THE TOTAL COORDINATE RING OF A NORMAL PROJECTIVE VARIETY

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1. Introduction

The total coordinate ring TC(X) of a variety is a generalization of the ring introduced and studied by Cox [Cox95] in connection with a toric variety. Consider a normal projective variety X with divisor class group $\mathrm{Cl}(X)$, and let us assume that it is a finitely generated free abelian group. We define the total coordinate ring of X to be

$$\mathrm{TC}(X) = \bigoplus_{D} H^0(X, \mathcal{O}_X(D)),$$

where the sum as above is taken over all Weil divisors of X contained in a fixed complete system of representatives of $\mathrm{Cl}(X)$. We refer to Definition 2.1 for a precise definition of $\mathrm{TC}(X)$.

Such rings grew out of an old problem of classical algebraic geometry, which we describe as follows. Let $p_1, \ldots, p_m$ be distinct m points in the projective space $\mathbb{P}^r$ over $\mathbb{C}$, and let $\pi : X \to \mathbb{P}^r$ be the blow up of $\mathbb{P}^r$ at $\{p_1, \ldots, p_m\}$. Put $E_i = \pi^{-1}(p_i)$ for $i = 1, \ldots, m$. Let H be a hyperplane in $\mathbb{P}^r$ and put $A = \pi^{-1}(H)$. Then $\mathrm{Cl}(X)$ is a free abelian group with basis $\overline{E_1}, \ldots, \overline{E_m}, \overline{A}$. Then, we may regard $H^0(X, \mathcal{O}_X(\sum_{i=1}^m n_i E_i + n_{m+1}A))$ as the linear system on $\mathbb{P}^r$ of degree n_{m+1} passing through p_i with multiplicity at least $-n_i$ for each i. The total coordinate ring

$$\mathrm{TC}(X) = \bigoplus_{n_1, \ldots, n_{m+1} \in \mathbb{Z}} H^0\Big(X, \mathcal{O}_X\Big(\sum_{i=1}^m n_i E_i + n_{m+1}A\Big)\Big)$$

facilitates the computation of $H^0(X, \mathcal{O}_X(\sum_{i=1}^m n_i E_i + n_{m+1}A))$, where $\mathbb{Z}$ denotes the ring of integers. In fact, if $\mathrm{TC}(X)$ is a Noetherian ring, then the dimension of $H^0(X, \mathcal{O}_X(\sum_{i=1}^m n_i E_i + n_{m+1}A))$ is a rational function in $n_1, \ldots, n_{m+1}$.

In the event that $\mathrm{TC}(X)$ is Noetherian for a normal projective variety with function field K whose divisor class group is finite free, the ring

$$R(D) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(nD))t^n \subset K[t, t^{-1}]$$

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is a Noetherian ring for any Weil divisor D on X (e.g., Elizondo and Srinivas [ES02]). Furthermore,

$$\bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{O}_X(E+nD))$$

is a finitely generated $R(D)$-module for any Weil divisors D and E.

The total coordinate ring sometimes appears as an invariant subring. Mukai exhibited numerous examples where the total coordinate rings are not Noetherian, this being related to the Hilbert's fourteenth problem. See [Muk01] and [Muk02].

In the present paper, we shall prove that the total coordinate ring is a unique factorization domain for any connected normal Noetherian scheme whose divisor class group is finitely generated free abelian group in Corollary 1.2.

The main theorem of the paper is the following:

Theorem 1.1. *Let X be a connected normal Noetherian scheme with function field K. Let $D_1, \ldots, D_r$ be Weil divisors on X, and let $t_1, \ldots, t_r$ be variables over K. We set*

$$R = \bigoplus_{n_1,\ldots,n_r\in\mathbb{Z}} H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_iD_i\Big)\Big) t_1^{n_1}\cdots t_r^{n_r} \subset S = K[t_1^{\pm 1}, \ldots, t_r^{\pm 1}].$$

1. *Then, R is a Krull domain.*
2. *Assume that $Q(R) = Q(S)$, where $Q(\ )$ stands for the field of fractions. Then there is a natural surjection*
$$\varphi : \mathrm{Cl}(X)\,/\langle \overline{D}_1, \ldots, \overline{D}_r\rangle \longrightarrow \mathrm{Cl}(R),$$
where $\overline{D}$ is the image in $\mathrm{Cl}(X)$ of a Weil divisor D on X.
3. *If there exist integers $n_1, \ldots, n_r$ such that $\sum_i n_iD_i$ is an ample divisor, then the map φ as above is an isomorphism.*

The following is an immediate consequence (ref. Remark 2.2) of the above theorem.

