therefore, we have $\xi(-\frac{1}{2}K_X) = \tau_1$. Since H and K_X are linearly independent in $\mathrm{CH}^1(X)_{\mathbb{Q}}$, we obtain $\tau_1 \neq 0$. Hence R is not a Roberts ring.

Remark 6.8 The author knows no example of a four-dimensional Gorenstein ring that is not a numerically Roberts ring.

Let R be a four-dimensional Gorenstein ring and set

$$\overline{\tau_R}(\overline{[R]}) = \overline{\tau_4} + \overline{\tau_3} + \overline{\tau_2} + \overline{\tau_1} + \overline{\tau_0}.$$

Since $\overline{\mathrm{A}_0(R)_{\mathbb{Q}}} = \overline{\mathrm{A}_1(R)_{\mathbb{Q}}} = 0$ by Proposition 3.7, we have $\overline{\tau_0} = \overline{\tau_1} = 0$. Since R is a Gorenstein ring, we have $\tau_3 = 0$. Therefore, R is a numerically Roberts ring if and only if $\overline{\tau_2} = 0$.

Furthermore, assume that R is the local ring (at the homogeneous maximal ideal) of an affine cone of a smooth projective variety X of dimension 3 over $\mathbb{C}$. In this case, it will be proved in Remark 7.12 that $\overline{\mathrm{A}_2(R)_{\mathbb{Q}}} = 0$. Therefore, R is a numerically Roberts ring in this case.

Remark 6.9 Assume that R is a numerically Roberts ring. Then, R/xR is also a numerically Roberts ring for any non-zero-divisor x of R, as will be proved in this remark. The author does not know whether $R_\mathfrak{p}$ is a numerically Roberts ring for a prime ideal $\mathfrak{p}$ of R.

Let $(R, \mathfrak{m})$ be a homomorphic image of a regular local ring S. Assume that $(A, \mathfrak{n})$ is a homomorphic image of R such that $\mathrm{pd}_R A < \infty$. Let $\mathbb{H}.$ be a finite R-free resolution of A. Then, we have the map

$$\chi_{\mathbb{H}.} : \mathrm{G}_0(R)_\mathbb{Q} \to \mathrm{G}_0(A)_\mathbb{Q}$$

defined by $\chi_{\mathbb{H}.}([M]) = \sum_i (-1)^i [H_i(\mathbb{H}. \otimes_R M)]$. Suppose that $\mathbb{F}.$ is a complex in $C^\mathfrak{n}(A)$. Then, since $\mathrm{pd}_R A < \infty$, there is a complex $\mathbb{G}. \in C^\mathfrak{m}(R)$ with a quasi-isomorphism $\mathbb{G}. \to \mathbb{F}.$ as in Lemma 1.10 in [3]. Using a spectral sequence argument, the diagram

$$\begin{tikzcd} \mathrm{G}_0(R)_\mathbb{Q} \arrow[r, "\chi_{\mathbb{H}.}"] \arrow[dr, "\chi_{\mathbb{G}.}"'] & \mathrm{G}_0(A)_\mathbb{Q} \arrow[d, "\chi_{\mathbb{F}.}"] \\ & \mathbb{Q} \end{tikzcd} \tag{6.10}$$

is commutative. Therefore, for $\beta \in \mathrm{N}\,\mathrm{G}_0(R)_\mathbb{Q}$, $\chi_{\mathbb{H}.}(\beta)$ is contained in $\mathrm{N}\,\mathrm{G}_0(A)_\mathbb{Q}$. Thus, we have an induced map $\overline{\chi_{\mathbb{H}.}} : \overline{\mathrm{G}_0(R)_\mathbb{Q}} \to \overline{\mathrm{G}_0(A)_\mathbb{Q}}$ that makes the following diagram commutative:

$$\begin{tikzcd} \mathrm{G}_0(R)_\mathbb{Q} \arrow[r, "\chi_{\mathbb{H}.}"] \arrow[d] & \mathrm{G}_0(A)_\mathbb{Q} \arrow[d] \\ \overline{\mathrm{G}_0(R)_\mathbb{Q}} \arrow[r, "\overline{\chi_{\mathbb{H}.}}"] & \overline{\mathrm{G}_0(A)_\mathbb{Q}} \end{tikzcd}$$

Since the diagram (6.10) is commutative, so is

$$\begin{tikzcd} \mathrm{A}_*(R)_\mathbb{Q} \arrow[r, "\mathrm{ch}(\mathbb{H}.)"] \arrow[dr, "\mathrm{ch}(\mathbb{G}.)"'] & \mathrm{A}_*(A)_\mathbb{Q} \arrow[d, "\mathrm{ch}(\mathbb{F}.)"] \\ & \mathbb{Q} \end{tikzcd}$$

by the local Riemann-Roch formula (Example 18.3.12 in [2]). Therefore, we have $\mathrm{ch}(\mathbb{H}.)(\mathrm{N}\,\mathrm{A}_*(R)_\mathbb{Q}) \subseteq \mathrm{N}\,\mathrm{A}_*(A)_\mathbb{Q}$ and the following commutative diagram:

$$\begin{tikzcd} \mathrm{A}_*(R)_\mathbb{Q} \arrow[r, "\mathrm{ch}(\mathbb{H}.)"] \arrow[d] & \mathrm{A}_*(A)_\mathbb{Q} \arrow[d] \\ \overline{\mathrm{A}_*(R)_\mathbb{Q}} \arrow[r, "\overline{\mathrm{ch}(\mathbb{H}.)}"] & \overline{\mathrm{A}_*(A)_\mathbb{Q}} \end{tikzcd} \tag{6.11}$$

Then, the following diagram is commutative:

$$\begin{tikzcd} \overline{\mathrm{G}_0(R)_\mathbb{Q}} \arrow[r, "\overline{\tau_R}"] \arrow[d, "\overline{\chi_{\mathbb{H}.}}"'] & \overline{\mathrm{A}_*(R)_\mathbb{Q}} \arrow[d, "\overline{\mathrm{ch}(\mathbb{H}.)}"] \\ \overline{\mathrm{G}_0(A)_\mathbb{Q}} \arrow[r, "\overline{\tau_A}"] & \overline{\mathrm{A}_*(A)_\mathbb{Q}} \end{tikzcd}$$

Therefore, we have

$$\overline{\mathrm{ch}(\mathbb{H}.)}\left(\overline{\tau_R}(\overline{[R]})\right) = \overline{\tau_A}(\overline{[A]}). \tag{6.12}$$

Now, assume that A coincides with R/xR for some non-zero-divisor x on R. Then, for $\gamma \in \mathrm{A}_i(R)_{\mathbb{Q}}$, $\mathrm{ch}(\mathbb{H}.)(\gamma)$ is contained in $\mathrm{A}_{i-1}(A)_{\mathbb{Q}}$ since $\mathrm{ch}(\mathbb{H}.) = \mathrm{ch}_1(\mathbb{H}.)$ by Corollary 18.1.2 in Fulton [2]. By the commutativity of the diagram (6.11), we have

$$\overline{\mathrm{ch}(\mathbb{H}.)}\left(\overline{\mathrm{A}_i(R)_{\mathbb{Q}}}\right) \subseteq \overline{\mathrm{A}_{i-1}(A)_{\mathbb{Q}}}$$

for each i. Here, assume that R is a d-dimensional numerically Roberts ring. Since $\overline{\tau_R}(\overline{[R]}) \in \overline{\mathrm{A}_d(R)_{\mathbb{Q}}}$, we have

$$\overline{\tau_R}(\overline{[A]}) = \overline{\mathrm{ch}(\mathbb{H}.)}\left(\overline{\tau_R}(\overline{[R]})\right) \in \overline{\mathrm{A}_{d-1}(A)_{\mathbb{Q}}}$$

by (6.12). Consequently A is a $(d-1)$-dimensional numerically Roberts ring.

Remark 6.13 Let $f : (A, \mathfrak{p}) \to (B, \mathfrak{q})$ be a flat local homomorphism of Noetherian local rings. Assume that $B/\mathfrak{p}B$ is a complete intersection, and $B/\mathfrak{q}$ is a finite (algebraic) separable extension of $A/\mathfrak{p}$. Then, A is a numerically Roberts ring if and only if B is a numerically Roberts ring.

The proof is omitted here.

7 Affine cones of smooth projective varieties

In this section, we treat affine cones of smooth projective varieties. Using a method and a result of Roberts and Srinivas [29], we shall calculate Chow group modulo numerical equivalence.

Let k be an algebraically closed field and let $A = \oplus_{n \geq 0} A_n$ be a Noetherian graded ring with $A_0 = k$ and $A = k[A_1]$. Assume that $X = \mathrm{Proj}(A)$ is smooth over k. Set $A_+ = \oplus_{n>0} A_n$, $R = A_{A_+}$ and $\mathfrak{m} = A_+ R$. Let $\pi : Z \to \mathrm{Spec}(R)$ be the blowing-up with center $\mathfrak{m}$. Then, $\pi^{-1}(\mathfrak{m})$ naturally coincides with X. Thus, we regard X as a closed subscheme of Z. Let h be the very ample divisor under the embedding $X = \mathrm{Proj}(A)$.