Corollary 1.2. *With the notation as in Theorem 1.1, R is a unique factorization domain if the set of the images of $D_1, \ldots, D_r$ generate $\mathrm{Cl}(X)$.*

In particular, the total coordinate ring is a unique factorization domain for a connected normal Noetherian scheme whose divisor class group is a finitely generated free abelian group.

We should mention that independently Berchtold and Hausen [BH02] proved that the total coordinate ring is a unique factorization domain for a locally factorial variety over an algebraically closed field whose Picard group is finitely generated free abelian group. Our method is purely algebraic, simple, and totally different from theirs. Here, we give a precise statement of the result of Berchtold and Hausen [BH02]. Let X be a normal variety over an algebraically closed field whose Picard group is finitely generated free abelian group. Let

$$\mathrm{A}(X) = \bigoplus_{\overline{D}\in\mathrm{Pic}(X)} H^0(X, \mathcal{O}_X(D))$$

be the ring defined in the same way as in Definition 2.1 using all Cartier divisors. Observe that if the divisor class group of X is finitely generated free abelian group, this is the subring of $\mathrm{TC}(X)$ [see (1) in Definition 2.1] consisting of those graded pieces corresponding to Cartier divisors. Then, Proposition 8.4 in [BH02] implies that the following four conditions are equivalent; (1) $\mathrm{A}(X)$ is a unique factorization domain, (2) X is locally factorial, (3) $\mathrm{Cl}(X) = \mathrm{Pic}(X)$, (4) $\mathrm{A}(X) = \mathrm{TC}(X)$. Let X_{reg} be the open subscheme consisting of non-singular points of X. Furthermore, assume that the divisor class group of X is finitely generated free abelian group. Since the codimension of $X \setminus X_{\mathrm{reg}}$ in X is at least 2, $\mathrm{TC}(X) = \mathrm{TC}(X_{\mathrm{reg}})$ is satisfied. Then, Proposition 8.4 in [BH02] implies that $\mathrm{TC}(X_{\mathrm{reg}})$ is a unique factorization domain since X_{reg} is locally factorial. Therefore we know that $\mathrm{TC}(X)$ is a unique factorization domain by the result of Berchtold and Hausen. Here, we remark that $\mathrm{A}(X_{\mathrm{reg}})$ coincides with $\mathrm{TC}(X_{\mathrm{reg}})$ since X_{reg} is locally factorial. On the other hand, $\mathrm{A}(X) = \mathrm{TC}(X)$ if and only if X is locally factorial.

We remark that we do not assume that a scheme is of finite type over an algebraically closed field in Corollary 1.2.

We shall prove Theorem 1.1 in the next section. In the final section, we give some examples of total coordinate rings. We prove that the total coordinate ring of the blow up of a projective space at a finite number of points on a line is finitely generated, see Example 3.3. Next we give an example of a normal projective variety X that has a non-finitely generated total coordinate ring in Example 3.4.

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2. The total coordinate ring

In this section we define the total coordinate ring and prove Theorem 1.1.

Definition 2.1. Let X be a connected normal Noetherian scheme with function field K. We assume that the divisor class group $\mathrm{Cl}(X)$ of X is isomorphic to $\mathbb{Z}^{\oplus s}$. Let $D_1, \ldots, D_s$ be Weil divisors on X such that the set of their images generate $\mathrm{Cl}(X)$. Let $t_1, \ldots, t_s$ be variables over K. We set

$$R(X; D_1, \ldots, D_s) = \bigoplus_{n_1, \ldots, n_s \in \mathbb{Z}} H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i\Big)\Big) t_1^{n_1} \cdots t_s^{n_s} \subset K[t_1^{\pm 1}, \ldots, t_s^{\pm 1}],$$

where we regard

$$H^0\Big(X, \mathcal{O}_X\Big(\sum_i n_i D_i\Big)\Big) = \{a \in K^\times \mid \mathrm{div}_X(a) + \sum_i n_i D_i \geq 0\} \cup \{0\}$$

as an additive subgroup of K. It is easily seen that $R(X; D_1, \ldots, D_s)$ is a subring of $K[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$. The ring $R(X; D_1, \ldots, D_s)$ is uniquely determined by X up to isomorphism, that is, it is independent of the choice of $D_1, \ldots, D_s$ up to isomorphism. We call it the *total coordinate ring* $\mathrm{TC}(X)$ of X. We may regard the total coordinate ring of X as a graded ring with the grading given by $\mathrm{Cl}(X)$. We sometimes write it as

$$\mathrm{TC}(X) = \bigoplus_{\overline{D} \in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D)), \tag{1}$$

where $\overline{D}$ denotes the class in $\mathrm{Cl}(X)$ represented by a Weil divisor D.