Note that $Z \setminus X = \mathrm{Spec}(R) \setminus \mathrm{Spec}(R/\mathfrak{m})$. Then, by the theory of localization sequences due to Thomason-Trobaugh [33], we have the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{K}_1(Z \setminus X)_{\mathbb{Q}} & \longrightarrow & \mathrm{K}_0^X(Z)_{\mathbb{Q}} & \stackrel{r}{\longrightarrow} & \mathrm{K}_0(Z)_{\mathbb{Q}} & \longrightarrow & \cdots \\
 & & \| & & \pi^* \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathrm{K}_1(\mathrm{Spec}(R) \setminus \mathrm{Spec}(R/\mathfrak{m}))_{\mathbb{Q}} & \longrightarrow & \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} & \stackrel{s}{\longrightarrow} & \mathrm{K}_0(R)_{\mathbb{Q}} & \longrightarrow & \cdots
\end{array}$$

where the horizontal sequences are exact, and $\mathrm{K}_0(R)_{\mathbb{Q}}$ (resp. $\mathrm{K}_0(Z)_{\mathbb{Q}}$) denotes the Grothendieck group of finitely generated projective R-modules (resp. locally free

$\mathcal{O}_Z$-modules of finite rank). We refer the reader to 1.5 in [3] for the definition of π^*. Since the map s coincides with 0 in this case, we have an exact sequence

$$\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} \xrightarrow{\pi^*} \mathrm{K}_0^X(Z)_{\mathbb{Q}} \xrightarrow{r} \mathrm{K}_0(Z)_{\mathbb{Q}}.$$

Since Z is a regular scheme, $\chi : \mathrm{K}_0^X(Z)_{\mathbb{Q}} \to \mathrm{G}_0(X)_{\mathbb{Q}}$ is an isomorphism by Lemma 1.9 in [3], where χ is defined by $\chi([\mathbb{F}.]) = \sum_i (-1)^i [H_i(\mathbb{F}.)]$. In particular, the natural map $\mathrm{K}_0(Z)_{\mathbb{Q}} \to \mathrm{G}_0(Z)_{\mathbb{Q}}$ is an isomorphism. Therefore, we have an exact sequence

$$\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} \xrightarrow{\chi\pi^*} \mathrm{G}_0(X)_{\mathbb{Q}} \xrightarrow{i_*} \mathrm{G}_0(Z)_{\mathbb{Q}},$$

where i_* is the map induced by the closed immersion $X \xrightarrow{i} Z$. Note that the diagram

$$\begin{array}{ccc} \mathrm{G}_0(X)_{\mathbb{Q}} & \xrightarrow{i_*} & \mathrm{G}_0(Z)_{\mathbb{Q}} \\ {\scriptstyle \tau_{X/Z}} \downarrow & & {\scriptstyle \tau_{Z/Z}} \downarrow \\ \mathrm{A}_*(X)_{\mathbb{Q}} & \xrightarrow{i_*} & \mathrm{A}_*(Z)_{\mathbb{Q}} \end{array}$$

is commutative, where $\tau_{X/Z}$ is the Riemann-Roch map of X with base regular scheme Z (see 18.2 and 20.1 in [2]). Since the vertical maps in the diagram as above are isomorphisms, the sequence

$$\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} \xrightarrow{\tau_{X/Z}\chi\pi^*} \mathrm{A}_*(X)_{\mathbb{Q}} \xrightarrow{i_*} \mathrm{A}_*(Z)_{\mathbb{Q}}$$

is exact.

Let Y be the blowing-up of $\mathrm{Spec}(A)$ with center A_+. Consider the following fibre squares:

$$\begin{array}{ccccc} X & \xrightarrow{i} & Z & \xrightarrow{j} & Y \\ {\scriptstyle \pi'} \downarrow & & {\scriptstyle \pi} \downarrow & & {\scriptstyle \pi"} \downarrow \\ \mathrm{Spec}(R/\mathfrak{m}) & \longrightarrow & \mathrm{Spec}(R) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Note that Y is a vector bundle on X with sheaf of sections $\mathcal{O}_X(-1)$. Let $p : Y \to X$ be the projection. By Theorem 3.3 (a) in [2], the pull-back map $p^* : \mathrm{A}_*(X) \to \mathrm{A}_*(Y)$ is an isomorphism. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{A}_*(X)_{\mathbb{Q}} & \xrightarrow{i_*} & \mathrm{A}_*(Z)_{\mathbb{Q}} \\ {\scriptstyle (ji)_*} \downarrow & {\scriptstyle j^*} \nearrow & {\scriptstyle i^*} \downarrow \\ \mathrm{A}_*(Y)_{\mathbb{Q}} & \xrightarrow{(ji)^*} & \mathrm{A}_*(X)_{\mathbb{Q}} \end{array} \tag{7.1}$$

Here, i_* (resp. $(ji)_*$) is the map induced by the closed immersion $i : X \to Z$ (resp. $ji : X \to Y$), j^* is the map induced by the flat map $j : Z \to Y$, and i^* (resp. $(ji)^*$) is the map taking the intersection with the Cartier divisor $i : X \to Z$ (resp. $ji : X \to Y$). We refer to Chapters 1 and 2 in [2] for these induced maps as above. Commutativity of the diagram (7.1) follows from Proposition 1.7 and Proposition 2.3 (d) in [2]. By Corollary 6.5 in [2], $(ji)^*$ coincides with p^{*-1}. In particular, $(ji)^*$ is an isomorphism. By the commutativity of the diagram (7.1), we have

$$\mathrm{Ker}\left(\mathrm{A}_*(X)_{\mathbb{Q}} \xrightarrow{i_*} \mathrm{A}_*(Z)_{\mathbb{Q}}\right) = \mathrm{Ker}\left(\mathrm{A}_*(X)_{\mathbb{Q}} \xrightarrow{(ji)^*(ji)_*} \mathrm{A}_*(X)_{\mathbb{Q}}\right).$$

Using Example 3.3.2 in [2], the map $(ji)^*(ji)_*$ coincides with the multiplication by $-h \in \mathrm{A}_{\dim X-1}(X)_{\mathbb{Q}}$. Let $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} = \oplus_{i=0}^{\dim X}\mathrm{CH}^i(X)_{\mathbb{Q}}$ be the Chow ring of X with rational coefficients. Note that $\mathrm{CH}^i(X) = \mathrm{A}_{\dim X-i}(X)$ for each i. Setting $\varphi = \tau_{X/Z}\chi\pi^*$, we obtain an exact sequence

$$\mathrm{K}_0^{\mathrm{m}}(R)_{\mathbb{Q}} \xrightarrow{\varphi} \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \xrightarrow{h} \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}, \tag{7.2}$$

where $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \xrightarrow{h} \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$ denotes the multiplication by h.

Set

$$\begin{aligned} K &= \mathrm{Ker}(\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \xrightarrow{h} \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}) \\ L &= \mathrm{Ker}(\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} \xrightarrow{h} \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}) \\ M &= \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}/h\cdot \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} \end{aligned}$$

where $\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}$ denotes the Chow ring of X (with rational coefficients) modulo numerical equivalence. Then, we have a commutative diagram

$$\begin{array}{ccccccccccc} & & \mathrm{K}_0^{\mathrm{m}}(R)_{\mathbb{Q}} & & & & & & & & \\ & & {\scriptstyle\varphi}\downarrow & & & & & & & & \\ 0 & \longrightarrow & K & \longrightarrow & \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} & \xrightarrow{h} & \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} & \xrightarrow{\xi} & \mathrm{A}_*(R)_{\mathbb{Q}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} & \xrightarrow{h} & \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are exact (e.g., [10]), and ξ is defined by $\xi([\mathrm{Proj}(A/P)]) = [\mathrm{Spec}(R/PR)]$ for each homogeneous prime ideal P of A such that $P \neq A_+$. Note that $\varphi : \mathrm{K}_0^{\mathrm{m}}(R)_{\mathbb{Q}} \to K$ is surjective by the exactness of the sequence (7.2). Let W be the image of the induced map $K \to L$. We denote by

$$\phi : \mathrm{K}_0^{\mathrm{m}}(R)_{\mathbb{Q}} \to W \tag{7.3}$$

the induced surjection. By the argument of Roberts-Srinivas [29], we know that

(7.4) $W = \overline{\mathrm{K}_0^{\mathrm{m}}(R)_{\mathbb{Q}}}$, that is, $0 \to \mathrm{N}\,\mathrm{K}_0^{\mathrm{m}}(R)_{\mathbb{Q}} \to \mathrm{K}_0^{\mathrm{m}}(R)_{\mathbb{Q}} \xrightarrow{\phi} W \to 0$ is exact, and

(7.5) there exists a map $\overline{\xi} : \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} \to \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$ such that the following diagram is commutative, where the vertical maps are the projections.

$$\begin{array}{ccc}
\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} & \xrightarrow{\xi} & \mathrm{A}_*(R)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} & \xrightarrow{\overline{\xi}} & \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}
\end{array}$$

Here, we give an outline of proofs of (7.4) and (7.5). For a homogeneous prime ideal P of A not equal to A_+, Z_P denotes the proper transform of $\mathrm{Spec}(R/PR)$, i.e., Z_P is the closed integral subscheme of Z such that $\pi(Z_P) = \mathrm{Spec}(R/PR)$. Note that the induced morphism $Z_P \to \mathrm{Spec}(R/PR)$ is a birational surjection. Consider the following fibre square:

$$\begin{array}{ccc}
X & \xrightarrow{i} & Z \\
{\scriptstyle \pi'} \downarrow & & {\scriptstyle \pi} \downarrow \\
\mathrm{Spec}(R/\mathfrak{m}) & \longrightarrow & \mathrm{Spec}(R)
\end{array}$$

For $\alpha \in \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}}$, we have

$$\mathrm{ch}(\alpha)\left([\mathrm{Spec}(R/PR)]\right) = \mathrm{ch}(\alpha)\left(\pi_*([Z_P])\right) = \pi'_*\left(\mathrm{ch}(\pi^*\alpha)\left([Z_P]\right)\right)$$

by Definition 17.1 in [2]. Since Z and X are regular schemes, we have isomorphisms

$$\mathrm{K}_0^X(Z)_{\mathbb{Q}} \xrightarrow{\chi} \mathrm{G}_0(X)_{\mathbb{Q}} \xleftarrow{\chi} \mathrm{K}_0^X(X)_{\mathbb{Q}}$$

by Lemma 1.9 in [3]. Take $\epsilon \in \mathrm{K}_0^X(X)_{\mathbb{Q}}$ such that $\chi(\pi^*\alpha) = \chi(\epsilon)$. Since the diagram

$$\begin{array}{ccc}
\mathrm{G}_0(X)_{\mathbb{Q}} & \xrightarrow{\tau_{X/X}} & \mathrm{A}_*(X)_{\mathbb{Q}} \\
{\scriptstyle \chi_\epsilon} \downarrow & & {\scriptstyle \mathrm{ch}(\epsilon)} \downarrow \\
\mathrm{G}_0(X)_{\mathbb{Q}} & \xrightarrow{\tau_{X/X}} & \mathrm{A}_*(X)_{\mathbb{Q}}
\end{array}$$

is commutative by the local Riemann-Roch formula (Example 18.3.12 in [2]), we have

$$\tau_{X/X}\chi(\pi^*\alpha) = \tau_{X/X}\chi(\epsilon) = \tau_{X/X}\chi_\epsilon([\mathcal{O}_X]) = \mathrm{ch}(\epsilon)\left(\tau_{X/X}([\mathcal{O}_X])\right) = \mathrm{ch}(\epsilon)\left([X]\right).$$

Note that, by the definition of the Riemann-Roch map (18.2 and 20.1 in [2]), we have $\tau_{X/X}([\mathcal{O}_X]) = X$.

On the other hand, by Corollary 18.1.2 in [2], we have

$$\pi'_*\left(\mathrm{ch}(\pi^*\alpha)\left([Z_P]\right)\right) = \pi'_*\left(\mathrm{ch}(\epsilon)\cdot \mathrm{td}(N)^{-1}\cdot i^*[Z_P]\right),$$

where N denotes the normal bundle of $i : X \to Z$, and $i^* : \mathrm{A}_*(Z)_{\mathbb{Q}} \to \mathrm{A}_*(X)_{\mathbb{Q}}$ denotes the map taking the intersection with X. Since $i^*[Z_P] = [\mathrm{Proj}(A/P)]$ and $\mathrm{ch}(\epsilon)\left([X]\right) = \tau_{X/X}\chi(\pi^*\alpha)$ as above, we have

$$\pi'_*\left(\mathrm{ch}(\epsilon)\cdot \mathrm{td}(N)^{-1}\cdot i^*[Z_P]\right) = \pi'_*\left(\tau_{X/X}\chi(\pi^*\alpha)\cdot \mathrm{td}(N)^{-1}\cdot [\mathrm{Proj}(A/P)]\right).$$

By Corollary 18.1.2 in [2] and the definition of the Riemann-Roch map (18.2 and 20.1 in [2]), we have

$$\tau_{X/Z} = \mathrm{td}(N)^{-1} \cdot \tau_{X/X}.$$

Hence, we have

$$\begin{array}{rcl} \pi'_* \left(\tau_{X/X}\chi(\pi^*\alpha) \cdot \mathrm{td}(N)^{-1} \cdot [\mathrm{Proj}(A/P)]\right) & = & \pi'_* \left(\tau_{X/Z}\chi(\pi^*\alpha) \cdot [\mathrm{Proj}(A/P)]\right) \\ & = & \pi'_* \left(\varphi(\alpha) \cdot [\mathrm{Proj}(A/P)]\right) \end{array}$$

by the definition of φ. Consequently, we have

$$\mathrm{ch}(\alpha)\left([\mathrm{Spec}(R/PR)]\right) = \pi'_* \left(\varphi(\alpha) \cdot [\mathrm{Proj}(A/P)]\right). \tag{7.6}$$

Recall that $\mathrm{A}_*(R)_{\mathbb{Q}}$ is generated by

$$\{[\mathrm{Spec}(R/PR)] \mid P \text{ is a homogeneous prime ideal of } A \text{ not equal to } A_+\}$$

because ξ is surjective (cf. Theorem 1.3 in [10]). Therefore, by (7.6),

α is in $\mathrm{N\,K}_0^{\mathrm{m}}(R)_{\mathbb{Q}}$ if and only if $\varphi(\alpha)$ coincides with 0 in $\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}$.

In other words, $\mathrm{N\,K}_0^{\mathrm{m}}(R)_{\mathbb{Q}} = \mathrm{Ker}(\phi)$, where ϕ is the map in (7.3). The statement (7.4) follows from this equality.

If $[\mathrm{Proj}(A/P)]$ is equal to 0 in $\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}$, then $[\mathrm{Spec}(R/PR)]$ is contained in $\mathrm{N\,A}_*(R)_{\mathbb{Q}}$ by (7.6). Thus, (7.5) holds. **q.e.d.**

By (7.5) above, we have a surjection $M \to \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$. Since $\dim \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} < \infty$, we have $\dim L = \dim M$. By (7.4) as above and Theorem 3.1, we have $\dim W = \dim \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$.

$$\begin{array}{ccc} \dim W & \leq & \dim L \\ \| & & \| \\ \dim \overline{\mathrm{A}_*(R)_{\mathbb{Q}}} & \leq & \dim M \end{array}$$

Therefore, the following three conditions are equivalent:

a) $W = L$,

b) the natural map $K \to L$ is surjective,

c) the natural map $\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}/h \cdot \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} \to \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$ is an isomorphism.

In particular, we have the following theorem:

Theorem 7.7 *If* $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$ *is isomorphic to* $\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}$*, then the natural map* $\mathrm{A}_*(R)_{\mathbb{Q}} \to \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$ *is an isomorphism.*

In Roberts-Srinivas [29], the following statements are proved:

1) Suppose $k = \mathbb{C}$. Then, there is an example such that $W \neq L$.

2) Suppose $k = \overline{\mathbb{Q}}$ or $\overline{\mathbb{F}_p}$. If some famous conjectures are true, then $W = L$.

Here, we give some examples.

Example 7.8 Suppose that $n \geq 2$. Let X be the blowing-up of the projective space $\mathbb{P}^n_{\mathbb{C}}$ at r distinct points. Let R be the local ring at the homogeneous maximal ideal of an affine cone of X. Using Proposition 6.7 in Fulton [2], it is easily verified that

$$\mathrm{A}_i(X)_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & (i = 0,\ n) \\ \mathbb{Q}^{r+1} & (i = 1, 2, \ldots, n-1) \\ 0 & (\text{otherwise}) \end{cases}$$

and $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$ is isomorphic to $\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}$. Then, the natural map $\mathrm{A}_*(R)_{\mathbb{Q}} \to \overline{\mathrm{A}_*(R)_{\mathbb{Q}}}$ is an isomorphism by Theorem 7.7. Therefore, we obtain the following:

$$\overline{\mathrm{A}_i(R)_{\mathbb{Q}}} = \begin{cases} \mathbb{Q} & (i = n+1) \\ \mathbb{Q}^{r} & (i = n) \\ 0 & (\text{otherwise}) \end{cases}$$

In this case, $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} = r + 1$.

Example 7.9 Let m and n be positive integers such that $n \geq m \geq 2$. Suppose

$$A = k[x_{ij} \mid i = 1, \ldots, m;\ j = 1, \ldots, n] / I_2(x_{ij}),$$

where $I_2(x_{ij})$ is the ideal of A generated by all of the 2×2 minors of the $m \times n$ matrix (x_{ij}). Then, $X = \mathrm{Proj}(A) = \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ and $\dim R = m + n - 1$. In this case, $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$ is isomorphic to $\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}$. By Theorem 7.7, we have

$$\mathrm{A}_i(R)_{\mathbb{Q}} = \overline{\mathrm{A}_i(R)_{\mathbb{Q}}} = \begin{cases} \mathbb{Q} & (\text{if } i = n, n+1, \ldots, m+n-1) \\ 0 & (\text{otherwise}) \end{cases}$$

and $\dim \overline{\mathrm{G}_0(R)_{\mathbb{Q}}} = m$.