Before proving the main theorem, we give a remark.

Remark 2.2. With the notation given in Theorem 1.1, it is easily seen that $Q(R)$ coincides with $Q(S)$ if one of the following two conditions are satisfied; (1) the set of the images of D_1, $\ldots$, D_s generate $\mathrm{Cl}(X)$, (2) there exist integers $n_1, \ldots, n_r$ such that $\sum_i n_i D_i$ is an ample divisor.

Even if $Q(R) = Q(S)$ is satisfied, the map φ in Theorem 1.1 is not an isomorphism in general, as can be seen in the following example. Let $\pi : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at a point p. Put $E = \pi^{-1}(p)$. Let H be a hyperplane in $\mathbb{P}^2$ and put $A = \pi^{-1}(H)$. Let K be the function field of X and let t be a variable over K. Set

$$R = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(nA))t^n \subset S = K[t^{\pm 1}].$$

Using the projection formula, we have $H^0(X, \mathcal{O}_X(nA)) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(nH))$. Therefore $Q(R) = Q(S)$ is satisfied. In this case, $\mathrm{Cl}(X)$ is a $\mathbb{Z}$-free module of rank 2 with basis $\overline{E}$ and $\overline{A}$. Since R is a polynomial ring over the base field, we have $\mathrm{Cl}(R) = 0$. Therefore the map φ in Theorem 1.1 is not an isomorphism in this case.

For the remainder of this section, we prove Theorem 1.1.

Let H_1 be the set of reduced and irreducible closed subschemes of X of codimension 1. Put

$$D_i = \sum_{F \in H_1} m_{i,F} F$$

for $i = 1, \ldots, r$. Let $\mathcal{O}_{X,F}$ be the local ring of X at F, and let $\mathcal{M}_{X,F}$ be the corresponding maximal ideal. For $F \in H_1$, we denote by v_F the normalized valuation of the discrete valuation ring $\mathcal{O}_{X,F}$. Then we have

$$R = \bigoplus_{n_1, \ldots, n_r \in \mathbb{Z}} \{a \in K \mid v_F(a) + \textstyle\sum_i n_i m_{i,F} \geq 0 \text{ for each } F \in H_1\} t_1^{n_1} \cdots t_r^{n_r}.$$

Here we set

$$R_F = \bigoplus_{n_1, \ldots, n_r \in \mathbb{Z}} \{a \in K \mid v_F(a) + \textstyle\sum_i n_i m_{i,F} \geq 0\} t_1^{n_1} \cdots t_r^{n_r}$$

for $F \in H_1$. It is easy to see that R_F is also a subring of $S = K[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ such that $R \subset R_F \subset S$ for any $F \in H_1$, and that $R = \cap_{F \in H_1} R_F$ is satisfied.

Since the local ring $(\mathcal{O}_{X,F}, \mathcal{M}_{X,F})$ is a discrete valuation ring, there exists an element $\alpha_F \in \mathcal{M}_{X,F}$ such that $\mathcal{M}_{X,F} = \alpha_F \mathcal{O}_{X,F}$. By the equality

$$\{a \in K \mid v_F(a) + \textstyle\sum_i n_i m_{i,F} \geq 0\} = \alpha_F^{-\sum_i n_i m_{i,F}} \mathcal{O}_{X,F},$$

we have

$$\begin{aligned} R_F &= \bigoplus_{n_1,\dots,n_r \in \mathbb{Z}} \alpha_F^{-\sum_i n_i m_{i,F}} \mathcal{O}_{X,F} t_1^{n_1} \cdots t_r^{n_r} \\ &= \bigoplus_{n_1,\dots,n_r \in \mathbb{Z}} \mathcal{O}_{X,F} (\alpha_F^{-m_{1,F}} t_1)^{n_1} \cdots (\alpha_F^{-m_{r,F}} t_r)^{n_r} \\ &= \mathcal{O}_{X,F} \left[(\alpha_F^{-m_{1,F}} t_1)^{\pm 1}, \dots, (\alpha_F^{-m_{r,F}} t_r)^{\pm 1} \right]. \end{aligned}$$

Therefore, R_F is a Noetherian normal domain. We remark that $\alpha_F R_F$ is the unique homogeneous prime ideal of height 1 of R_F. Let $\{Q_\lambda \mid \lambda \in \Lambda\}$ be the set of non-homogeneous prime ideals of height 1 of R_F. Since R_F is a Krull domain,

$$R_F = (R_F)_{\alpha_F R_F} \bigcap \left(\bigcap_{\lambda \in \Lambda} (R_F)_{Q_\lambda} \right)$$

is satisfied (see Theorem 12.3 in Matsumura [Mat90]). Since $S = R_F[\alpha_F^{-1}]$, we have

$$S = \bigcap_{\lambda \in \Lambda} (R_F)_{Q_\lambda}$$

and

$$R = \cap_{F \in H_1} R_F = \left(\bigcap_{F \in H_1} (R_F)_{\alpha_F R_F} \right) \cap S. \tag{2}$$

Here for $F \in H_1$, we set $P_F = R \cap \alpha_F R_F$. Then we have

$$\begin{aligned} P_F &= \bigoplus_{n_1,\dots,n_r \in \mathbb{Z}} \left\{ a \in K \;\middle|\; \begin{array}{l} v_F(a) + \sum_i n_i m_{i,F} > 0, \text{ and} \\ v_G(a) + \sum_i n_i m_{i,G} \geq 0 \text{ for each } G \in H_1 \end{array} \right\} t_1^{n_1} \cdots t_r^{n_r} \\ &= \bigoplus_{n_1,\dots,n_r \in \mathbb{Z}} H^0(X, \mathcal{O}_X(\sum_i n_i D_i - F)) t_1^{n_1} \cdots t_r^{n_r}. \quad (3) \end{aligned}$$

We shall prove that R is a Krull domain. By equation (2), it is enough to show that for any $a \in R \setminus \{0\}$, a is a unit in $(R_F)_{\alpha_F R_F}$ except for finitely many F's. (Here, we remark that $Q(R)$ does not have to coincides with $Q(S)$.) We note that

$$\alpha_F (R_F)_{\alpha_F R_F} \cap R = \alpha_F R_F \cap R = P_F.$$

By equation (3) as above, it is easy to see that there exist only finitely many F's such that a is contained in P_F. We have thus proven that R is a Krull domain.

By Theorem 12.3 in Matsumura [Mat90], the set $\{P_F \mid F \in H_1\}$ includes the set of homogeneous prime ideals of height 1 of R. Set

$$H_1' = \{F \in H_1 \mid \mathrm{ht}_R P_F = 1\}.$$

Thus $\{P_F \mid F \in H'_1\}$ is the set of homogeneous prime ideals of height 1 of R. (Let E be the divisor given in Remark 2.2. Then, it is easy to see that $\mathrm{ht}_R P_E = 2$. Therefore $E \in H_1 \setminus H'_1$ in this case.) By Theorem 12.3 in [Mat90], we have

$$R = \left(\bigcap_{F \in H'_1} R_{P_F} \right) \cap S.$$

We denote by $\mathrm{Div}(X)$ the group of Weil divisors on X. It is the free abelian group generated by H_1. Let $\mathrm{P}(X) \subset \mathrm{Div}(X)$ be the subgroup of principal divisors

$$\{\mathrm{div}_X(a) \mid a \in K^\times\},$$

where

$$\mathrm{div}_X(a) = \sum_{F \in H_1} v_F(a) F.$$

By definition, $\mathrm{Cl}(X)$ is the quotient group $\mathrm{Div}(X)/\mathrm{P}(X)$. Let v'_F be the normalized valuation of the discrete valuation ring R_{P_F} for $F \in H'_1$.

For the remainder of the proof, we assume that $Q(R)$ coincides with $Q(S)$. We remark that $R_{P_F} = (R_F)_{\alpha_F R_F}$ is satisfied for $F \in H'_1$ since $Q(R) = Q(S)$. We denote by $\mathrm{HDiv}(R)$ the set of homogeneous Weil divisors on R. That is, it is the free abelian group generated by $\{P_F \mid F \in H'_1\}$. Let $\mathrm{HP}(R) \subset \mathrm{HDiv}(R)$ denote the subgroup generated by

$$\{\mathrm{div}_R(b) \mid b \text{ is a non-zero homogeneous element of } R\},$$

where

$$\mathrm{div}_R(b) = \sum_{F \in H'_1} v'_F(b) P_F.$$

Then, it is known that the natural map

$$\mathrm{HDiv}(R)/\mathrm{HP}(R) \to \mathrm{Cl}(R)$$

is an isomorphism (e.g., see Samuel [Sam64]).