Example 7.10 Set

$$\begin{aligned} A_{2n-1} &= k[x_1, y_1, \ldots, x_n, y_n]/(x_1 y_1 + \cdots + x_n y_n) \quad \text{and} \\ A_{2n} &= k[x_1, y_1, \ldots, x_n, y_n, z]/(z^2 + x_1 y_1 + \cdots + x_n y_n). \end{aligned}$$

By Swan [32] (see Section 4 in Levine [17]), we have

$$\mathrm{CH}^i(\mathrm{Proj}(A_{2n-1})) = \begin{cases} \mathbb{Q} & (i = 0, 1, \ldots, n-2, n, \ldots, 2n-2) \\ \mathbb{Q} \oplus \mathbb{Q} & (i = n-1) \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$\mathrm{CH}^i(\mathrm{Proj}(A_{2n})) = \begin{cases} \mathbb{Q} & (i = 0, 1, \ldots, 2n-1) \\ 0 & (\text{otherwise}). \end{cases}$$

Let R_l denote the localization of A_l at the homogeneous maximal ideal. Then, we can show

$$\mathrm{A}_i(R_{2n-1})_{\mathbb{Q}} = \overline{\mathrm{A}_i(R_{2n-1})_{\mathbb{Q}}} = \begin{cases} \mathbb{Q} & (\text{if } i = n, 2n-1) \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$\mathrm{A}_i(R_{2n})_{\mathbb{Q}} = \overline{\mathrm{A}_i(R_{2n})_{\mathbb{Q}}} = \begin{cases} \mathbb{Q} & (\text{if } i = 2n) \\ 0 & (\text{otherwise}) \end{cases}$$

since $\mathrm{CH}^{\cdot}(\mathrm{Proj}(A_l))_{\mathbb{Q}}$ is isomorphic to $\mathrm{CH}^{\cdot}_{\mathrm{num}}(\mathrm{Proj}(A_l))_{\mathbb{Q}}$ in this case.

Example 7.11 Let k be a field and set

$$A = k[x_{ij} \mid i = 1, \ldots, m;\ j = 1, \ldots, n]_{(x_{ij} \mid i,\ j)} / I_t(x_{ij}),$$

where $n \geq m \geq t$ are positive integers, and $I_t(x_{ij})$ is the ideal of A generated by all the $t \times t$ minors of the $m \times n$ matrix (x_{ij}). Let R be the local ring at the homogeneous maximal ideal of A. Then, by Example 6.2 in [12], the following three conditions are equivalent:

1. R is a Roberts ring,

2. R is a complete intersection,

3. $t = 1$ or $m = n = t$.

In the case of $t = 2$, R is a numerically Roberts ring if and only if $m = n = 2$ (see Example 7.9). In the case of $t \geq 3$, the author does not know when R is a numerically Roberts ring.

Let $A_d(n)$ be the local ring at the homogeneous maximal ideal of the affine cone of the Grassmann variety $G_d(n)$ under the Plücker embedding, where d and n are integers such that $0 < d < n$. In [15], it is proved that $A_d(n)$ is a Roberts ring if and only if one of the following conditions are satisfied,

1. $d = 1$;

2. $d = n - 1$;

3. $d = 2$ and $n = 4$;

4. $d = 3$ and $n = 6$.

It is known that $\mathrm{CH}^{\cdot}(G_d(n))_{\mathbb{Q}} = \mathrm{CH}^{\cdot}_{\mathrm{num}}(G_d(n))_{\mathbb{Q}}$ (see Chapter 14 in [2]). Therefore, $A_d(n)$ is a numerically Roberts ring if and only if $A_d(n)$ is a Roberts ring.

Set

$$B_m(n) = k[z_{ij} \mid 1 \le i < j \le n]_{(z_{ij} \mid i,\ j)} \big/ \mathrm{Pf}_m(z_{ij}),$$

where $\mathrm{Pf}_m(z_{ij})$ is the pfaffian ideal of degree m of the $n \times n$ generic anti-symmetric matrix (z_{ij}) (see Section 5 in [15]). By Theorem 5.1 in [15], the following conditions are equivalent:

1. $B_m(n)$ is a Roberts ring,
2. $B_m(n)$ is a complete intersection,
3. $m = 1$ or $n = 2m$.

In the case of $m = 2$, $B_2(n)$ coincides with $A_2(n)$. Therefore, $B_2(n)$ is a numerically Roberts ring if and only if $n = 4$. In the case of $m \ge 3$, the author does not know when $B_m(n)$ is a numerically Roberts ring.

Remark 7.12 Let R be the local ring at the homogeneous maximal ideal of an affine cone A of a smooth projective variety X over $\mathbb{C}$. Set $k = \dim X$. Assume that the natural map

$$\mathrm{CH}^{\cdot}_{\mathrm{hom}}(X)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} \tag{7.13}$$

is an isomorphism, where $\mathrm{CH}^{\cdot}_{\mathrm{hom}}(X)_{\mathbb{Q}}$ is the Chow ring (with rational coefficients) modulo homological equivalence (Chapter 19 in [2]). Then, the fact that $\overline{\mathrm{A}_j(R)_{\mathbb{Q}}} = 0$ for $j \le (k+1)/2 = \dim R/2$ can be proven as follows. (The map (7.13) is an isomorphism in the case in which $\dim X$ is at most 3 or X is an abelian variety (Example 19.3.2 in [2]). If we assume that Grothendieck's standard conjectures are true, then the map (7.13) is an isomorphism.)

We have an injective map

$$cl^X : \mathrm{CH}^i_{\mathrm{hom}}(X)_{\mathbb{Q}} \longrightarrow H^{2i}(X, \mathbb{Q})$$

for $i = 0, 1, \ldots, k$ called the *cycle map* (see Chapter 19 in [2]). Let h be the very ample divisor of the embedding $X = \mathrm{Proj}(A)$. Since the map (7.13) is an isomorphism, we have the following commutative diagram for $i > k/2$:

$$\begin{array}{ccc}
\mathrm{CH}^{k-i}_{\mathrm{num}}(X)_{\mathbb{Q}} & \xrightarrow{cl^X} & H^{2(k-i)}(X, \mathbb{Q}) \\
{\scriptstyle h^{2i-k}} \downarrow & & {\scriptstyle h^{2i-k}} \downarrow \\
\mathrm{CH}^{i}_{\mathrm{num}}(X)_{\mathbb{Q}} & \xrightarrow{cl^X} & H^{2i}(X, \mathbb{Q})
\end{array}$$

Note that the horizontal maps are injective. By the hard Lefschetz theorem (cf. [17]), the right vertical map in the above diagram is an isomorphism. Since

$\dim \mathrm{CH}^{k-i}_{\mathrm{num}}(X)_{\mathbb{Q}} = \dim \mathrm{CH}^{i}_{\mathrm{num}}(X)_{\mathbb{Q}}$, the left vertical map is also an isomorphism. In particular, the map

$$\mathrm{CH}^{i-1}_{\mathrm{num}}(X)_{\mathbb{Q}} \xrightarrow{h} \mathrm{CH}^{i}_{\mathrm{num}}(X)_{\mathbb{Q}}$$

is surjective for $i > k/2$.

Recall that there exist surjective maps

$$\mathrm{CH}^{i}_{\mathrm{num}}(X)_{\mathbb{Q}}/h \cdot \mathrm{CH}^{i-1}_{\mathrm{num}}(X)_{\mathbb{Q}} \to \overline{\mathrm{A}_{\dim R - i}(R)_{\mathbb{Q}}}$$

for each i by (7.5). Therefore, we have $\overline{\mathrm{A}_j(R)_{\mathbb{Q}}} = 0$ for $j \leq (k+1)/2 = \dim R/2$. By Remark 3.6 and (8.7), this is equivalent to the condition that $\chi_{\mathbb{F}.}(M)$ is equal to 0 for any $\mathbb{F}. \in C^{\mathfrak{m}}(R)$ and any finitely generated R-module M with $\dim M \leq \dim R/2$.

8 A vanishing of Chow group modulo numerical equivalence

In this section, we attempt to prove the following theorem:

Theorem 8.1 *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local domain that is a homomorphic image of a regular local ring. Assume that there exists a regular alteration $\pi : Z \to \mathrm{Spec}(R)$. Then, we have $\overline{\mathrm{A}_t(R)_{\mathbb{Q}}} = 0$ for $t < d - \dim \pi^{-1}(\mathfrak{m})$.*

By this theorem, we know that, for any regular alteration $\pi' : Z' \to \mathrm{Spec}(R)$, we have

$$\dim \pi'^{-1}(\mathfrak{m}) \geq d - \min\{t \mid \overline{\mathrm{A}_t(R)_{\mathbb{Q}}} \neq 0\}.$$

Before proving the theorem, we give some examples.

Example 8.2 Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local domain and let $f : W \to \mathrm{Spec}(R)$ be a resolution of singularities of $\mathrm{Spec}(R)$. Then, by the above argument, we have

$$\dim f^{-1}(\mathfrak{m}) \geq d - \min\{t \mid \overline{\mathrm{A}_t(R)_{\mathbb{Q}}} \neq 0\}.$$

Let R be a ring in Example 7.9. In this case, d is equal to $m + n - 1$ and we have

$$\min\{t \mid \overline{\mathrm{A}_t(R)_{\mathbb{Q}}} \neq 0\} = n.$$

On the other hand, let $g : V \to \mathrm{Spec}(R)$ be the blowing-up of $\mathrm{Spec}(R)$ with center $(x_{11}, x_{21}, \ldots, x_{m1})$. Then, g is a resolution of singularities of $\mathrm{Spec}(R)$ with $\dim g^{-1}(\mathfrak{m}) = m - 1$. Therefore, in this case,

$$\dim g^{-1}(\mathfrak{m}) = d - \min\{t \mid \overline{\mathrm{A}_t(R)_{\mathbb{Q}}} \neq 0\}.$$

Next, we assume that R is the local ring at the homogeneous maximal ideal of an affine cone of an abelian variety X over the field of complex numbers. Then, it is known that, for any regular alteration $\pi : Z \to \mathrm{Spec}(R)$, $\dim \pi^{-1}(\mathfrak{m})$ is equal to $\dim X = d - 1$ (cf. Example 3.9 in Ishii-Milman [6]). On the other hand, by Remark 7.12, we have

$$\min\{t \mid \overline{\mathrm{A}_t(R)_{\mathbb{Q}}} \neq 0\} > d/2.$$

Therefore, if $d \geq 3$, we have

$$\dim \pi^{-1}(\mathfrak{m}) > d - \min\{t \mid \overline{\mathrm{A}_t(R)_{\mathbb{Q}}} \neq 0\}$$

for any regular alteration $\pi : Z \to \mathrm{Spec}(R)$.