Let $\xi : \mathrm{Div}(X) \to \mathrm{HDiv}(R)$ be the surjective homomorphism given by $\xi(F) = P_F$ for $F \in H'_1$ and $\xi(F) = 0$ for $F \in H_1 \setminus H'_1$. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{P}(X) & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & \mathrm{Cl}(X) & \longrightarrow 0 \\ & & & & \xi\downarrow & & & \\ 0 & \longrightarrow & \mathrm{HP}(R) & \longrightarrow & \mathrm{HDiv}(R) & \longrightarrow & \mathrm{Cl}(R) & \longrightarrow 0 \end{array}$$

For $F \in H'_1$ and $a \in K^\times$, $v_F(a) = v'_F(a)$ is satisfied since $R_{P_F} = (R_F)_{\alpha_F R_F}$. Therefore for $a \in K^\times$,

$$\xi(\mathrm{div}_X(a)) = \mathrm{div}_R(a)$$

is satisfied. Hence we have $\xi(\mathrm{P}(X)) \subset \mathrm{HP}(R)$.

By definition, the additive group $\mathrm{HP}(R)$ is generated by

$$\{\mathrm{div}_R(a) \mid a \in K^\times\} \quad \text{and} \quad \{\mathrm{div}_R(t_i) \mid i = 1, \ldots, r\},$$

since $Q(R) = Q(S)$. For $F \in H_1'$, since $R_{P_F} = (R_F)_{\alpha_F R_F}$ and

$$R_F = \mathcal{O}_{X,F}\left[(\alpha_F^{-m_{1,F}} t_1)^{\pm 1}, \dots , (\alpha_F^{-m_{r,F}} t_r)^{\pm 1}\right],$$

$v'_F(\alpha_F^{-m_{i,F}} t_i) = 0$ is satisfied. Therefore we have $v'_F(t_i) = m_{i,F}$. Hence for each i,

$$\xi(D_i) = \xi(\sum_{F \in H_1} m_{i,F} F) = \sum_{F \in H_1'} v'_F(t_i) P_F = \mathrm{div}_R(t_i)$$

is satisfied. Since $\mathrm{HP}(R)$ is generated by $\xi(\mathrm{P}(X))$ and $\{\xi(D_i) \mid i = 1, \dots , r\}$, we have an isomorphism

$$\begin{aligned} \mathrm{Cl}(R) &\simeq \mathrm{Div}(X)\,/\mathrm{P}(X) + \langle D_1, \dots , D_r\rangle + \langle F \mid F \in H_1 \setminus H_1'\rangle \\ &= \mathrm{Cl}(X)\,/\langle \overline{D_1}, \dots , \overline{D_r}\rangle + \langle \overline{F} \mid F \in H_1 \setminus H_1'\rangle\,. \end{aligned}$$

By this isomorphism, we arrive at the natural surjection

$$\varphi : \mathrm{Cl}(X)\,/\langle \overline{D_1}, \dots , \overline{D_r}\rangle \longrightarrow \mathrm{Cl}(R).$$

For the rest of this section, we shall prove that φ is an isomorphism if the subgroup $\langle D_1, \dots , D_r\rangle$ of $\mathrm{Div}(X)$ contains an ample divisor. We remark that $Q(R) = Q(S)$ is satisfied in this case. It is enough to show $H_1 = H_1'$. In order to show this, we need only show that $R_{P_F} = (R_F)_{\alpha_F R_F}$ for each $F \in H_1$. It is sufficient to show $R_F \subset R_{P_F}$ for each $F \in H_1$.

Let f be a homogeneous element of R_F. Since R_F is a graded ring such that $R \subset R_F \subset Q(R)$, it is easy to see that there exist $\alpha, \beta \in K^\times$ and integers $p_1, \dots , p_r, q_1, \dots , q_r$ such that

$$\alpha t_1^{p_1} \cdots t_r^{p_r}, \beta t_1^{q_1} \cdots t_r^{q_r} \in R \quad \text{and} \quad f = \frac{\beta t_1^{q_1} \cdots t_r^{q_r}}{\alpha t_1^{p_1} \cdots t_r^{p_r}} \in R_F.$$

Thus we have

$$\begin{aligned} \mathrm{div}_X(\alpha) + \sum_i p_i D_i &\geq 0 \\ \mathrm{div}_X(\beta) + \sum_i q_i D_i &\geq 0 \\ v_F(\beta/\alpha) + \sum_i (q_i - p_i) m_{i,F} &\geq 0. \end{aligned}$$

We want to show $f \in R_{P_F}$.

If $v_F(\alpha) + \sum_i p_i m_{i,F} = 0$, then $\alpha t_1^{p_1} \cdots t_r^{p_r} \in R \setminus P_F$ is satisfied. Therefore in this case, we have $f \in R_{P_F}$.

Assume that $v_F(\alpha) + \sum_i p_i m_{i,F} > 0$. Put $p = v_F(\alpha) + \sum_i p_i m_{i,F}$. We remark that $v_F(\beta) + \sum_i q_i m_{i,F} \geq p$. Since $\langle D_1, \dots , D_r\rangle$ contains an ample divisor, there exist $\gamma \in K^\times$ and integers $s_1, \dots , s_r$ such that

$$\begin{aligned} \mathrm{div}_X(\gamma) + \sum_i s_i D_i + pF &\geq 0 \\ v_F(\gamma) + \sum_i s_i m_{i,F} + p &= 0. \end{aligned}$$

Then we have

$$\begin{aligned}
\mathrm{div}_X(\beta\gamma) + \sum_i (q_i + s_i) D_i &\geq 0 \\
\mathrm{div}_X(\alpha\gamma) + \sum_i (p_i + s_i) D_i &\geq 0 \\
v_F(\alpha\gamma) + \sum_i (p_i + s_i) m_{i,F} &= 0.
\end{aligned}$$

Therefore we have $\beta\gamma t_1^{q_1+s_1} \cdots t_r^{q_r+s_r} \in R$ and $\alpha\gamma t_1^{p_1+s_1} \cdots t_r^{p_r+s_r} \in R \setminus P_F$. Since

$$f = \frac{\beta t_1^{q_1} \cdots t_r^{q_r}}{\alpha t_1^{p_1} \cdots t_r^{p_r}} = \frac{\beta\gamma t_1^{q_1+s_1} \cdots t_r^{q_r+s_r}}{\alpha\gamma t_1^{p_1+s_1} \cdots t_r^{p_r+s_r}},$$

we have $f \in R_{P_F}$.

This completes the proof of Theorem 1.1.

Remark 2.3. Yi Hu and Seán Keel [HKe00, Theorem 2.9] proved the following.

Let X be a $\mathbb{Q}$-factorial projective variety such that $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{N}^1(X)$. Then X is a Mori dream space if and only if TC(X) is finitely generated. If X is a Mori dream space then X is a GIT quotient of $V = spec(\mathrm{TC}(X))$ by the torus $G = \mathrm{Hom}(\mathbb{N}^r, \mathbb{G}_m)$

Here a Mori dream space is a variety with nice geometric properties. For example, the nef cone $\mathrm{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles, and there exist small $\mathbb{Q}$-factorial modifications of X.

3. Some examples

We give some examples of total coordinate rings in the section.

Example 3.1. It is well known that the divisor class group is a finitely generated free abelian group for a smooth complete toric variety (e.g., see 63p in [Ful93]). Furthermore in this case, Cox [Cox95] proved that TC(X) is a homogeneous polynomial ring. He called TC(X) the *homogeneous coordinate ring* of X.

Remark 3.2. Total coordinate rings have a deep relation with invariant theory as Mukai [Muk01] has shown. Here we present some of his results.

Let $\mathbb{P}^r$ be the projective space of dimension r over the field of complex numbers $\mathbb{C}$. Let X be the blow up of $\mathbb{P}^r$ at m distinct points. Then we have $\mathrm{Cl}(X) \simeq \mathbb{Z}^{m+1}$. Assume that the m points are not on any hyperplane in $\mathbb{P}^r$. Then with a suitable linear action of $G = \mathbf{G}_a^{m-r-1}$ on a polynomial ring S over $\mathbb{C}$ with $2m$ variables, the invariant subring S^G is isomorphic to $\mathrm{TC}(X)$.

On the other hand, Nagata [Nag61] proved that the effective cone $\mathrm{Eff}(X)$ in $\mathrm{Cl}(X)$ is not finitely generated as a semi-group if X is a blow up of $\mathbb{P}^2$ at 9 general points. Therefore, $\mathrm{TC}(X)$ is not Noetherian in this case. It is a counterexample of Hilbert's 14-th problem.

Mukai also generalized in [Muk02] a result of Dolgachev and realized the root system $T_{p,q,r}$ in the cohomology group of a certain rational variety of Picard number $p + q + r - 1$. As an application he proved that the invariant subring of a tensor product with an actions of Nagata type is infinitely generated if the Weyl group of the corresponding root system $T_{p,q,r}$ is infinite.

Example 3.3. Let $p_1, \dots, p_m$ be m points on a projective line contained in the projective r-space $\mathbb{P}^r$ over an algebraically closed field k. Then the total coordinate ring TC(X) of the blow up $X = Bl_{p_1,\dots,p_m}(\mathbb{P}^r)$ of $\mathbb{P}^r$ at $\{p_1, \dots, p_m\}$ is a Noetherian ring.