In order to prove Theorem 8.1, we need the following lemma:

Lemma 8.3 *Let X be a d-dimensional integral scheme that is of finite type over a regular scheme. Assume that there is a regular alteration $\pi : Z \to X$. Let Y be a closed subset of X. Let i be a non-negative integer and take $c \in \mathrm{A}^i(Y \to X)_{\mathbb{Q}}$. Then, the following conditions are equivalent:*

1) $c = 0$ in $\mathrm{A}^i(Y \to X)_{\mathbb{Q}}$.

2) $c([Z]) = 0$ in $\mathrm{A}_{d-i}(\pi^{-1}(Y))_{\mathbb{Q}}$.

In particular, if $d - i > \dim \pi^{-1}(Y)$, then $\mathrm{A}^i(Y \to X)_{\mathbb{Q}}$ coincides with 0.

Before proving Lemma 8.3, recall bivariant classes and the bivariant group $\mathrm{A}^i(Y \to X)$ defined in Chapter 17 in Fulton [2]. A bivariant class c in $\mathrm{A}^i(Y \to X)$ is a collection of homomorphisms

$$c_g^{(k)} : \mathrm{A}_k(X') \longrightarrow \mathrm{A}_{k-i}(X' \times_X Y)$$

for all $g : X' \to X$ and all $k \in \mathbb{Z}$ compatible with proper push-forward, flat pull-back, and intersection products, that is, $c_g^{(k)}$'s satisfy (C_1), (C_2) and (C_3) in Definition 17.1 in [2].

Proof. By definition, 1) implies 2).

We shall prove that 2) implies 1). Let V be an s-dimensional integral scheme and let $g : V \to X$ be a morphism of finite type. We want to show $c([V]) = 0$ in $\mathrm{A}_{s-i}(g^{-1}(Y))_{\mathbb{Q}}$.

Take a closed integral subscheme V' of $V \times_X Z$ such that the composite map $V' \to V \times_X Z \to V$ is proper surjective and generically finite. Then, by the compatibility with proper push-forward (Definition 17.1 (C_1) in [2]), $c([V']) = 0$ implies that $c([V]) = 0$. Replacing V by V', we may assume that there is a morphism $h : V \to Z$ such that $\pi \cdot h = g$.

Setting $W = \pi^{-1}(Y)$, we have a natural map

$$\mathrm{A}^i(Y \to X)_{\mathbb{Q}} \to \mathrm{A}^i(W \to Z)_{\mathbb{Q}}$$

as in 17.2 (P_3) in [2]. The image of c will be denoted by c again. Then, $c \in \mathrm{A}^i(W \to Z)_{\mathbb{Q}}$ satisfies $c([Z]) = 0$ by the assumption. In this situation, we want to show that $c([V]) = 0$.

By Nagata's compactification ([18], [23]), there exist an integral scheme $\overline{V}$, a proper morphism $\overline{h} : \overline{V} \to Z$ and an open immersion $j : V \to \overline{V}$ such that $h = \overline{h} \cdot j$. Then, by the compatibility with flat pull-back (Definition 17.1 (C_2) in [2]), $c([\overline{V}]) = 0$ implies $c([V]) = 0$. Replacing V by $\overline{V}$, we may assume that the morphism $h : V \to Z$ is proper.

Furthermore, using Chow's lemma (cf. [4]), we may assume that $h : V \to Z$ is projective by the compatibility with proper push-forward. Suppose that V is a closed subscheme of $\mathbb{P}^n_Z$ and the composite map

$$V \subseteq \mathbb{P}^n_Z \to Z$$

coincides with h. Set $t = \operatorname{codim}_{\mathbb{P}^n_Z} V$. Since $\mathbb{P}^n_Z$ is a regular scheme, there exists a locally-free $\mathcal{O}_{\mathbb{P}^n_Z}$-resolution $\mathbb{F}.$ of $\mathcal{O}_V$. Then, we have

$$\mathrm{ch}_t(\mathbb{F}.)([\mathbb{P}^n_Z]) = [V]$$

by the top term property (Theorem 18.3 (5) in [2]). Therefore, we have

$$\begin{aligned} c([V]) &= c(\mathrm{ch}_t(\mathbb{F}.)([\mathbb{P}^n_Z])) \\ &= \mathrm{ch}_t(\mathbb{F}.)(c([\mathbb{P}^n_Z])) \end{aligned}$$

since c and $\mathrm{ch}_t(\mathbb{F}.)$ commute by a theorem of Roberts [27]. Since $\mathbb{P}^n_Z \to Z$ is a flat map, $c([Z]) = 0$ implies $c([\mathbb{P}^n_Z]) = 0$ by the compatibility with flat pull-back. Therefore, we have $c([V]) = 0$. **q.e.d.**

Corollary 8.4 *Let X be a d-dimensional integral scheme that is of finite type over a regular scheme. Assume that there exists a regular alteration $\pi : Z \to X$. Let Y be a closed subset of X and set $W = \pi^{-1}(Y)$. Let $\mathbb{F}.$ be a bounded $\mathcal{O}_X$-locally free complex with support in Y. Then, for a non-negative integer k, the following conditions are equivalent:*

1) $\mathrm{ch}_i(\mathbb{F}.) = 0$ *in* $\mathrm{A}^i(Y \to X)_{\mathbb{Q}}$ *for* $i = 0, 1, \ldots, k-1$,

2) $\sum_i (-1)^i [H_i(\pi^* \mathbb{F}.)] \in F_{d-k}\,\mathrm{G}_0(W)_{\mathbb{Q}}$,

where $F_{d-k}\,\mathrm{G}_0(W)_{\mathbb{Q}}$ *is a vector subspace of* $\mathrm{G}_0(W)_{\mathbb{Q}}$ *spanned by*

$$\{[\mathcal{M}] \mid \mathcal{M} \text{ is a coherent } \mathcal{O}_W\text{-module with } \dim \operatorname{Supp} \mathcal{M} \le d-k\}.$$

Note that condition 1) above is independent of the choice of a regular alteration $\pi : Z \to X$.

Proof. By Lemma 8.3, condition 1) is equivalent to the condition that $\mathrm{ch}_i(\pi^*\mathbb{F}.)([Z]) = 0$ in $\mathrm{A}_{d-i}(W)_{\mathbb{Q}}$ for $i = 0, 1, \ldots, k-1$. Furthermore, this is equivalent to

$$\mathrm{ch}(\pi^*\mathbb{F}.)([Z]) \in \bigoplus_{j=0}^{d-k} \mathrm{A}_j(W)_{\mathbb{Q}}.$$

Since the diagram

$$\begin{array}{ccc} \mathrm{G}_0(Z)_{\mathbb{Q}} & \stackrel{\tau_{Z/Z}}{\longrightarrow} & \mathrm{A}_*(Z)_{\mathbb{Q}} \\ {\scriptstyle \chi_{\pi^*\mathbb{F}.}} \downarrow & & {\scriptstyle \mathrm{ch}(\pi^*\mathbb{F}.)} \downarrow \\ \mathrm{G}_0(W)_{\mathbb{Q}} & \stackrel{\tau_{W/Z}}{\longrightarrow} & \mathrm{A}_*(W)_{\mathbb{Q}} \end{array}$$

is commutative by the local Riemann-Roch theorem [2], we have

$$\tau_{W/Z}\left(\sum_i (-1)^i [H_i(\pi^*\mathbb{F}.)]\right) = \mathrm{ch}(\pi^*\mathbb{F}.)(\tau_{Z/Z}([\mathcal{O}_Z])) = \mathrm{ch}(\pi^*\mathbb{F}.)([Z]).$$

Recall that, since Z is a regular base scheme, we have $\tau_{Z/Z}([\mathcal{O}_Z]) = [Z]$.

By the top term property, we have $\tau_{W/Z}(F_{d-k}\,\mathrm{G}_0(W)_{\mathbb{Q}}) = \oplus_{j=0}^{d-k} \mathrm{A}_j(W)_{\mathbb{Q}}$. Therefore, 1) is equivalent to 2). **q.e.d.**

Next, we define two invariants for complexes in $C^{\mathfrak{m}}(R)$.

Definition 8.5 Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. For a complex $\mathbb{F}. \in C^{\mathfrak{m}}(R)$, we define

$$\begin{aligned} b(\mathbb{F}.) &= \min\{s \mid \mathrm{ch}_s(\mathbb{F}.) \neq 0 \text{ in } \mathrm{A}^s(\mathrm{Spec}(R/\mathfrak{m}) \to \mathrm{Spec}(R))_{\mathbb{Q}}\}, \\ n(\mathbb{F}.) &= \min\{s \mid \mathrm{ch}_s(\mathbb{F}.) : \mathrm{A}_s(R)_{\mathbb{Q}} \to \mathbb{Q} \text{ does not coincide with } 0\}. \end{aligned}$$

If $\mathrm{ch}_s(\mathbb{F}.)$ is 0 in $\mathrm{A}^s(\mathrm{Spec}(R/\mathfrak{m}) \to \mathrm{Spec}(R))_{\mathbb{Q}}$ for all s, we set $b(\mathbb{F}.) = \infty$. If the map $\mathrm{ch}_s(\mathbb{F}.) : \mathrm{A}_s(R)_{\mathbb{Q}} \to \mathbb{Q}$ is equal 0 for all s, we set $n(\mathbb{F}.) = \infty$.

By definition, either $0 \le n(\mathbb{F}.) \le d$ or $n(\mathbb{F}.) = \infty$.

It is easily verified that

$$0 \le b(\mathbb{F}.) \le n(\mathbb{F}.). \tag{8.6}$$

Furthermore, note that

$$\min\{t \mid \overline{\mathrm{A}_t(R)_{\mathbb{Q}}} \neq 0\} = \min\{n(\mathbb{F}.) \mid \mathbb{F}. \in C^{\mathfrak{m}}(R)\}. \tag{8.7}$$

Remark 8.8 Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local domain with regular alteration $\pi : Z \to \mathrm{Spec}(R)$. Set $X = \pi^{-1}(\mathfrak{m})$ and $\chi(\pi^*\mathbb{F}.) = \sum_i (-1)^i [H_i(\pi^*\mathbb{F}.)] \in \mathrm{G}_0(X)_{\mathbb{Q}}$. Then, for each $\mathbb{F}. \in C^{\mathfrak{m}}(R)$, we have

$$\chi(\pi^*\mathbb{F}.) \in F_{d-b(\mathbb{F}.)}\, \mathrm{G}_0(X)_{\mathbb{Q}} \setminus F_{d-b(\mathbb{F}.)-1}\, \mathrm{G}_0(X)_{\mathbb{Q}}$$

by Corollary 8.4. This is equivalent to

$$\tau_{X/Z}(\chi(\pi^*\mathbb{F}.)) \in \bigoplus_{i \le d-b(\mathbb{F}.)} \mathrm{A}_i(X)_{\mathbb{Q}} \setminus \bigoplus_{i < d-b(\mathbb{F}.)} \mathrm{A}_i(X)_{\mathbb{Q}}. \tag{8.9}$$

In particular, either $0 \le b(\mathbb{F}.) \le d$ or $b(\mathbb{F}.) = \infty$.

Now we return to the proof of Theorem 8.1. Let $(R, \mathfrak{m})$ be a local ring that satisfies the assumptions in Theorem 8.1. Then, for $\mathbb{F}. \in C^{\mathfrak{m}}(R)$, we have

$$\sum_i (-1)^i [H_i(\pi^*\mathbb{F}.)] \in F_{d-b(\mathbb{F}.)}\, \mathrm{G}_0(\pi^{-1}(\mathfrak{m}))_{\mathbb{Q}} \setminus F_{d-b(\mathbb{F}.)-1}\, \mathrm{G}_0(\pi^{-1}(\mathfrak{m}))_{\mathbb{Q}}.$$

In particular, we have

$$d - b(\mathbb{F}.) \le \dim \pi^{-1}(\mathfrak{m})$$

for any $\mathbb{F}. \in C^{\mathfrak{m}}(R)$. It follows that

$$d - \dim \pi^{-1}(\mathfrak{m}) \le \min\{b(\mathbb{F}.) \mid \mathbb{F}. \in C^{\mathfrak{m}}(R)\} \le \min\{n(\mathbb{F}.) \mid \mathbb{F}. \in C^{\mathfrak{m}}(R)\} = \min\{t \mid \overline{\mathrm{A}_t(R)_{\mathbb{Q}}} \ne 0\}$$

by (8.6) and (8.7).

We have completed the proof of Theorem 8.1.

Remark 8.10 Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring.

Let $\mathbb{K}.$ be a Koszul complex with respect to a system of parameters for R. Then, $b(\mathbb{K}.) = n(\mathbb{K}.) = d$.

Suppose that R is a homomorphic image of a regular local ring $(S, \mathfrak{n})$. Assume that $\mathbb{F}. \in C^{\mathfrak{m}}(R)$ is liftable to S, that is, there exists an S-free complex $\mathbb{G}.$ such that $\mathbb{F}. = \mathbb{G}. \otimes_S R$. (Note that a Koszul complex of a system of parameters for R is always liftable.) Then,

$$b(\mathbb{F}.) = d \text{ or } \infty.$$

In fact, let Y be the support of the complex $\mathbb{G}.$, i.e., $Y = \cup_i \mathrm{Supp}(H_i(\mathbb{G}.)) \subset \mathrm{Spec}(S)$. Then, $\mathrm{Spec}(R) \cap Y = \{\mathfrak{n}\}$. Since $\dim R + \dim Y \le \dim S$ by Serre's theorem [30], we have $\dim Y \le \dim S - d$. Since $\mathrm{Spec}(S)$ itself is a regular alteration of $\mathrm{Spec}(S)$, we have $\mathrm{ch}_i(\mathbb{G}.) = 0$ in $\mathrm{A}^i(Y \to \mathrm{Spec}(S))_{\mathbb{Q}}$ for $i = 0, 1, \cdots, d-1$ by Corollary 8.4. It follows that $\mathrm{ch}_i(\mathbb{F}.)$ coincides with 0 in $\mathrm{A}^i(\mathrm{Spec}(R/\mathfrak{m}) \to \mathrm{Spec}(R))_{\mathbb{Q}}$ for $i = 0, 1, \cdots, d-1$.

Remark 8.11 We discuss affine cones of smooth projective varieties. With notation as in Section 7, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} & \stackrel{\varphi}{\longrightarrow} & K & \subseteq & \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \\ & {\scriptstyle \phi} \searrow & \downarrow & & \downarrow \\ & & W & \subseteq & \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}} \end{array}$$

Then, by the definition of φ (see (7.2)) and Remark 8.8, we have

$$\varphi([\mathbb{F}.]) \in \bigoplus_{i \geq b(\mathbb{F}.)-1} \mathrm{CH}^i(X)_{\mathbb{Q}} \setminus \bigoplus_{i > b(\mathbb{F}.)-1} \mathrm{CH}^i(X)_{\mathbb{Q}}$$

for each $\mathbb{F}. \in C^{\mathfrak{m}}(R)$.

On the other hand, using the method described in Section 7, we can prove

$$\phi([\mathbb{F}.]) \in \bigoplus_{i \geq n(\mathbb{F}.)-1} \mathrm{CH}^i_{\mathrm{num}}(X)_{\mathbb{Q}} \setminus \bigoplus_{i > n(\mathbb{F}.)-1} \mathrm{CH}^i_{\mathrm{num}}(X)_{\mathbb{Q}}.$$

Therefore, the difference between $b(\mathbb{F}.)$ and $n(\mathbb{F}.)$ corresponds to the difference between rational equivalence and numerical equivalence on X.

Example 8.12 Let R be a ring as in Example 7.9. In this case, since $\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$ is isomorphic to $\mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbb{Q}}$, $b(\mathbb{F}.)$ coincides with $n(\mathbb{F}.)$ for any $\mathbb{F}. \in C^{\mathfrak{m}}(R)$. By (8.7), we have $b(\mathbb{F}.) = n(\mathbb{F}.) \geq n$.

Let $\mathbb{H}\{s\}.$ be the complex as in Remark 2.5. Then, $b(\mathbb{H}\{s\}.) = n(\mathbb{H}\{s\}.) = s$ for $s = n, n+1, \ldots, m+n-1$.

Example 8.13 We give an example of $\mathbb{F}. \in C^{\mathfrak{m}}(R)$ with $b(\mathbb{F}.) \neq n(\mathbb{F}.)$.

Set

$$A = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3).$$

We regard A as a graded ring with $\deg(x) = \deg(y) = \deg(z) = 1$. Set $X = \mathrm{Proj}(A)$, and let $(R, \mathfrak{m})$ be the local ring of A at (x, y, z). With notation as in Section 7, the map

$$\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} \stackrel{\varphi}{\longrightarrow} K = \mathrm{Ker}(\mathrm{CH}^{\cdot}(X)_{\mathbb{Q}} \stackrel{h}{\longrightarrow} \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}})$$

is surjective. Take $0 \neq \alpha \in \mathrm{CH}^1(X)_{\mathbb{Q}}$ such that $0 = \overline{\alpha} \in \mathrm{CH}^1_{\mathrm{num}}(X)_{\mathbb{Q}}$. (Since X is an elliptic curve over $\mathbb{C}$, it is easy to take such α.) Then, by the surjectivity of φ, there exists $\mathbb{F}. \in C^{\mathfrak{m}}(R)$ such that $\varphi([\mathbb{F}.]) = n\alpha$ with some positive integer n. Then, by Remark 8.11, we have

$$b(\mathbb{F}.) = 2 \neq \infty = n(\mathbb{F}.).$$

9 Applications of Lemma 8.3

In this section, we give two applications of Lemma 8.3. The first is a sufficient condition of the vanishing property of intersection multiplicities (see Theorem 9.1 below). The second gives another proof of the vanishing theorem (see Theorem 9.4 below) of the first localized Chern characters due to Roberts [28].