We give a proof:

Let $p_1, \dots, p_m$ be m distinct points in $\mathbb{P}^r$. Let $\pi : X \to \mathbb{P}^r$ be the blow up of $\mathbb{P}^r$ at $\{p_1, \dots, p_m\}$. Put $E_i = \pi^{-1}(p_i)$ for $i = 1, \dots, m$. Let H be a hyperplane in $\mathbb{P}^r$ and put $A = \pi^{-1}(H)$. Then $\mathrm{Cl}(X)$ is a free abelian group with basis $\overline{E_1}, \dots, \overline{E_m}, \overline{A}$.

Let $B = k[Z_0, \dots, Z_r]$ be the homogeneous coordinate ring of $\mathbb{P}^r$. We denote by I_i the homogeneous prime ideal of B corresponding to the point p_i. For $i = 1, \dots, m$ and $s \in \mathbb{Z}$, we define

$$F_{is} = \begin{cases} I_i^s & \text{if } s \geq 0 \\ B & \text{if } s < 0. \end{cases}$$

Then we have

$$[F_{1b_1} \cap \cdots \cap F_{mb_m}]_a = H^0(X, \mathcal{O}_X(aA - b_1E_1 - \cdots - b_mE_m)),$$

where $[\ ]_a$ denotes the homogeneous component of degree a. Therefore we have

$$\begin{aligned} \mathrm{TC}(X) &= \bigoplus_{a,b_1,\dots,b_m \in \mathbb{Z}} H^0(X, \mathcal{O}_X(aA - b_1E_1 - \cdots - b_mE_m)) \\ &= \bigoplus_{b_1,\dots,b_m \in \mathbb{Z}} (F_{1b_1} \cap \cdots \cap F_{mb_m}) T_1^{b_1} \cdots T_m^{b_m} \\ &\subset B[T_1^{\pm 1}, \dots, T_m^{\pm m}] \end{aligned}$$

For the rest of the proof, we assume that $\{p_1, \dots, p_m\}$ lie on a line in $\mathbb{P}^r$. We may assume

$$\begin{aligned} I_1 &= (f_1, Z_2, \dots, Z_r) \\ I_2 &= (f_2, Z_2, \dots, Z_r) \\ &\vdots \\ I_m &= (f_m, Z_2, \dots, Z_r), \end{aligned}$$

where $f_1, \cdots, f_m \in k[Z_0, Z_1]$ are linear forms such that any two elements in $\{f_1, \cdots, f_m\}$ are linearly independent over k.

Here, for $i = 1, \dots, m$ and $s \in \mathbb{Z}$, we put

$$f_{is} = \begin{cases} f_i^s & \text{if } s \geq 0 \\ 1 & \text{if } s < 0. \end{cases}$$

Let g be a polynomial in B. Put

$$g = \sum_{c_2,\dots,c_r \ge 0} g_{c_2\cdots c_r} Z_2^{c_2} \cdots Z_r^{c_r},$$

where $g_{c_2\cdots c_r} \in k[Z_0, Z_1]$. Then g is contained in F_{ib_i} if and only if $g_{c_2\cdots c_r} \in f_{i,b_i-c_2-\cdots-c_r} k[Z_0, Z_1]$ for any integers $c_2, \dots, c_r \ge 0$. Here, b_i or $b_i - c_2 - \cdots - c_r$ are possibly negative. Then g is contained in $F_{1b_1} \cap \cdots \cap F_{mb_m}$ if and only if $g_{c_2\cdots c_r} \in f_{i,b_i-c_2-\cdots-c_r} k[Z_0, Z_1]$ for any integers $c_2, \dots, c_r \ge 0$ and $i = 1, \dots, m$. Furthermore, this is equivalent to

$$g_{c_2\cdots c_r} \in f_{1,b_1-c_2-\cdots-c_r} \cdots f_{m,b_m-c_2-\cdots-c_r} k[Z_0, Z_1]$$

for any integers $c_2, \dots, c_r \ge 0$. In particular, $g \in F_{1b_1} \cap \cdots \cap F_{mb_m}$ if and only if $g_{c_2\cdots c_r} Z_2^{c_2} \cdots Z_r^{c_r} \in F_{1b_1} \cap \cdots \cap F_{mb_m}$ for any $c_2, \dots, c_r \ge 0$.