Theorem 9.1 *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local domain. Assume that R is a homomorphic image of an excellent regular local ring S and there exists a regular alteration $\pi : Z \to \mathrm{Spec}(R)$. Let Y be a closed subset of $\mathrm{Spec}(R)$ such that*

$$\pi|_{Z\setminus\pi^{-1}(Y)} : Z \setminus \pi^{-1}(Y) \to \mathrm{Spec}(R) \setminus Y$$

is finite. If $\dim \pi^{-1}(Y) \leq d/2$, then R satisfies the vanishing property, that is,

$$\sum_i (-1)^i \ell_R(\mathrm{Tor}_i^R(M, N)) = 0$$

for finitely generated R-modules M and N such that $\mathrm{pd}_R M < \infty$, $\mathrm{pd}_R N < \infty$, $\ell_R(M \otimes_R N) < \infty$ and $\dim M + \dim N < d$, where pd_R denotes the projective dimension as an R-module.

Proof. Let M and N be R-modules that satisfy the assumptions in the theorem. We may assume $\dim M \geq \dim N$. Then, $\dim N < d/2$. Let $\mathbb{F}.$ and $\mathbb{G}.$ be finite free resolutions of M and N, respectively.

Set $\tau_{R/S}([R]) = \tau_d + \tau_{d-1} + \cdots + \tau_0$, where $\tau_i \in \mathrm{A}_i(R)_{\mathbb{Q}}$. Then, by the local Riemann-Roch formula, we have

$$\begin{aligned} \sum_i (-1)^i \ell_R(\mathrm{Tor}_i^R(M, N)) &= \mathrm{ch}(\mathbb{F}. \otimes \mathbb{G}.) \cap \tau_{R/S}([R]) \\ &= \sum_{i,\ j\geq 0} \mathrm{ch}_i(\mathbb{F}.)\mathrm{ch}_j(\mathbb{G}.) \cap \tau_{i+j}. \end{aligned}$$

We want to prove $\mathrm{ch}_i(\mathbb{F}.)\mathrm{ch}_j(\mathbb{G}.) \cap \tau_{i+j} = 0$ for any $i, j \geq 0$.

Assume that $\mathrm{ch}_i(\mathbb{F}.)\mathrm{ch}_j(\mathbb{G}.) \cap \tau_{i+j} \neq 0$ for some $i, j \geq 0$. By Lemma 8.3, we have

(9.2) $$\dim \pi^{-1}(\mathrm{Supp}(M)) \geq d - i$$

since $\mathrm{ch}_i(\mathbb{F}.) \neq 0$ in $\mathrm{A}^i(\mathrm{Supp}(M) \to \mathrm{Spec}(R))_{\mathbb{Q}}$. Furthermore, since

$$0 \neq \mathrm{ch}_j(\mathbb{G}.) \cap \tau_{i+j} \in \mathrm{A}_i(\mathrm{Supp}(N))_{\mathbb{Q}},$$

we have $0 \leq i \leq \dim N$ and

$$\dim M + i \leq \dim M + \dim N < d.$$

Therefore,

$$\dim M < d - i.$$

We have

$$\begin{aligned} \pi^{-1}(\mathrm{Supp}(M)) \subseteq \pi^{-1}(Y \cup \mathrm{Supp}(M)) &= \pi^{-1}(Y) \cup \pi^{-1}(\mathrm{Supp}(M) \setminus Y) \\ &= \pi^{-1}(Y) \cup \overline{\pi^{-1}(\mathrm{Supp}(M) \setminus Y)}, \end{aligned}$$

where $\overline{\pi^{-1}(\mathrm{Supp}(M) \setminus Y)}$ denotes the closure of $\pi^{-1}(\mathrm{Supp}(M) \setminus Y)$. It is easy to see that

$$\dim \overline{\pi^{-1}(\mathrm{Supp}(M) \setminus Y)} \leq \dim M,$$

since R is an excellent ring. Therefore, we have

$$\dim \pi^{-1}(\mathrm{Supp}(M)) \leq \max\{\dim \pi^{-1}(Y), \dim M\}.$$

Since $i \leq \dim N < d/2$, we obtain

$$\dim \pi^{-1}(Y) \leq d/2 < d - i.$$

Then, we have

$$\dim \pi^{-1}(\mathrm{Supp}(M)) \leq \max\{\dim \pi^{-1}(Y), \dim M\} < d - i.$$

This inequality contradicts (9.2). **q.e.d.**

Example 9.3 Let m, n, q be positive integers such that $2 \leq m \leq n$. Set

$$R = k\left[\{x_{ij} \mid i = 1, \cdots, m;\ j = 1, \ldots, n\},\ y_1, \ldots, y_q\right]_{(\underline{x}, \underline{y})} \Big/ I_2(x_{ij}).$$

Then, we have $\dim R = m + n + q - 1$. Let $\pi : Z \to \mathrm{Spec}(R)$ be the blowing-up of $\mathrm{Spec}(R)$ along $(x_{11}, x_{21}, \ldots, x_{m1})$. Then, Z is a resolution of singularities of $\mathrm{Spec}(R)$ such that $\dim \pi^{-1}((x_{11}, x_{21}, \ldots, x_{m1})) = m + q - 1$. Therefore, if $m + q \leq n + 1$, then $\pi : Z \to \mathrm{Spec}(R)$ satisfies the assumptions in Theorem 9.1.

Note that the dimension of the singular locus of $\mathrm{Spec}(R)$ is equal to q. (For a local ring such that the dimension of its singular locus is at most 1, the vanishing property was proved by Roberts [27].)

In the remainder of this section, we shall give another proof to the vanishing theorem of the first localized Chern characters due to Roberts [28] as follows:

Theorem 9.4 *Let X be a scheme that is of finite type over an excellent regular scheme. Let $X = X_1 \cup \cdots \cup X_t$ be the irreducible decomposition. Assume that each $(X_i)_{\mathrm{red}}$ has a regular alteration. Let Y be a closed subset of X such that* $\mathrm{codim}_{X_i} X_i \cap Y \geq 2$ *for $i = 1, \ldots, t$. Then, for each $\mathbb{F}. \in C^Y(X)$, we have*

$$\mathrm{ch}_1(\mathbb{F}.) = 0$$

in $\mathrm{A}^1(Y \to X)_{\mathbb{Q}}$.

Note that Roberts proved the vanishing theorem of the first localized Chern characters without assuming that $(X_i)_{\mathrm{red}}$'s have regular alterations.

Proof. Let V be an s-dimensional integral scheme with a morphism $f : V \to X$ of finite type. We want to show that $\mathrm{ch}_1(f^*\mathbb{F}.) : \mathrm{A}_s(V)_{\mathbb{Q}} \to \mathrm{A}_{s-1}(f^{-1}(Y))_{\mathbb{Q}}$ coincides with 0.

Since V is irreducible, some irreducible component X_i contains $f(V)$. Let $f_i : V \to X_i$ be the induced map. We have only to show that

$$\mathrm{ch}_1(f_i^*(\mathbb{F}. \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i})) : \mathrm{A}_s(V)_{\mathbb{Q}} \to \mathrm{A}_{s-1}(f_i^{-1}(X_i \cap Y))_{\mathbb{Q}}$$

coincides with 0. Therefore, we may assume that X is an integral scheme with a regular alteration.

Set $d = \dim X$. Let $\pi : Z \to X$ be a regular alteration. By Proposition 18.1 (a) in [2], $\mathrm{ch}_0(\mathbb{F}.)$ is the multiplication by $\sum_i (-1)^i \operatorname{rank}_{\mathcal{O}_X} F_i$. Since Y is a proper closed subset of X, we have $\mathrm{ch}_0(\mathbb{F}.) = 0$ in $\mathrm{A}^0(Y \to X)_{\mathbb{Q}}$. Set $W = \pi^{-1}(Y)$. By Corollary 8.4, $\mathrm{ch}_1(\mathbb{F}.) = 0$ in $\mathrm{A}^1(Y \to X)_{\mathbb{Q}}$ if and only if

$$\tau_{W/Z}(\chi_{\pi^*\mathbb{F}.}([\mathcal{O}_Z])) \in \bigoplus_{i=0}^{d-2} \mathrm{A}_i(W)_{\mathbb{Q}}.$$

Since W is a proper closed subset of Z, we have $\dim W \leq d-1$. If $\dim W \leq d-2$, we have nothing to prove.

Assume that $\dim W = d-1$. Let $W_1, \ldots, W_l$ be the irreducible components of W of dimension $d-1$. Then, $\mathrm{A}_{d-1}(W)_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space with basis $[W_1]$, $\ldots$, $[W_l]$. It is sufficient to show that the coefficient of $[W_i]$ in $\tau_{W/Z}(\chi_{\pi^*\mathbb{F}.}([\mathcal{O}_Z]))$ is equal to 0 for $i = 1, \ldots, l$.

Assume that the coefficient of $[W_1]$ in $\tau_{W/Z}(\chi_{\pi^*\mathbb{F}.}([\mathcal{O}_Z]))$ is not 0.

Let U be an affine open set of X such that $F_i|_U$ is $\mathcal{O}_U$-free for each i, and $\pi^{-1}(U) \cap W_1 \neq \emptyset$. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{K}_0^Y(X)_{\mathbb{Q}} & \longrightarrow & \mathrm{K}_0^{U\cap Y}(U)_{\mathbb{Q}} \\
{\scriptstyle \pi^*}\downarrow & & {\scriptstyle {\pi_U}^*}\downarrow \\
\mathrm{K}_0^W(Z)_{\mathbb{Q}} & \longrightarrow & \mathrm{K}_0^{\pi^{-1}(U)\cap W}(\pi^{-1}(U))_{\mathbb{Q}} \\
{\scriptstyle \chi}\downarrow & & {\scriptstyle \chi}\downarrow \\
\mathrm{G}_0(W)_{\mathbb{Q}} & \longrightarrow & \mathrm{G}_0(\pi^{-1}(U) \cap W)_{\mathbb{Q}} \\
{\scriptstyle \tau_{W/Z}}\downarrow & & {\scriptstyle \tau_{\pi^{-1}(U)\cap W/\pi^{-1}(U)}}\downarrow \\
\mathrm{A}_*(W)_{\mathbb{Q}} & \longrightarrow & \mathrm{A}_*(\pi^{-1}(U) \cap W)_{\mathbb{Q}}
\end{array}$$

The horizontal maps are restrictions to U. By the commutativity, the coefficient of $[\pi^{-1}(U) \cap W_1]$ in

$$\tau_{\pi^{-1}(U)\cap W/\pi^{-1}(U)} \chi {\pi_U}^*([\mathbb{F}.|_U])$$

is equal to the coefficient of $[W_1]$ in $\tau_{W/Z}(\chi_{\pi^*\mathbb{F}.}([\mathcal{O}_Z]))$.

Replacing X by U, we may assume that X is an affine integral scheme and $\mathbb{F}.$ is a complex of free modules.

Setting $\overline{X} = \operatorname{Spec}(H^0(Z, \mathcal{O}_Z))$, we take the Stein factorization

$$\begin{array}{ccc} Z & & \\ \downarrow & \searrow \pi & \\ \overline{X} & \xrightarrow{g} & X \end{array}$$

of $\pi : Z \to X$.

Since X is an excellent scheme, $\operatorname{codim}_{\overline{X}} g^{-1}(Y)$ is at least 2. Therefore, replacing X by $\overline{X}$, we may assume that X is an affine normal scheme and $\pi : Z \to X$ is birational.

We denote by i the inclusion $W \to Z$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{K}_0^Y(X)_{\mathbb{Q}} & \xrightarrow{\pi^*} & \mathrm{K}_0^W(Z)_{\mathbb{Q}} & \xrightarrow{\chi} & \mathrm{G}_0(W)_{\mathbb{Q}} & \xrightarrow{i_*} & \mathrm{G}_0(Z)_{\mathbb{Q}} \\ & & & & \tau_{W/Z} \downarrow & & \downarrow \tau_{Z/Z} \\ & & & & \mathrm{A}_*(W)_{\mathbb{Q}} & \xrightarrow{i_*} & \mathrm{A}_*(Z)_{\mathbb{Q}} \\ & & & & p \downarrow & & \downarrow p \\ & & & & \mathrm{A}_{d-1}(W)_{\mathbb{Q}} & \xrightarrow{i_*} & \mathrm{A}_{d-1}(Z)_{\mathbb{Q}} \end{array}$$

where i_* is induced by the proper morphism $i : W \to Z$ and p's are the projections. Set $\alpha_{d-1} = p \cdot \tau_{W/Z} \cdot \chi(\pi^* \mathbb{F}.)$. By our assumption, we have $\alpha_{d-1} \neq 0$.

On the other hand, we have

$$i_* \chi(\pi^* \mathbb{F}.) = \sum_j (-1)^j [\pi^* F_j] = \left(\sum_j (-1)^j \operatorname{rank} F_j \right) \cdot [\mathcal{O}_Z] = 0,$$

because each F_j is a free $\mathcal{O}_X$-module and Y is a proper closed subset of X. Therefore, $i_*(\alpha_{d-1}) = 0$. This equality contradicts the following claim:

Claim 9.5 *The map* $i_* : \mathrm{A}_{d-1}(W)_{\mathbb{Q}} \longrightarrow \mathrm{A}_{d-1}(Z)_{\mathbb{Q}}$ *is injective.*

In the remainder of this section, we shall prove the claim.

Let $W_1, \ldots, W_l$ be the irreducible components of W of dimension $d-1$. Then, $\mathrm{A}_{d-1}(W)_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space with basis $[W_1], \ldots, [W_l]$. Suppose that $i_*(q_1[W_1] + \cdots + q_l[W_l]) = 0$. Then, there exists a positive integer n and $a \in R(Z)^\times = R(X)^\times$ such that

$$\operatorname{div}_Z(a) = nq_1[W_1] + \cdots + nq_l[W_l].$$

Then, we have

$$\operatorname{div}_X(a) = \pi_*(\operatorname{div}_Z(a)) = nq_1\pi_*([W_1]) + \cdots + nq_l\pi_*([W_l])$$

by Proposition 1.4 in [2]. Since $\pi(W_i) \subset Y$, $\operatorname{codim}_X \pi(W_i)$ is at least 2 for each i. Therefore, we have $\operatorname{div}_X(a) = 0$. Since X is an affine normal scheme, a is a unit of $\Gamma(X, \mathcal{O}_X)$. Hence a is also a unit of $\Gamma(Z, \mathcal{O}_Z)$ and $\operatorname{div}_Z(a) = 0$. It follows that $q_1 = \cdots = q_l = 0$. **q.e.d.**

References

[1] S. P. Dutta, M. Hochster and J. E. MacLaughlin, *Modules of finite projective dimension with negative intersection multiplicities*, Invent. Math. **79** (1985), 253–291.

[2] W. Fulton, *Intersection Theory, 2nd Edition*, Springer-Verlag, Berlin, New York, 1997.

[3] H. Gillet and C. Soulé, *Intersection theory using Adams operation*, Invent. Math. **90** (1987), 243–277.

[4] R. Hartshorne, *Algebraic Geometry,* Graduate Texts in Math., No. 52, Springer-Verlag, Berlin and New York, 1977.

[5] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. **79** (1964), 109–326.

[6] S. Ishii and P. Milman, *The geometric minimal models of analitic spaces*, Math. Ann. **323** (2002), 437–451.

[7] A. J. de Jong, *Smoothness, semi-stability and alterations*, Publ. Math. IHES **83** (1996), 51–93.

[8] Y. Kamoi and K. Kurano, *On maps of Grothendieck groups induced by completion*, J. Alg. **254** (2002), 21–43.

[9] K. Kurano, *An approach to the characteristic free Dutta multiplicities*, J. Math. Soc. Japan, **45** (1993), 369–390.

[10] K. Kurano, *A remark on the Riemann-Roch formula for affine schemes associated with Noetherian local rings*, Tohoku Math. J. **48** (1996), 121–138.

[11] K. Kurano, *Test modules to calculate Dutta Multiplicities*, J. Alg. **236** (2001), 216-235.

[12] K. Kurano, *On Roberts rings*, J. Math. Soc. Japan. **53** (2001), 333–355.

[13] K. Kurano, *Roberts rings and Dutta multiplicities*, "Geometric and combinatorial aspects of commutative algebra", 273–287. Lect. Notes in Pure and Applied Math., **217**, Marcel Dekker, 2001.

[14] K. Kurano and P. C. Roberts, *Adams operations, localized Chern characters, and the positivity of Dutta multiplicity in characteristic 0*, Trans. Amer. Math. Soc. **352** (2000), 3103–3116.

[15] K. Kurano and A. K. Singh, *Todd classes of affine cones of Grassmannians*, Int. Math. Res. Notices **35** (2002), 1841–1855.

[16] M. Levine, *Localization on singular varieties*, Invent. Math. **91** (1988), 423–464.

[17] J. D. Lewis, *A survey of the Hodge conjecture, 2nd edition*, CRM mono. ser. 10, Amer. Math. Soc., Providence, RI, 1999.

[18] W. Lütkebohmert, *On compactification of schemes*, Manuscripta Math. **80** (1993), 95–111.

[19] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Adv. Math., 8, Cambridge Univ. Press, Cambridge-New York, 1989.

[20] C. M. Miller and A. K. Singh, *Intersection multiplicities over Gorenstein rings*, Math Ann **317** (2000), 155–171.

[21] P. Monsky, *The Hilbert-Kunz function*, Math. Ann. **263** (1983), 43–49.

[22] M. Nagata, *Local Rings*, Interscience Tracts in Pure and Appl. Math., Wiley, New York, 1962.

[23] M. Nagata, *A generalization of the imbedding problem of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **3** (1963), 89–102.

[24] T. Ogoma, *General Néron desingularization based on the idea of Popescu*, J. Alg. **167** (1994), 57–84.

[25] D. Popescu, *General Néron desingularization and approximation*, Nagoya Math. J. **104** (1986), 85–115.

[26] P. C. Roberts, *The vanishing of intersection multiplicities and perfect complexes*, Bull. Amer. Math. Soc. **13** (1985), 127–130.

[27] P. C. Roberts, *Local Chern characters and intersection multiplicities*, Proc. of Symposia in Pure Math., **46** (1987), 389–400.

[28] P. C. Roberts, *MacRae invariant and the first local chern character*, Trans. Amer. Math. Soc., **300** (1987), 583–591.

[29] P. C. Roberts and V. Srinivas, *Modules of finite length and finite projective dimension*, Invent. Math., **151** (2003), 1–27

[30] J.-P. Serre, *Algèbre locale, Multiplicités*, Lecture Notes in Math. 11, Springer-Verlag, Berlin, New York, 1965.

[31] V. Srinivas, *Algebraic K-theory, Second edition*, Progress in Mathematics, 90. Birkhauser Boston, Inc., Boston, MA, 1996

[32] R. Swan, *K-theory of quadratic hypersurfaces*, Ann. Math. **122** (1985), 113–154.

[33] R. W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories.* "The Grothendieck Festschrift", Vol. III, 247–435, Progr. Math., 88, Birkhauser Boston, Boston, MA, 1990.

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