Here we claim that

$$\mathrm{TC}(X) = B[T_1^{-1}, \dots, T_m^{-1}, Z_2 T_1 \cdots T_m, \dots, Z_r T_1 \cdots T_m, f_1 T_1, \dots, f_m T_m]. \tag{4}$$

It is easy to see that the right-hand side is included in the left side. Assume that $g \in F_{1b_1} \cap \cdots \cap F_{mb_m}$. Then $g T_1^{b_1} \cdots T_m^{b_m}$ is contained in the left side. We want to show that the right side also contains it. We may assume that $g = g_1 Z_2^{c_2} \cdots Z_r^{c_r}$, where $g_1 \in k[Z_0, Z_1]$ and $c_1, \dots, c_r$ are non-negative integers. Since $g_1 Z_2^{c_2} \cdots Z_r^{c_r} \in F_{1b_1} \cap \cdots \cap F_{mb_m}$, we may assume that

$$g_1 = g_2 f_{1,b_1-c_2-\cdots-c_r} \cdots f_{m,b_m-c_2-\cdots-c_r},$$

where $g_2 \in k[Z_0, Z_1]$. Then

$$\begin{aligned} & g T_1^{b_1} \cdots T_m^{b_m} \\ = \ & g_2 f_{1,b_1-c_2-\cdots-c_r} \cdots f_{m,b_m-c_2-\cdots-c_r} Z_2^{c_2} \cdots Z_r^{c_r} T_1^{b_1} \cdots T_m^{b_m} \\ = \ & g_2 (Z_2 T_1 \cdots T_m)^{c_2} \cdots (Z_r T_1 \cdots T_m)^{c_r} \\ & \times (f_{1,b_1-c_2-\cdots-c_r} T_1^{b_1-c_2-\cdots-c_r}) \cdots (f_{m,b_m-c_2-\cdots-c_r} T_m^{b_m-c_2-\cdots-c_r}). \end{aligned}$$

Here, if $b_i - c_2 - \cdots - c_r < 0$, then

$$f_{i,b_i-c_2-\cdots-c_r} T_i^{b_i-c_2-\cdots-c_r} = (T_i^{-1})^{c_2+\cdots+c_r-b_i}.$$

If $b_i - c_2 - \cdots - c_r \ge 0$, then

$$f_{i,b_i-c_2-\cdots-c_r} T_i^{b_i-c_2-\cdots-c_r} = (f_i T_i)^{b_i-c_2-\cdots-c_r}.$$

Thus $g T_1^{b_1} \cdots T_m^{b_m}$ is contained in the ring on the right-hand side in (4), and this completes the proof.

Observe that if $m \ge 2$, then we obtain

$$\begin{aligned} & B[T_1^{-1}, \dots, T_m^{-1}, Z_2 T_1 \cdots T_m, \dots, Z_r T_1 \cdots T_m, f_1 T_1, \dots, f_m T_m] \\ = \ & k[T_1^{-1}, \dots, T_m^{-1}, Z_2 T_1 \cdots T_m, \dots, Z_r T_1 \cdots T_m, f_1 T_1, \dots, f_m T_m]. \end{aligned}$$

We shall give an example of a normal projective variety with infinitely generated total coordinate ring.

Example 3.4. Let us consider the weighted polynomial ring B_k over a field k in three variables x, y, z of degree a, b, c, respectively. Take the weighted projective plane

$$\mathbb{P}^2_k(a,b,c) = \mathrm{Proj}(B_k)$$

and consider the blow up $\pi : X_k(a,b,c) \to \mathbb{P}^2_k(a,b,c)$ at the smooth point

$$\mathfrak{p}_k(a,b,c) := \mathrm{Ker}\big(k[x,y,z] \xrightarrow{\varphi} k[t]\big),$$

where φ is the homomorphism of k-algebras defined by $\varphi(x) = t^a$, $\varphi(y) = t^b$ and $\varphi(z) = t^c$. We denote $X_k(a,b,c)$ and $\mathfrak{p}_k(a,b,c)$ simply by X_k and $\mathfrak{p}_k$, respectively. Using intersection theory on X_k, Cutkosky [Cut91] studied the finite generation of the symbolic Rees ring $R_s(\mathfrak{p}_k) = \oplus_{n\geq 0}\mathfrak{p}_k^{(n)}T^n \subset B_k[T]$. We remark that the total coordinate ring $\mathrm{TC}(X_k)$ is equal to $R_s(\mathfrak{p}_k)[T^{-1}]$. Furthermore, $\mathrm{TC}(X_k)$ is Noetherian ring if and only if $R_s(\mathfrak{p}_k)$ is.

Assume that $a = 7n-3, b = n(5n-2), c = 8n-3$ with $n \geq 4$ and $(n,3) = 1$. In this case, the total coordinate ring $\mathrm{TC}(X_k)$ is a Noetherian ring if and only if the characteristic of k is positive (see Goto, Nishida and Watanabe [GNW94]).

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